PEAKLESS MOTZKIN PATHS OF BOUNDED HEIGHT

HELMUT PRODINGER

ABSTRACT. There was recent interest in Motzkin paths without peaks (peak: up-step followed immediately by down-step); additional results about this interesting family is worked out. The new results are the enumeration of such paths that live in a strip $[0..\ell]$, and as consequence the asymptotics of the average height, which is given by $2\cdot 5^{-1/4}\sqrt{\pi n}$. Methods include the kernel method and singularity analysis of generating functions.

1. INTRODUCTION

Motzkin paths are cousins of the more famous Dyck paths. They appear first in [7]. In the encyclopedia [12] they are sequence A001006, with many references given. They consist of up-steps $U = (1, 1)$, down-steps $D = (1, -1)$ and horizontal (flat) steps $F = (1, 0)$. Slightly different notations are also in use. They start at the origin and must never go below the x-axis. Usually one requires the path to end on the x-axis as well, but occasionally one uses the term *Motzkin path* also for paths that end on a different level. Figure 1 shows all Motzkin paths of 4 steps (=length 4).

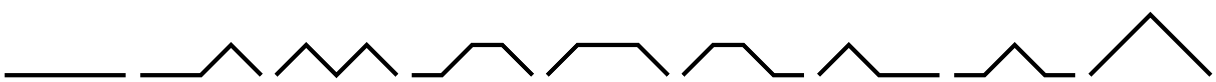

FIGURE 1. All 9 Motzkin of 4 steps (length 4).

An important concept is the *height* of a path. It is the maximal y-coordinate when scanning the path (from left to right, say). For the paths in Figure 1, the heights are (in this order) $0, 1, 1, 1, 1, 1, 1, 1, 2$. The average height of all Motzkin paths of length n was computed in an early paper of the present writer [9].

Recently, I learned from the paper [2] that there is interest in *peakless Motzkin paths.* A peak in a Motzkin path is a sequence of an up-step followed immediately by a down-step. Figure 2 indicates all peaks in the list of Motzkin paths of length 4. The enumerating sequence is A004148 in [12], where one can find several references; one recent paper about the subject is [3]. A general discussion about forbidden patterns, concentrating on analytic aspects, is in [1].

The paths without peaks are called peakless, and there are four of them, as shown in Figure 3.

2010 *Mathematics Subject Classification.* 05A15.
Key words and phrases. Motzkin paths, peakless, height, generating functions, asymptotics.

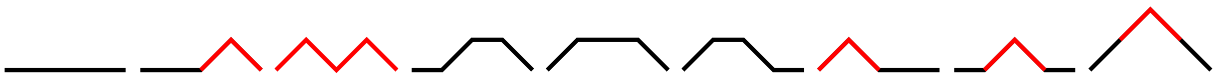

FIGURE 2. Motzkin paths with peaks indicated.

FIGURE 3. Peakless Motzkin paths of length 4.

2. A warmup: Enumeration of peakless Motzkin paths via the kernel method

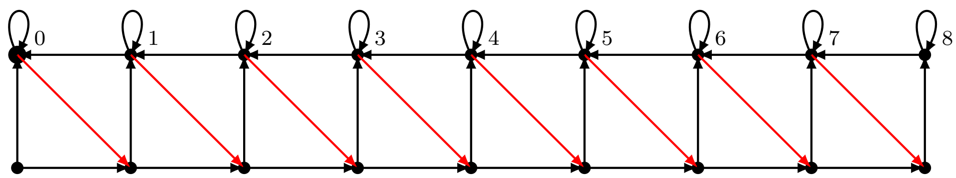

FIGURE 4. Graph (automaton) to recognize peakless Motzkin paths. Starting at the origin and ending at nodes labelled 0 corresponds to Motzkin paths, and ending at a node labelled k to a path that ends at level k.

We will use the following generating functions: $[z^n]f_i(z)$ is the number of peakless paths ending at state i in the top layer (Figure 4), $[z^n]g_i(z)$ is the number of peakless paths ending at state i in the bottom layer; for convenience, we mostly write f_i and g_i. The following recursions can be read off the automaton, by considering the last step separately.

$$\begin{aligned} f_0 &= 1 + zf_0 + zf_1 + zg_0, \\ f_i &= zf_i + zf_{i+1} + zg_i, \quad i \geq 1, \\ g_0 &= 0, \\ g_{i+1} &= zf_i + zg_i, \quad i \geq 0. \end{aligned}$$

To solve this system, one introduces double generating functions

$$F(u,z) = \sum_{i \geq 0} u^i f_i(z) \quad \text{and} \quad G(u,z) = \sum_{i \geq 0} u^i g_i(z).$$

Again, for convenience, we mostly write $F(u)$ and $G(u)$. By summing the recursions, we find

$$\begin{aligned} F(u) &= 1 + zF(u) + \frac{z}{u}(F(u) - F(0)) + zG(u), \\ G(u) &= zuF(u) + zuG(u) = \frac{zuF(u)}{1 - zu}. \end{aligned}$$

Eliminating one function, we are left to solve (note that $F(0) = f_0$)

$$F(u) = 1 + zF(u) + \frac{z}{u}(F(u) - F(0)) + \frac{z^2uF(u)}{1 - zu}.$$

Rewriting the functional equation, we find

$$F(u) = \frac{(-u + zF(0))(1 - zu)}{zu^2 + (z - z^2 - 1)u + z} = \frac{(-u + zF(0))(1 - zu)}{z(u - s_1)(u - s_2)}$$

with

$$s_1 = \frac{1 - z + z^2 + \sqrt{(1 + z + z^2)(1 - 3z + z^2)}}{2z} \tag{1}$$

and

$$s_2 = \frac{1 - z + z^2 - \sqrt{(1 + z + z^2)(1 - 3z + z^2)}}{2z}. \tag{2}$$

Note that $s_1s_2 = 1$. Plugging $u = 0$ into that equation does not help, but one of the factors from the denominator can be cancelled. This is (a simple instance of) the kernel method. Some twenty years ago, I collected various related examples [10], and many more just recently [11].

Since $s_2 = z + z^2 + \cdots$, the factor $(u - s_2)$ must cancel, since $1/(u - s_2)$ would not have a power series expansion around u, z close to zero. Performing the cancellation, we get

$$F(u) = \frac{zs_2 - 1 + zu - z^2F(0)}{z(u - s_1)} \quad \text{and} \quad F(0) = \frac{zs_2 - 1 - z^2F(0)}{-zs_1}.$$

Solving,

$$\begin{aligned} F(0) = f_0 = \frac{s_2}{z} &= \frac{1 - z + z^2 - \sqrt{(1 + z + z^2)(1 - 3z + z^2)}}{2z^2} \\ &= 1 + z + z^2 + 2z^3 + 4z^4 + 8z^5 + 17z^6 + \cdots \end{aligned}$$

We are mostly interested in

$$H(u) := F(u) + G(u) = \frac{-u + zF(0)}{zu + z - z^2u - u + zu^2},$$

although $F(u)$ and $G(u)$ could be computed separately as well. The coefficients of $[u^k]H(u)$ are just enumerating all peakless Motzkin paths ending on level k. Hence

$$H(u) = \frac{-u + zF(0)}{zu + z - z^2u - u + zu^2} = \frac{-1}{z(u - s_1)}.$$

Expanding and noting that $s_1 = 1/s_2$, we find

$$[u^k]H(u) = \frac{s_2^{k+1}}{z}.$$

This could be seen as well by a canonical decomposition of a peakless Motzkin path according to the last return to the x-axis.

The denominator of $H(u)$ has a special significance; if we write $h_k = [u^k]H(u)$, the recursion for these quantities can be read off from the denominator:

$$zh_k + (z - z^2 - 1)h_{k-1} + zh_{k-2} = 0, \tag{3}$$

this can be checked directly as well by inserting $h_k = s_2^{k+1}/z$ and simplifying.

The sequence enumerating peakless Motzkin paths is A004148 in [12]. If we call them $m(n) = [z^n]s_2/z$, then the software Gfun, implemented in Maple, produces the recursion

$$nm(n) - (2n+3)m(n+1) - (n+3)m(n+2) - (2n+9)m(n+3) + (n+6)m(n+4) = 0,$$

with initial values $m(0) = 1$, $m(1) = 1$, $m(2) = 1$, $m(3) = 2$.

A second order linear recursion with constant coefficients is driven by the characteristic equation and its two roots. Not surprisingly, they are s_1 and s_2.

For completeness, we mention the asymptotics of the coefficients $m(n)$ of

$$\frac{s_2}{z} = \frac{1 - z + z^2 - \sqrt{(1+z+z^2)(1-3z+z^2)}}{2z^2}.$$

This is a standard application of *singularity analysis of generating functions*, as described in [5]. First, we must consider the closest singularity to the origin. The candidates are the solutions of $(1+z+z^2)(1-3z+z^2) = 0$. There are two complex solutions of absolute value 1, which are irrelevant, and then $\frac{3\pm\sqrt{5}}{2}$. The relevant value is $\varrho = \frac{3-\sqrt{5}}{2}$; note that $1/\varrho = \frac{3+\sqrt{5}}{2}$, which is the square of the golden ratio $\phi = \frac{1+\sqrt{5}}{2}$ from the Fibonacci fame.

The local expansion around $z \sim \varrho$ looks like

$$\frac{s_2}{z} \sim \frac{1}{\varrho} - \frac{5^{1/4}}{\varrho}\sqrt{1 - \frac{z}{\varrho}},$$

and following the principles of singularity analysis we might translate this to the coefficients:

$$[z^n]\frac{s_2}{z} \sim \frac{5^{1/4}\varrho^{-n-1}}{2\sqrt{\pi}n^{3/2}}.$$

3. Peakless Motzkin paths of bounded height

We fix a parameter $\ell \ge 0$ and postulate that $[u^j]H(u) = 0$ for $j > \ell$. This means that states $\ell+1, \ell+2, \dots$ (on both layers) can never be reached. The recursion, compare (3)

$$zh_{k+1} + (z - z^2 - 1)h_k + zh_{k-1} = 0$$

is then best written as a matrix equation:

$$\begin{pmatrix} z-z^2-1 & z & 0 & 0 & \dots & \\ z & z-z^2-1 & z & 0 & 0 & \dots \\ 0 & z & z-z^2-1 & z & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \\ & & & & z & z-z^2-1 \end{pmatrix} \begin{pmatrix} h_0 \\ h_1 \\ h_2 \\ \vdots \\ h_\ell \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

Let $\mathscr{D}_\ell$ be the determinant of the $(\ell+1)\times(\ell+1)$ matrix. $\mathscr{D}_0 = z - z^2 - 1$, $\mathscr{D}_1 = (z-z^2-1)^2 - z^2 = (1-z)^2(1+z^2)$. It is a bit easier to work with $\mathscr{D}_{-1} = 1$ instead of $\mathscr{D}_1$. The solution is, by standard methods,

$$\mathscr{D}_\ell = -\frac{1}{W}(-zs_1)^{\ell+2} + \frac{1}{W}(-zs_2)^{\ell+2},$$

where we use the abbreviation $W = \sqrt{(1+z+z^2)(1-3z+z^2)}$.

The quantity h_0, describing *all* paths, restricted as described, can, by Cramer's rule, be written as

$$h_0 = \frac{-\mathscr{D}_{\ell-1}}{\mathscr{D}_\ell} = -\frac{-\frac{1}{W}(-zs_1)^{\ell+1} + \frac{1}{W}(-zs_2)^{\ell+1}}{-\frac{1}{W}(-zs_1)^{\ell+2} + \frac{1}{W}(-zs_2)^{\ell+2}}$$
$$= \frac{1}{z}\frac{-(-s_1)^{\ell+1} + (-s_2)^{\ell+1}}{(-s_1)^{\ell+2} - (-s_2)^{\ell+2}} = \frac{1}{z}\frac{s_1^{\ell+1} - s_2^{\ell+1}}{s_1^{\ell+2} - s_2^{\ell+2}}.$$

In the limit $\ell \to \infty$, $h_0 = \dfrac{1}{zs_1} = \dfrac{s_2}{z}$. This is, as we have seen already, the generating function of peakless Motzkin paths without boundary. The other functions h_i could be computed by Cramer's rule as well, but we concentrate only on the paths that return to the origin and are bounded by ℓ. They live in the strip $[0..\ell]$. At this stage, we drop the '0' from the notation and make the 'ℓ' explicit by writing $A_{n,\ell}$. We summarize the results:

Theorem 1. *The number of peakless Motzkin paths of length n (returning to the x-axis), bounded by ℓ, is given by*

$$A_{n,\ell} = [z^n]\frac{1}{z}\frac{s_1^{\ell+1} - s_2^{\ell+1}}{s_1^{\ell+2} - s_2^{\ell+2}},$$

with the functions s_1 and s_2 given in (1) and (2). The formula is only correct for $\ell \ge 1$; if $\ell = 0$, there is only one such path of length of n, namely consisting of flat steps only; the generating function must then be replaced by $\dfrac{1}{1-z}$.

We will use the notations ($\ell \ge 1$)

$$A_\ell(z) = \frac{1}{z}\frac{s_1^{\ell+1} - s_2^{\ell+1}}{s_1^{\ell+2} - s_2^{\ell+2}} \quad \text{and} \quad A_\infty(z) = \frac{s_2}{z}.$$

As can be checked directly (best with a computer), there is the recursion of the continued fraction type

$$A_\ell = \frac{1}{1 - z + z^2 - z^2 A_{\ell-1}},$$

and thus

$$A_1 = \cfrac{1}{1 - z + z^2 - \cfrac{z^2}{1 - z + z^2}}, \quad A_2 = \cfrac{1}{1 - z + z^2 - \cfrac{z^2}{1 - z + z^2 - \cfrac{z^2}{1 - z + z^2}}}$$

and so on. As discussed, the formula for $A_0 = \frac{1}{1-z}$ is slightly different.

Continued fraction expansions are very common in the context of generating functions of lattice paths of bounded height; the first example is (perhaps) [4].

The total generating function A_∞ has a pretty expansion as well, viz.

$$A_\infty = 1 + \cfrac{z}{1 - \cfrac{z}{1 - \cfrac{z}{1 - \cfrac{z^3}{1 - \cfrac{z}{1 - \cfrac{z}{1 - \cfrac{z^3}{1 - \ddots}}}}}}}$$

this can be checked directly. I am wondering whether this might have an easy combinatorial interpretation.

Now we consider the *average height* of peakless Motzkin paths of length n, assuming that all of them are equally likely. As always, when enumerating the average height, the relevant formula is

$$\frac{[z^n]\sum_{\ell\ge0}\big(A_\infty(z) - A_\ell(z)\big)}{[z^n]A_\infty(z)}.$$

The difference $A_\infty(z) - A_\ell(z)$ enumerates paths of height $> \ell$, or

$$\frac{s_2}{z} - \frac{1}{z}\frac{s_1^{\ell+1} - s_2^{\ell+1}}{s_1^{\ell+2} - s_2^{\ell+2}} = \frac{1}{z}\frac{(1-s_2^2)s_2^{\ell+1}}{s_1^{\ell+2} - s_2^{\ell+2}} = \frac{W}{z^2}\frac{s_2^{\ell+2}}{s_1^{\ell+2} - s_2^{\ell+2}}$$

This must be expanded around $z = \varrho$; we decrease ℓ by one, since then we enumerate paths of height $\ge \ell$. Since $s_2 \sim 1$, we find the approximation

$$\frac{W}{\varrho^2}\frac{1}{s_1^{\ell+1} - 1} = \frac{W}{\varrho^2}\frac{s_2^{\ell+1}}{1 - s_2^{\ell+1}} = \frac{W}{\varrho^2}\sum_{k\ge1} s_2^{(\ell+1)k}$$

A local expansion yields

$$\frac{W}{\varrho^2} \sim \frac{2\cdot 5^{1/4}}{\varrho}\Big(1 - \frac{z}{\varrho}\Big)^{1/2}.$$

From our earlier computation,

$$s_2 \sim 1 - 5^{1/4}\Big(1 - \frac{z}{\varrho}\Big)^{1/2}.$$

From [6] we conclude that

$$\sum_{k\ge1}\frac{s_2^k}{1 - s_2^k} \sim -\frac{\log(1 - s_2)}{1 - s_2},$$

as $s_2 \to 1$. Our paper [6] has many more technical details about a similar scenario. Putting both expansions together (we don't care about a missing term in the sum as we only want to work out the leading term),

$$\begin{aligned}\frac{W}{\varrho^2}\sum_{k,\ell\ge 1} s_2^{(\ell+1)k} &\sim \frac{2\cdot 5^{1/4}}{\varrho}\Big(1-\frac{z}{\varrho}\Big)^{1/2}\cdot\frac{-\log(1-s_2)}{1-s_2}\\ &\sim \frac{2\cdot 5^{1/4}}{\varrho}\Big(1-\frac{z}{\varrho}\Big)^{1/2}\cdot\frac{-\log\Big(5^{1/4}\big(1-\frac{z}{\varrho}\big)^{1/2}\Big)}{5^{1/4}\big(1-\frac{z}{\varrho}\big)^{1/2}}\\ &\sim -\frac{2}{\varrho}\log\Big(5^{1/4}\big(1-\frac{z}{\varrho}\big)^{1/2}\Big) \sim -\frac{2}{\varrho}\log\Big(1-\frac{z}{\varrho}\Big)^{1/2} \sim -\frac{1}{\varrho}\log\Big(1-\frac{z}{\varrho}\Big).\end{aligned}$$

By singularity analysis (transfer theorem) we find that

$$[z^n]\frac{W}{\varrho^2}\sum_{k,\ell\ge 1} s_2^{(\ell+1)k} \sim -[z^n]\frac{1}{\varrho}\log\Big(1-\frac{z}{\varrho}\Big) \sim \frac{\varrho^{-n-1}}{n}.$$

As discussed before, the total number of peakless Motzkin paths of length n is asymptotic to

$$[z^n]\frac{s_2}{z} \sim \frac{5^{1/4}\varrho^{-n-1}}{2\sqrt{\pi}n^{3/2}}.$$

and for the average height we have to consider the quotient of the last two expressions, which is

$$\frac{\varrho^{-n-1}}{n}\frac{2\sqrt{\pi}n^{3/2}}{5^{1/4}\varrho^{-n-1}} = \frac{2\sqrt{\pi n}}{5^{1/4}}.$$

The numerical constant $2/5^{1/4} = 1.337480610$. This can be compared with the average height of *all* Motzkin paths of length n, which is asymptotic to $\sqrt{\frac{\pi n}{3}}$, see [8]; $3^{-1/2} = 0.5773502693$.

References

[1] Andrei Asinowski, Axel Bacher, Cyril Banderier, and Bernhard Gittenberger. Analytic combinatorics of lattice paths with forbidden patterns, the vectorial kernel method, and generating functions for pushdown automata. *Algorithmica*, 82(3):386–428, 2020.

[2] Jean-Luc Baril and José Luis Ramírez. Fibonacci and Catalan paths in a wall. 2023.

[3] Naiomi Cameron and Everett Sullivan. Peakless Motzkin paths with marked level steps at fixed height. *Discrete Mathematics*, 344.1:112154, 2021.

[4] Nicolaas G. de Bruijn, Donald E. Knuth, and Stephen O. Rice. The average height of planted plane trees. In *Graph Theory and Computing*, pages 15–22. Academic Press, New York, 1972.

[5] Philippe Flajolet and Robert Sedgewick. *Analytic Combinatorics*. Cambridge University Press, Cambridge, 2009.

[6] Clemens Heuberger, Helmut Prodinger, and Stephan Wagner. The height of multiple edge plane trees. *Aequationes Math.*, 90(3):625–645, 2016.

[7] Theodore S. Motzkin. Relations between hypersurface cross ratios, and a combinatorial formula for partitions of a polygon, for permanent preponderance, and for non-associative products. *Bulletin of the American Mathematical Society*, 54 (4):352–360, 1948.
[8] Helmut Prodinger. The average height of a stack where three operations are allowed and some related problems. *J. Combin. Inform. System Sci.*, 5(4):287–304, 1980.
[9] Helmut Prodinger. The height of planted plane trees revisited. *Ars Combin.*, 16(B):51–55, 1983.
[10] Helmut Prodinger. The kernel method: a collection of examples. *Sém. Lothar. Combin.*, 50:Art. B50f, 19, 2003/04.
[11] Helmut Prodinger. A walk through my lattice path garden. *Sém. Lothar. Combin.*, 87b:Art. B87b,49, 2023.
[12] Neil J. A. Sloane and The OEIS Foundation Inc. The On-Line Encyclopedia of Integer Sequences, 2023.

Department of Mathematics, University of Stellenbosch 7602, Stellenbosch, South Africa and NITheCS (National Institute for Theoretical and Computational Sciences), South Africa.
Email address: hproding@sun.ac.za