SKEW DYCK PATHS WITHOUT UP–DOWN–LEFT

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ABSTRACT. Skew Dyck paths without up–down–left are enumerated. In a second step, the number of contiguous subwords 'up–down–left' are counted. This explains and extends results that were posted in the Encyclopedia of Integer Sequences.

1. INTRODUCTION

The entry A128729 in [5] is a bit mysterious. It presents some results about skew Dyck paths without a (contiguous) sequence 'up–down–left,' but without justification, and without reference to a research paper. The present note is designed to fill these gaps. First, skew Dyck paths are Dyck paths with additional left steps $(-1,-1)$ but without overlapping itself. We found it more convenient [4] to replace a left step by a red down-step $(1,-1)$, labelled red. So we might say that up–down–red is forbidden. In a second step, it won't be forbidden, but counted how often this happens. If it does not happen, it is the same as 'forbidden'. Probably the first paper that deals with skew Dyck paths is [1].

The Figure 1 describes skew Dyck paths. One can compute more: Paths that end at level k, by any of the 3 types of steps. Note that up–red and red–up are forbidden, as they would overlap itself.

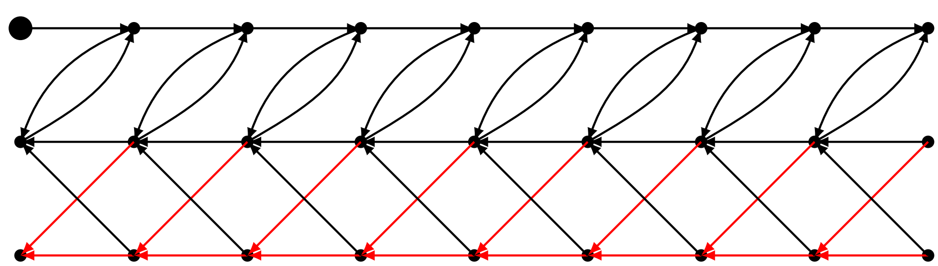

FIGURE 1. Three layers of states according to the type of steps leading to them (up, down-black, down-red).

The next step is to take care of up–down–red. Such a red step is depicted in Figure 2 in a special color.

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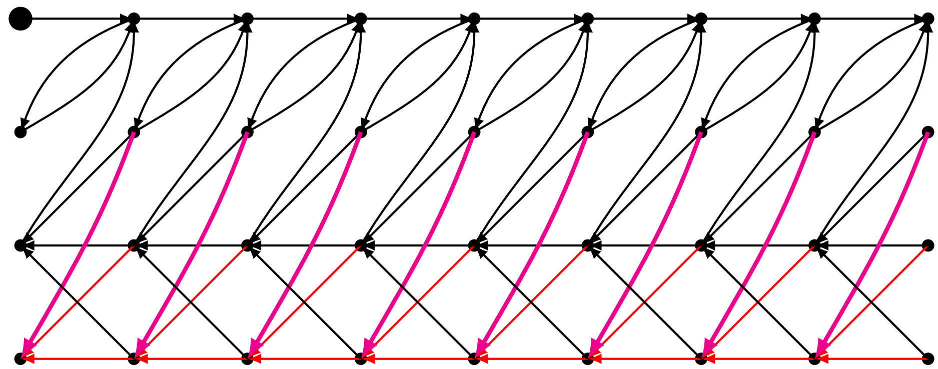

FIGURE 2. Four layers of states according to the type of steps leading to them.

2. The magenta colored steps are forbidden

We introduce 4 sequences of generating functions, according to the 4 layers in the defining graph. The variable z counts the steps. The recursions can be seen immediately from the graph:

$$\begin{aligned}
f_{n+1} &= zf_n + zg_n + zh_n,\ \ f_0 = 1,\\
g_n &= zf_{n+1},\\
h_n &= zg_{n+1} + zh_{n+1} + zk_{n+1},\\
k_n &= zh_{n+1} + zk_{n+1}.
\end{aligned}$$

And now we use a second variable u to be able to work with bivariate generating functions;

$$\begin{aligned}
\sum_{n\ge0} u^{n+1}f_{n+1} &= \sum_{n\ge0} zu^{n+1}f_n + \sum_{n\ge0} zu^{n+1}g_n + \sum_{n\ge0} zu^{n+1}h_n,\\
\sum_{n\ge0} u^{n+1}g_n &= \sum_{n\ge0} zu^{n+1}f_{n+1},\\
\sum_{n\ge0} u^{n+1}h_n &= \sum_{n\ge0} zu^{n+1}g_{n+1} + \sum_{n\ge0} zu^{n+1}h_{n+1} + \sum_{n\ge0} zu^{n+1}k_{n+1},\\
\sum_{n\ge0} u^{n+1}k_n &= \sum_{n\ge0} zu^{n+1}h_{n+1} + \sum_{n\ge0} zu^{n+1}k_{n+1}.
\end{aligned}$$

Translating this into bivariate generating functions $F(u) = F(u, z)$ etc., leads to

$$\begin{aligned}
F(u) &= 1 + zuF(u) + zuG(u) + zuH(u),\\
uG(u) &= zF(u) - z,\\
uH(u) &= zG(u) - zg_0 + zH(u) - zh_0 + zK(u) - zk_0,\\
uK(u) &= zH(u) - zh_0 + zK(u) - zk_0.
\end{aligned}$$

Solving the system leads to

$$F(u) = \frac{u^2z^2g_0 + u^2z^2h_0 + u^2z^2k_0 - uz^3 + 2zu - u^2 + u^2z^2 - uz^3g_0 - z^4}{-u^2 + zu^3 + 2zu - u^2z^2 - uz^3 - z^4},$$

$$G(u) = \frac{z^2\left(uzg_0 + uzh_0 + uzk_0 + 2zu - z^2g_0 - u^2\right)}{-u^2 + zu^3 + 2zu - u^2z^2 - uz^3 - z^4},$$

$$H(u) = \frac{-z\left(uz^2 - z^3 - g_0u - h_0u - k_0u + zg_0 + zu^2g_0 + u^2zh_0 + u^2zk_0 + uz^2h_0 + uz^2k_0 - z^3g_0\right)}{-u^2 + zu^3 + 2zu - u^2z^2 - uz^3 - z^4},$$

$$K(u) = \frac{-\left(u^2zk_0 + u^2zh_0 - h_0u + uz^2g_0 - k_0u + uz^2k_0 + uz^2h_0 - zg_0 + z^3 + z^3g_0 + k_0z^3 + h_0z^3\right)z}{-u^2 + zu^3 + 2zu - u^2z^2 - uz^3 - z^4},$$

One cannot just plug in $u = 0$ to identify the constants. However,

$$-u^2 + zu^3 + 2zu - u^2z^2 - uz^3 - z^4 = z(u - u_1)(u - u_2)(u - u_3),$$

and the bad factors $(u - u_2)(u - u_3)$ can be cancelled out, both, in the denominator and the numerator. This is essential for the kernel method [3]. This leads to

$$F(u) = \frac{z^2g_0 + z^2h_0 + z^2k_0 - 1 + z^2}{z(u - u_1)},$$

$$G(u) = \frac{-z}{(u - u_1)},$$

$$H(u) = \frac{-z(g_0 + h_0 + k_0)}{(u - u_1)},$$

$$K(u) = \frac{-z(h_0 + k_0)}{(u - u_1)}.$$

Now we can plug in $u = 0$ and solve to get

$$g_0 = \frac{z}{u_1},$$

$$h_0 = \frac{1 - z^2 - zu_1}{zu_1},$$

$$k_0 = \frac{1 - z^2 - zu_1}{u_1(u_1 - z)}.$$

In total

$$1 + g_0 + h_0 + k_0 = \frac{u_1 - z^2u_1 - z^3}{zu_1(u_1 - z)} = \frac{1 - zu_1}{z^2}$$
$$= 1 + z^2 + 2z^4 + 6z^6 + 20z^8 + 71z^{10} + 262z^{12} + 994z^{14} + 3852z^{16} + \cdots.$$

The last simplification was done by noticing that $u_1 = \mathsf{RootOf}(-u^2 + zu^3 + 2zu - u^2z^2 - uz^3 - z^4, u)$ and using Maple's evala command. (One can also use the algeqtoseries command to go from the algebraic equation to the series expansion.)

We further find

$$F + G + H + K = -\frac{u_1(1 - zu_1)}{z^2(u - u_1)}$$

and

$$[u^k](F + G + H + K) = \frac{1 - zu_1}{z^2 u_1^k}.$$

We summarize the results of this section:

Theorem 1. *The generating function of skew Dyck paths without up–down–left ending on level k is given by*

$$\frac{1 - zu_1}{z^2 u_1^k}$$

where

$$u_1 = \frac{1}{z} - z - z^3 - 2z^5 - 6z^7 - 20z^9 - 71z^{11} - 262z^{13} - \cdots$$

is the solution of the algebraic equation $-u^2 + zu^3 + 2zu - u^2z^2 - uz^3 - z^4 = 0$ that does not contain imaginary numbers in its series expansion around $z = 0$. Equivalently, the other solutions u_2, u_3 are such that $1/(u - u_2)$ and $1/(u - u_3)$ have no power series expansion around $(0, 0)$.

Noticing that one can only return to level 0 in an even number of steps, one can write an alternative form for $f_0 + g_0 + h_0 + k_0$:

$$\begin{aligned} &\mathsf{RootOf}(z^2u^3 - z(2 - z)u^2 + (1 - z^2)u - 1 + z + z^2, u) \\ &\quad = 1 + z + 2z^2 + 6z^3 + 20z^4 + 71z^5 + 262z^6 + 994z^7 + 3852z^8 + \cdots . \end{aligned}$$

We show how the algebraic equation for u_1 is transformed into one for $\frac{1-zu_1}{z^2}$; in the last step, $z^2 = Z$ was substituted:

$$\begin{aligned} u_1 &\longrightarrow -u^2 + zu^3 + 2zu - u^2z^2 - uz^3 - z^4 = 0, \\ zu_1 &\longrightarrow -(zu)^2 + (zu)^3 + 2z^2(zu) - z^2(zu)^2 - z^4(zu) - z^6 = 0, \\ -zu_1 &\longrightarrow -(-zu)^2 - (-zu)^3 - 2z^2(-zu) - z^2(-zu)^2 + z^4(-zu) - z^6 = 0, \\ 1 - zu_1 &\longrightarrow -(1 - zu - 1)^2 - (1 - zu - 1)^3 - 2z^2(1 - zu - 1) \\ &\qquad -z^2(1 - zu - 1)^2 + z^4(1 - zu - 1) - z^6 = 0, \\ &= 2(1 - zu)^2 - (1 - zu) - (1 - zu)^3 + z^2 - z^2(1 - zu)^2 + z^4(1 - zu) - z^4 - z^6 = 0, \\ \frac{1 - zu_1}{z^2} = U_1 &\longrightarrow 2z^4U^2 - z^2U - z^6U^3 + z^2 - z^6U^2 + z^6U - z^4 - z^6 = 0, \\ U_1 &\longrightarrow 2z^2U^2 - U - z^4U^3 + 1 - z^4U^2 + z^4U - z^2 - z^4 = 0, \\ U_1 &\longrightarrow 2ZU^2 - U - Z^2U^3 + 1 - Z^2U^2 + Z^2U - Z - Z^2 = 0. \end{aligned}$$

So, writing $S(z) = f_0 + g_0 + h_0 + k_0$, the algebraic equation is

$$z^2S^3 - z(2 - z)S^2 + (1 - z^2)S - 1 + z + z^2 = 0,$$

this is the form given in A128729 [5], and z counts only the half-length of the skew Dyck path, and Maple's gfun translates that into a differential equation

$$31z-8-15zS-(2z-1)(44z^3+15z^2-48z+8)S'-z(11z^2+16z-4)(2z-1)^2S''=0,$$

from which gfun derives a linear recursion with polynomial coefficients:

$$\begin{aligned}&-44n(n+1)s_n-2(n+1)(10n-7)s_{n+1}\\&\qquad+3(115+106n+23n^2)s_{n+2}-32(n+4)(n+3)s_{n+3}+4(n+5)(n+4)s_{n+4}=0,\end{aligned}$$

where

$$S(z)=\sum_{n\ge0}s_nz^n.$$

Asymptotics. The technique is *singularity analysis of generating functions*, as described in [2].

We consider $eq:=z^2S^3-z(2-z)S^2+(1-z^2)S-1+z+z^2$ and find the dominant singularity (z_0,S_0). For that, we consider the equation

$$\frac{eq}{dS}=3z^2S^2-2z(2-z)S+1-z^2=0$$

and thus $S_0=\frac{z+1}{3z}$. Plugging this into eq, we find $z_0=\frac{2}{11}(3\sqrt{3}-4)$ as closest solution to the origin. From this, $S_0=1+\frac{\sqrt{3}}{2}$. Expanding eq locally,

$$z-z_0\sim\frac{216-129\sqrt{3}}{121}\Big(S-1-\frac{\sqrt{3}}{2}\Big)^2,$$

or

$$(z_0-z)\Big(8+\frac{43}{9}\sqrt{3}\Big)=\Big(2+\frac{8}{9}\sqrt{3}\Big)\Big(1-\frac{z}{z_0}\Big)\sim\Big(S-1-\frac{\sqrt{3}}{2}\Big)^2.$$

This gives us the local expansion around the square-root singularity:

$$S\sim1+\frac{\sqrt{3}}{2}-\sqrt{2+\frac{8}{9}\sqrt{3}}\,\sqrt{1-\frac{z}{z_0}}.$$

Therefore

$$[z^n]S\sim\sqrt{2+\frac{8}{9}\sqrt{3}}\,\frac{1}{2\sqrt{\pi}}\Big(2+\frac{3}{2}\sqrt{3}\Big)^nn^{-3/2}.$$

3. Counting the up–down–red configurations

This is not too different from what we did before, but now we use another variable, t, which is attached to the magenta colored edges. In this way, the exponent of t refers to the number of up–down–red configurations;

$$\begin{aligned}f_{n+1}&=zf_n+zg_n+zh_n,\ f_0=1,\\g_n&=zf_{n+1},\\h_n&=zg_{n+1}+zh_{n+1}+zk_{n+1},\\k_n&=ztg_{n+1}+zh_{n+1}+zk_{n+1}.\end{aligned}$$

We use again a second variable u to be able to work with bivariate generating functions;

$$\begin{aligned}\sum_{n\geq 0} u^{n+1} f_{n+1} &= \sum_{n\geq 0} zu^{n+1} f_n + \sum_{n\geq 0} zu^{n+1} g_n + \sum_{n\geq 0} zu^{n+1} h_n,\\ \sum_{n\geq 0} u^{n+1} g_n &= \sum_{n\geq 0} zu^{n+1} f_{n+1},\\ \sum_{n\geq 0} u^{n+1} h_n &= \sum_{n\geq 0} zu^{n+1} g_{n+1} + \sum_{n\geq 0} zu^{n+1} h_{n+1} + \sum_{n\geq 0} zu^{n+1} k_{n+1},\\ \sum_{n\geq 0} u^{n+1} k_n &= \sum_{n\geq 0} ztu^{n+1} g_{n+1} + \sum_{n\geq 0} zu^{n+1} h_{n+1} + \sum_{n\geq 0} zu^{n+1} k_{n+1}.\end{aligned}$$

Translating this into bivariate generating functions $F(u) = F(u, z)$ etc., leads to

$$\begin{aligned}F(u) &= 1 + zuF(u) + zuG(u) + zuH(u),\\ uG(u) &= zF(u) - z,\\ uH(u) &= zG(u) - zg_0 + zH(u) - zh_0 + zK(u) - zk_0,\\ uK(u) &= ztG(u) - ztg_0 + zH(u) - zh_0 + zK(u) - zk_0.\end{aligned}$$

Solving the system leads to

$$\begin{aligned}F(u) &= \frac{\mathscr{F}}{2zu - z^4 - u^2 + z^4 t - uz^3 + zu^3 - u^2 z^2},\\ G(u) &= \frac{\mathscr{G}}{2zu - z^4 - u^2 + z^4 t - uz^3 + zu^3 - u^2 z^2},\\ H(u) &= \frac{\mathscr{H}}{2zu - z^4 - u^2 + z^4 t - uz^3 + zu^3 - u^2 z^2},\\ K(u) &= \frac{\mathscr{K}}{2zu - z^4 - u^2 + z^4 t - uz^3 + zu^3 - u^2 z^2},\end{aligned}$$

with $\mathscr{F} = u^2z^2h_0 + z^3utg_0 + 2zu - u^2 + z^4t - z^4 + u^2z^2k_0 + u^2z^2 - uz^3 + u^2z^2g_0 - uz^3g_0$, $\mathscr{G} = z^3uh_0 + z^4g_0t + z^3uk_0 + 2uz^3 + uz^3g_0 - z^4g_0 - u^2z^2$, $\mathscr{H} = z^2tg_0 - z^3utg_0 - z^4g_0t + z^4 - uz^3 - u^2z^2h_0 - z^4t + uzg_0 + uzh_0 + uzk_0 - z^2g_0 - u^2z^2k_0 - u^2z^2g_0 - z^3uh_0 - z^3uk_0 + z^4g_0$, $\mathscr{K} = uztg_0 - z^2u^2tg_0 - z^2tg_0 + z^4g_0t + z^4h_0t + z^4k_0t - z^4 - z^3ut + uzh_0 + uzk_0 + z^2g_0 - z^4g_0 - uz^3g_0 - z^4h_0 - z^3uh_0 - u^2z^2h_0 - z^4k_0 - z^3uk_0 - u^2z^2k_0 + z^4t$.

The denominator $2zu - z^4 - u^2 + z^4t - uz^3 + zu^3 - u^2z^2$ still factors as $z(u-u_1)(u-u_2)(u-u_3)$ where u_1, u_2, u_3 now depend on t. Again, the factor $(u-u_2)(u-u_3)$ can be cancelled out:

$$\begin{aligned}F(u) &= \frac{z^2g_0 + z^2h_0 + z^2k_0 - 1 + z^2}{z(u-u_1)},\\ G(u) &= \frac{-z}{(u-u_1)},\\ H(u) &= \frac{-z(g_0 + h_0 + k_0)}{(u-u_1)},\end{aligned}$$

$$K(u) = \frac{-z(tg_0 + h_0 + k_0)}{(u - u_1)}.$$

Now we can plug in $u = 0$ and compute the constants:

$$g_0 = \frac{z}{u_1},$$
$$h_0 = \frac{1 - z^2 - zu_1}{zu_1},$$
$$k_0 = \frac{(1 - z^2 - zu_1)(tu_1 - tz + z)}{zu_1(u_1 + tz - z)}.$$

In total

$$1 + g_0 + h_0 + k_0 = \frac{-tzu_1^2 + tu_1 + z^3t - z^2u_1 - z^3 + u_1}{zu_1(u_1 + tz - z)} = \frac{1 - zu_1}{z^2}.$$

The series starts as

$$1 + z^2 + (2+t)z^4 + (4t+6)z^6 + (16t+20)z^8 + (71+64t+2t^2)z^{10} + (262+261t+20t^2)z^{12} + \cdots .$$

One can compute the triple generating function, where z refers to the length of the walk, u to the level at the end, and t to the number of up–down–red configurations:

$$F(u)+G(u)+H(u)+K(u) = \frac{z^4t - u_1^2 + tz^2u_1^2 + z^3u_1 + z^2u_1^2 - 2u_1tz - t^2z^4}{u_1(u_1 + tz - z)z(u - u_1)} = \frac{u_1(zu_1 - 1)}{z^2(u - u_1)}.$$

Consequently

$$[u^k](F(u) + G(u) + H(u) + K(u)) = \frac{1 - zu_1}{z^2u_1^k}.$$

Since only an even number of steps brings us to level 0, we have

$$z^2R^3 - z(2 - z)R^2 + (1 - z^2)R - 1 + z + z^2 - tz^2 = 0$$

for $1 + g_0 + h_0 + k_0$ where z counts the half-length of the skew Dyck path. This is the form given in sequence A128728 [5].

References

[1] Emeric Deutsch, Emanuele Munarini, Simone Rinaldi. Skew Dyck paths. *J. Stat. Plann. Infer.*, 140 (8) (2010) 2191–2203.

[2] P. Flajolet and A. M. Odlyzko. Singularity analysis of generating functions. SIAM J. Discret. Math. 3:216–240, 1990.

[3] H. Prodinger, The kernel method: A collection of examples, *Sém. Lothar. Combin.*, **B50f** (2004), 19 pages.

[4] H. Prodinger, Partial skew Dyck paths—a kernel method approach, preprint, 2021.

[5] N. J. A. Sloane et al., The On-line Encyclopedia of Integer Sequences, 2022.

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