Lecture Notes on the course q–series in Combinatorics and Number Theory by Helmut Prodinger first given in the winter semester 2000 at Wits University, Johannesburg

This course was attended by Andrew van Antwerpen, Aubrey Blecher, Stewart Gebbie, Arnold Knopfmacher, Albert Tsifhumulo. Thanks for the patience.

1. Notation

$$(x;q)_n := \prod_{k=0}^{n-1}(1-xq^k), \quad (x;q)_\infty := \prod_{k\geq 0}(1-xq^k).$$

Note that

$$(x;q)_n = \frac{(x;q)_\infty}{(xq^n;q)_\infty}$$

2. First identities

Theorem 1 (Cauchy's formula)**.**

$$\sum_{n\geq 0}\frac{(a;q)_n t^n}{(q;q)_n} = \frac{(at;q)_\infty}{(t;q)_\infty}.$$

We don't consider convergence, work formally most of the time. Nevertheless, it is customary to assume $|q|<1$.

Often humans write $(x)_n$ instead of $(x;q)_n$.

Proof. Write

$$F(t) = \frac{(at;q)_\infty}{(t;q)_\infty} = \sum_{n\geq 0} A_n t^n.$$

Note that

$$F(t) = \frac{1-at}{1-t}F(tq) \quad \text{or} \quad (1-t)F(t) = (1-at)F(tq).$$

Hence

$$A_n - A_{n-1} = q^n A_n - aq^{n-1}A_{n-1}, \quad \text{or} \quad A_n = \frac{1-aq^{n-1}}{1-q^n}A_{n-1}.$$

Iteration together with $A_0 = 1$ gives the result

$$A_n = \frac{(a;q)_n}{(q;q)_n}.$$

□

Theorem 2 (Euler's partition identities)**.**

$$\sum_{n\geq 0}\frac{t^n}{(q;q)_n} = \frac{1}{(t;q)_\infty}$$

$$\sum_{n\geq 0}\frac{t^n q^{\binom{n}{2}}}{(q;q)_n} = (-t;q)_\infty$$

Proof. Plug in $a = 0$ and obtain the first one. For the second one, replace a by a/b and t by bt:

$$\sum_{n\geq 0} \frac{(b-a)(b-aq)\dots(b-aq^{n-1})t^n}{(q;q)_n} = \prod_{n\geq 0} \frac{1-atq^n}{1-btq^n}.$$

Now set $b = 0$ and $a = -1$,

$$\sum_{n\geq 0} \frac{1\cdot q\dots q^{n-1}t^n}{(q;q)_n} = \prod_{n\geq 0}(1+tq^n),$$

and we have it. □

Heine's formula:

$$\begin{aligned}
\sum_{n\geq 0} \frac{(a;q)_n(b;q)_n t^n}{(q;q)_n(c;q)_n} &= \frac{(b;q)_\infty}{(c;q)_\infty}\sum_{n\geq 0}\frac{(a;q)_n t^n}{(q;q)_n}\frac{(cq^n;q)_\infty}{(bq^n;q)_\infty} \\
&= \frac{(b;q)_\infty}{(c;q)_\infty}\sum_{n\geq 0}\frac{(a;q)_n t^n}{(q;q)_n}\sum_{m\geq 0}\frac{(c/b;q)_m b^m q^{nm}}{(q;q)_m} \qquad \text{(by Cauchy)} \\
&= \frac{(b;q)_\infty}{(c;q)_\infty}\sum_{m\geq 0}\frac{(c/b;q)_m b^m}{(q;q)_m}\sum_{n\geq 0}\frac{(a;q)_n t^n q^{nm}}{(q;q)_n} \\
&= \frac{(b;q)_\infty}{(c;q)_\infty}\sum_{m\geq 0}\frac{(c/b;q)_m b^m}{(q;q)_m}\frac{(atq^m;q)_\infty}{(tq^m;q)_\infty} \qquad \text{(by Cauchy)} \\
&= \frac{(b;q)_\infty(at;q)_\infty}{(c;q)_\infty(t;q)_\infty}\sum_{m\geq 0}\frac{(c/b;q)_m(t;q)_m b^m}{(q;q)_m(at;q)_m}
\end{aligned}$$

Heine's corollary:

$$\sum_{n\geq 0} \frac{(a;q)_n(b;q)_n(c/ab)^n}{(q;q)_n(c;q)_n} = \prod_{m\geq 0}\frac{(1-cq^m/a)(1-cq^m/b)}{(1-cq^m)(1-cq^m/ab)} = \frac{(c/a;q)_\infty(c/b;q)_\infty}{(c;q)_\infty(c/ab;q)_\infty}.$$

We use the previous corollary for the proof:

$$\begin{aligned}
\sum_{n\geq 0} \frac{(a;q)_n(b;q)_n(c/ab)^n}{(c;q)_n(q;q)_n} &= \frac{(b;q)_\infty(c/b;q)_\infty}{(c;q)_\infty(c/ab;q)_\infty}\sum_{n\geq 0}\frac{(c/ab;q)_n b^n}{(q;q)_n} \\
&= \frac{(b;q)_\infty(c/b;q)_\infty}{(c;q)_\infty(c/ab;q)_\infty}\cdot\frac{(c/a;q)_\infty}{(b;q)_\infty} = \frac{(c/a;q)_\infty(c/b;q)_\infty}{(c;q)_\infty(c/ab;q)_\infty}
\end{aligned}$$

Bailey's corollary:

$$\sum_{n\geq 0} \frac{(a;q)_n(b;q)_n(-q/b)^n}{(q;q)_n(aq/b;q)_n} = \prod_{m\geq 0}\frac{(1-aq^{2m+1})(1+q^{m+1})(1+aq^{2m+2}/b^2)}{(1-aq^{m+1}/b)(1+aq^{m+1}/b)}$$

We start with the corollary of Heine [his transformation formula] (interchanging a and b):

$$\begin{aligned}\sum_{n\geq 0}\frac{(b;q)_n(a;q)_n(-q/b)^n}{(q;q)_n(aq/b;q)_n} &= \frac{(a;q)_\infty(-q;q)_\infty}{(aq/b;q)_\infty(-q/b;q)_\infty}\sum_{m\geq 0}\frac{(q/b;q)_m(-q/b;q)_m a^m}{(q;q)_m(-q;q)_m}\\ &= \frac{(a;q)_\infty(-q;q)_\infty}{(aq/b;q)_\infty(-q/b;q)_\infty}\sum_{m\geq 0}\frac{(q^2/b^2;q^2)_m a^m}{(q^2;q^2)_m}\\ &= \frac{(a;q)_\infty(-q;q)_\infty}{(aq/b;q)_\infty(-q/b;q)_\infty}\frac{(aq^2/b^2;q^2)_\infty}{(a;q^2)_\infty}\\ &= \frac{(aq;q^2)_\infty(-q;q)_\infty(aq^2/b;q^2)_\infty}{(aq/b^2;q)_\infty(-q/b;q)_\infty}\end{aligned}$$

Note that we used

$$\frac{(a;q)_\infty}{(a;q^2)_\infty} = (aq;q^2)_\infty.$$

Cauchy's corollary:

$$\sum_{n\geq 0}\frac{q^{n(n-1)}z^n}{(q;q)_n(z;q)_n} = \prod_{m\geq 0}\frac{1}{1-zq^m} = \frac{1}{(z;q)_\infty}$$

In Heine's corollary, set $a=\alpha^{-1}$, $b=\beta^{-1}$, $c=z$:

$$\sum_{n\geq 0}\frac{(\alpha-1)(\alpha-q)\dots(\alpha-q^{n-1})(\beta-1)(\beta-q)\dots(\beta-q^{n-1})z^n}{(q;q)_n(z;q)_n} = \frac{(z\alpha;q)_\infty(z\beta;q)_\infty}{(z;q)_\infty(z\alpha\beta;q)_\infty}$$

Now set $\alpha=\beta=0$, and we have it.

Euler's corollary:

$$\sum_{n\geq 0}\frac{q^{n^2}}{\big((q;q)_n\big)^2} = \prod_{m\geq 1}\frac{1}{1-q^m} = \frac{1}{(q;q)_\infty}$$

This follows by setting $z=q$ in Cauchy's corollary.

Another corollary:

$$\sum_{n\geq 0}\frac{(a;q)_n q^{\binom{n+1}{2}}}{(q;q)_n} = \prod_{m\geq 1}(1-aq^{2m-1})(1+q^m)$$

Use Bailey's corollary and set $b=\beta^{-1}$:

$$\sum_{n\geq 0}\frac{(a;q)_n(\beta-1)(\beta-q)\dots(\beta-q^{n-1})(-q)^n}{(q;q)_n(aq\beta;q)_n}=\prod_{m\geq 0}\frac{(1-aq^{2m+1})(1+q^{m+1})(1+a\beta^2q^{2m+2})}{(1-a\beta q^{m+1})(1+a\beta q^{m+1})}$$

Now set $\beta=0$:

$$\sum_{n\geq 0}\frac{(a;q)_n q^{\binom{n}{2}+n}}{(q;q)_n}=\prod_{m\geq 0}(1-aq^{2m+1})(1+q^{m+1})$$

3. Jacobi's triple product

$$\sum_{n\in\mathbb{Z}}z^nq^{n^2}=\prod_{n\geq 0}(1-q^{2n+2})(1+zq^{2n+1})(1+z^{-1}q^{2n+1})$$

Proof: We start from Euler's second partition identity.

$$\begin{aligned}\prod_{n\geq 0}(1+zq^{2n+1})&=\sum_{m\geq 0}\frac{z^mq^{m^2}}{(q^2;q^2)_m}\\&=\frac{1}{(q^2;q^2)_\infty}\sum_{m\geq 0}z^mq^{m^2}(q^{2m+2};q^2)_\infty\\&=\frac{1}{(q^2;q^2)_\infty}\sum_{m\in\mathbb{Z}}z^mq^{m^2}(q^{2m+2};q^2)_\infty\end{aligned}$$

This, because all the new terms are actually zero.

We continue by expanding the product by Euler's identity:

$$\begin{aligned}\prod_{n\geq 0}(1+zq^{2n+1})&=\frac{1}{(q^2;q^2)_\infty}\sum_{m\in\mathbb{Z}}z^mq^{m^2}\sum_{r\geq 0}\frac{(-q^{2m+2})^rq^{r(r-1)}}{(q^2;q^2)_r}\\&=\frac{1}{(q^2;q^2)_\infty}\sum_{m\in\mathbb{Z}}z^mq^{m^2}\sum_{r\geq 0}\frac{(-1)^rq^{2mr+r+r^2}}{(q^2;q^2)_r}\\&=\frac{1}{(q^2;q^2)_\infty}\sum_{r\geq 0}\frac{(-1)^rq^{r+r^2}}{(q^2;q^2)_r}\sum_{m\in\mathbb{Z}}z^mq^{m^2}q^{2mr}\\&=\frac{1}{(q^2;q^2)_\infty}\sum_{r\geq 0}\frac{(-1)^rq^rz^{-r}}{(q^2;q^2)_r}\sum_{m\in\mathbb{Z}}z^{m+r}q^{(m+r)^2}\\&=\frac{1}{(q^2;q^2)_\infty}\sum_{r\geq 0}\frac{(-1)^rq^rz^{-r}}{(q^2;q^2)_r}\sum_{m\in\mathbb{Z}}z^mq^{m^2}\end{aligned}$$

$$= \frac{1}{(q^2;q^2)_\infty}\frac{1}{(-q/z;q^2)_\infty}\sum_{m\in\mathbb{Z}} z^m q^{m^2}$$

This is the desired result.

Replace q by $q^{k+\frac{1}{2}}$ and then set $z = -q^{k+\frac{1}{2}-i}$:

$$\sum_{n\in\mathbb{Z}}(-1)^n q^{n(k+\frac{1}{2}-i)} q^{(k+\frac{1}{2})n^2}$$
$$= \prod_{n\geq 0}(1-q^{(2k+1)(n+1)})(1-q^{k+\frac{1}{2}-i}q^{(k+\frac{1}{2})(2n+1)})(1-q^{-k-\frac{1}{2}+i}q^{(k+\frac{1}{2})(2n+1)})$$

Let us simplify:

$$\sum_{n\in\mathbb{Z}}(-1)^n q^{(2k+1)n(n+1)/2-in} = \prod_{n\geq 0}(1-q^{(2k+1)(n+1)})(1-q^{(2k+1)n+i})(1-q^{(2k+1)(n+1)-i})$$

And now

$$\sum_{n\geq 0}(-1)^n q^{(2k+1)n(n+1)/2-in}(1-q^{(2n+1)i})$$
$$= \sum_{n\geq 0}(-1)^n q^{(2k+1)n(n+1)/2-in} - \sum_{n\geq 0}(-1)^n q^{(2k+1)n(n+1)/2-in}q^{(2n+1)i}$$
$$= \sum_{n\geq 0}(-1)^n q^{(2k+1)n(n+1)/2-in} + \sum_{n\geq 1}(-1)^n q^{(2k+1)n(n-1)/2+in}$$
$$= \sum_{n\geq 0}(-1)^n q^{(2k+1)n(n+1)/2-in} + \sum_{n\leq -1}(-1)^n q^{(2k+1)n(n+1)/2-in}$$
$$= \sum_{n\in\mathbb{Z}}(-1)^n q^{(2k+1)n(n+1)/2-in}$$

(This is not really used! [at the moment!])

Let us now set $k = i = 1$ in:

$$\sum_{n\in\mathbb{Z}}(-1)^n q^{(2k+1)n(n+1)/2-in} = \prod_{n\geq 0}(1-q^{(2k+1)(n+1)})(1-q^{(2k+1)n+i})(1-q^{(2k+1)(n+1)-i})$$

We get

$$\sum_{n\in\mathbb{Z}}(-1)^n q^{\frac{n(3n+1)}{2}} = \prod_{n\geq 0}(1-q^{3n+3})(1-q^{3n+1})(1-q^{3n+2}) = \prod_{n\geq 1}(1-q^n)$$

Replacing n by $-n$ in the first sum we get the celebrated pentagonal theorem of Euler:

$$\sum_{n\in\mathbb{Z}}(-1)^n q^{\frac{n(3n-1)}{2}} = \prod_{n\geq 1}(1-q^n)$$

Put $z = -1$ in Jacobi's triple product to obtain (Gauss):

$$\begin{aligned}\sum_{n\in\mathbb{Z}}(-1)^n q^{n^2} &= \prod_{n\geq 0}(1-q^{2n+2})(1-q^{2n+1})(1-q^{2n+1})\\ &= (q^2;q^2)_\infty (q;q^2)_\infty (q;q^2)_\infty\\ &= (q;q)_\infty (q;q^2)_\infty\\ &= (q;q)_\infty \frac{(q;q)_\infty}{(q^2;q^2)_\infty}\\ &= (q;q)_\infty \frac{(q;q)_\infty}{(q;q)_\infty(-q;q)_\infty}\\ &= \frac{(q;q)_\infty}{(-q;q)_\infty} = \prod_{m\geq 1}\frac{1-q^m}{1+q^m}\end{aligned}$$

Here is another corollary of Gauss:

$$\begin{aligned}\sum_{n\geq 0} q^{n(n+1)/2} &= \tfrac{1}{2}\sum_{n\in\mathbb{Z}} q^{n(n+1)/2}\\ &= \tfrac{1}{2}\sum_{n\in\mathbb{Z}} (\sqrt{q}\,)^n (\sqrt{q}\,)^{n^2}\\ &= \tfrac{1}{2}\prod_{n\geq 0}(1-q^{n+1})(1+q^{n+1})(1+q^n)\\ &= \prod_{n\geq 0}(1-q^{n+1})(1+q^{n+1})(1+q^{n+1})\\ &= \prod_{n\geq 1}(1-q^n)(1+q^n)(1+q^n)\\ &= \prod_{n\geq 1}(1-q^{2n})\frac{1-q^{2n}}{1-q^n}\\ &= \prod_{n\geq 1}\frac{1-q^{2n}}{1-q^{2n-1}}\end{aligned}$$

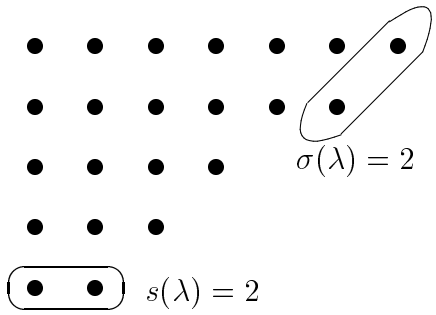
4. Franklin's proof of Euler's pentagonal theorem

Triangular numbers $1+2+3+\cdots+n$, square numbers $1+3+5+\cdots+2n-1$, pentagonal numbers $1+4+7+\cdots+(3n-2)=\frac{n(3n-1)}{2}$. Also $2+5+8+\cdots+(3n-1)=\frac{n(3n+1)}{2}$. Let $p_{\mathrm{e}}(\mathcal{D},n)$ denote the number of partitions of n into an even number of distinct parts. Let $p_{\mathrm{o}}(\mathcal{D},n)$ denote the number of partitions of n into an odd number of distinct parts. Then

$$p_{\mathrm{e}}(\mathcal{D},n)-p_{\mathrm{o}}(\mathcal{D},n)=\begin{cases}(-1)^m & \text{if } n=\frac{1}{2}m(3m\pm 1)\\ 0 & \text{otherwise.}\end{cases}$$

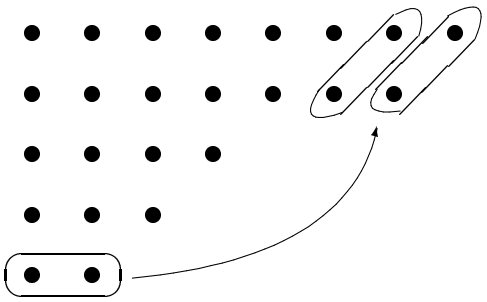
A bijection will be constructed, which is is almost always successful, but not when $n = \frac{1}{2}m(3m \pm 1)$; a single exceptional case will arise.

To each partition λ we will associate two numbers $s(\lambda)$ and $\sigma(\lambda)$. Instead of a formal definition, let us draw a picture for $\lambda = (76432)$. (This is often called a Ferrer's diagram.)

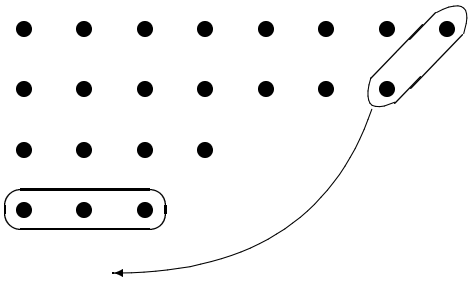


Observe that the number of parts is the number of rows.

We transform partitions as follows: If $s(\lambda) \leq \sigma(\lambda)$, then we do this:



If $s(\lambda) > \sigma(\lambda)$, then we do this:



This procedure changes the parity of the number of parts. Exactly one rule is applicable, so we have a one–to–one correspondence. However, there are certain partitions for which the mapping will not work.

Case 2 breaks down when the partition has r parts, being $2r, 2r-1, \ldots, r+1$. The number is then

$$(r+1)+(r+2)+\cdots+(2r)=\frac{1}{2}r(3r+1).$$

Case 1 breaks down when the partition has r parts, being $2r-1, 2r-2, \ldots, r$. The number is then

$$r+(r+1)+\cdots+(2r-1)=\frac{1}{2}r(3r-1).$$

This partition has no counterpart and is either in the even or odd count, depending on $(-1)^r$.

$$\begin{aligned}\sum_{m\in\mathbb{Z}}(-1)^m q^{\frac{1}{2}m(3m-1)} &= 1+\sum_{m\geq 1}(-1)^m q^{\frac{1}{2}m(3m-1)}+\sum_{m\leq -1}(-1)^m q^{\frac{1}{2}m(3m-1)}\\ &= 1+\sum_{m\geq 1}(-1)^m q^{\frac{1}{2}m(3m-1)}+\sum_{m\geq 1}(-1)^m q^{\frac{1}{2}m(3m+1)}\\ &= 1+\sum_{n\geq 1}\Big(p_{\mathrm{e}}(\mathcal{D},n)-p_{\mathrm{o}}(\mathcal{D},n)\Big)q^n.\end{aligned}$$

It remains to show that

$$1+\sum_{n\geq 1}\Big(p_{\mathrm{e}}(\mathcal{D},n)-p_{\mathrm{o}}(\mathcal{D},n)\Big)q^n=\prod_{n\geq 1}(1-q^n).$$

Well, that is not too hard. If we expand

$$\prod_{n\geq 1}(1-q^n)$$

we get all partitions with distinct parts, and they come with a sign. This sign depends on the parity of the number of parts, and that is all we need. —

Euler now does this:

$$1=\sum_{n\geq 0}p(n)q^n\left(1+\sum_{m\geq 1}(-1)^m q^{\frac{1}{2}m(3m-1)}+\sum_{m\geq 1}(-1)^m q^{\frac{1}{2}m(3m+1)}\right);$$

he compares coefficients!

$$\begin{aligned}p(n)&-p(n-1)-p(n-2)\\ &+p(n-5)+p(n-7)\\ &-p(n-12)-p(n-15)+\cdots=0\end{aligned}$$

That is a useful recursion formula to compute the partition numbers.

5. Combinatorial proof of the triple product after R. Chapman

It will be proved in the form

$$\prod_{n\geq 1}(1+xq^{2n-1})(1+x^{-1}q^{2n-1}) = \prod_{n\geq 1}\frac{1}{1-q^{2n}}\sum_{m\in\mathbb{Z}}x^m q^{m^2}. \tag{1}$$

Let $p(r)$ denote the number of partitions of the number r. The rhs of (1) is

$$\sum_{r\geq 0}\sum_{m\in\mathbb{Z}}p(r)x^m q^{m^2+2r}.$$

The lhs of (1) is

$$\sum_{m,t}A(m,t)x^m q^t$$

Then $A(m,t) = |\mathcal{A}(m,t)|$ where $\mathcal{A}(m,t)$ is the set of ordered pairs (B,C) of finite sets of positive odd integers with $|B|-|C| = m \in \mathbb{Z}$ and $\Sigma(B)+\Sigma(C) = t$, where $\Sigma(X)$ denotes the sum of the elements of X. Note

$$|B|-|C| \equiv \Sigma(B)+\Sigma(C) \mod 2.$$

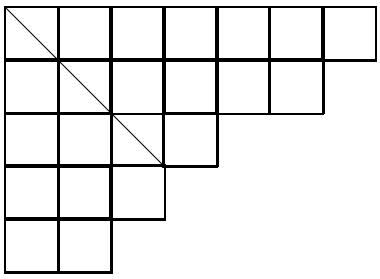
Also, if $|B|-|C| = m$, then at least one of the sets B, C has $\geq |m|$ elements. The smallest sum that can be formed is then $1+3+\ldots(2m-1) = m^2$. Thus $A(m,t) = 0$ unless $m \equiv t \mod 2$ and $t \geq m^2$.

We must thus prove that

$$A(m, m^2+2r) = p(r), \qquad m \in \mathbb{Z}, r \geq 0.$$

A bijection will be constructed between $\mathcal{P}(r)$, the set of partitions of r and $\mathcal{A}(m, m^2+2r)$.

Start with the instance $m = 0$. Let us represent $\mathcal{P}(r)$ by their Ferrer's diagram.

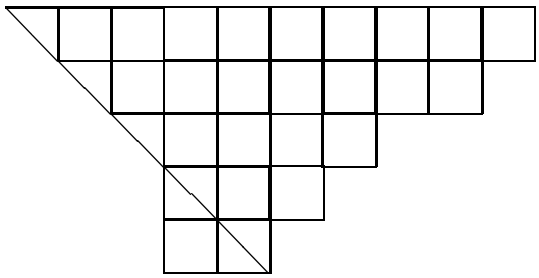


The following examples refers to $7+6+4+3+2$, a partition of 22. Draw the south–east diagonal as in the example. It divides the diagram into 2 regions. To the right, we have k strips of unit height (in the example $k = 3$). Call the areas from top to bottom $b_1/2, b_2/2, \ldots, b_k/2$. (In the example $b_1 = 13$, $b_2 = 9$, $b_3 = 3$.)

Do a similar thing with the left part, considering the k vertical strips of unit width, giving as the numbers $c_1/2, c_2/2, \ldots, c_k/2$. (In the example $c_1 = 9$, $c_2 = 7$, $c_3 = 3$.)

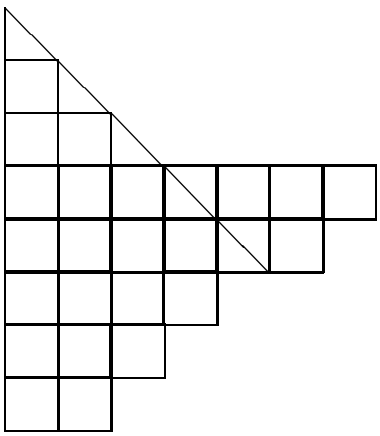
Set $B = \{3, 9, 13\}$ and $C = \{3, 7, 9\}$. These sets do the job, also in general.

Now let $m \geq 1$, as an example $m = 3$. Then we do this (see the example):



Let B have now 5 elements, in the example $\{1, 5, 9, 15, 19\}$. And $C = \{1, 3\}$. Twice the area of this diagram is $\Sigma(B) + \Sigma(C)$ plus $1 + 3 + \cdots + (2m - 1) = m^2$. Clearly $|B| - |C| = 3$.

If $m < 0$, then we do this:



(We might also think about interchanging the roles of B and C by mirroring the diagram along the south–east diagonal.)

6. Triple product and constant terms

Andrews does this:

$$\frac{1}{(q;q)_\infty} \sum_{n \in \mathbb{Z}} z^n q^{\binom{n+1}{2}} = (-zq;q)_\infty (-z^{-1};q)_\infty .$$

This form is obtained by replacing z by zq in

$$\frac{1}{(q;q)_\infty} \sum_{n \in \mathbb{Z}} z^n q^{\binom{n}{2}} = (-z;q)_\infty (-q/z;q)_\infty .$$

How is this form related to the traditional triple product? Replace q by q^2 in

$$\sum_{n \in \mathbb{Z}} z^n q^{\binom{n}{2}} = (q;q)_\infty (-z;q)_\infty (-q/z;q)_\infty .$$

to get

$$\sum_{n\in\mathbb{Z}} z^n q^{n^2-n} = \prod_{n\geq 0}(1-q^{2n+2})(1+zq^{2n})(1+z^{-1}q^{2n+2}).$$

Now replace z by zq (the steps are reversible):

$$\sum_{n\in\mathbb{Z}} z^n q^{n^2} = \prod_{n\geq 0}(1-q^{2n+2})(1+zq^{2n+1})(1+z^{-1}q^{2n+1}).$$

Denoting either side by $\varphi(z)$, it is claimed that the functional equation

$$\varphi(z) = zq\varphi(zq)$$

is satisfied. Let us check that:

$$\begin{aligned} zq\varphi(zq) &= \frac{zq}{(q;q)_\infty}\sum_{n\in\mathbb{Z}} z^n q^{n+\binom{n+1}{2}} \\ &= \frac{1}{(q;q)_\infty}\sum_{n\in\mathbb{Z}} z^{n+1} q^{\binom{n+2}{2}} \\ &= \frac{1}{(q;q)_\infty}\sum_{n\in\mathbb{Z}} z^n q^{\binom{n+1}{2}} = \varphi(z). \end{aligned}$$

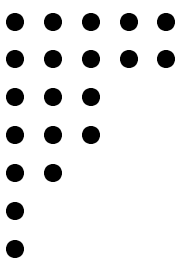
Also,

$$\begin{aligned} zq\varphi(zq) &= zq(-zq^2;q)_\infty(-1/zq;q)_\infty \\ &= zq\frac{1+\frac{1}{zq}}{1+zq}(-zq;q)_\infty(-1/z;q)_\infty \\ &= (-zq;q)_\infty(-1/z;q)_\infty = \varphi(z). \end{aligned}$$

It follows that the Laurent series of such a function is uniquely determined in a deleted neighbourhood of $z=0$ up to a constant factor. Hence we must check that the constant factors coincide. So we have to look at the coefficients of z^0. Find

$$[z^0](-zq;q)_\infty(-z^{-1};q)_\infty.$$

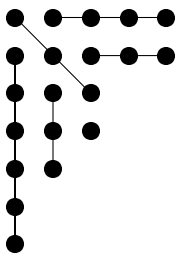
Now it is very similar to what Chapman does, but even attributed to Frobenius! Draw the Ferrer's diagram a bit differently:



And now like this:

To this associate its "Frobenius symbol"

$$\begin{pmatrix} 4 & 3 & 0 \\ 6 & 3 & 1 \end{pmatrix}.$$



Note that the sum of all the entries plus 3 (the number of elements in the diagonal is 20, the number that is partitioned.)

Every partition has thus a representation by a Frobenius symbol

$$\begin{pmatrix} a_1 & a_2 & \dots & a_r \\ b_1 & b_2 & \dots & b_r \end{pmatrix},$$

with $a_1 > a_2 > \cdots > a_r \geq 0$ and $b_1 > b_2 > \cdots > b_r \geq 0$.

Look at the constant term of

$$\prod_{n\geq 1}(1+zq^n)\prod_{m\geq 1}(1+z^{-1}q^{m-1}):$$

Multiply r terms from the first product and r terms from the second product:

$$q^{a_1+\cdots+a_r}\cdot q^{b_1+\cdots+b_r}.$$

This is just a Frobenius symbol

$$\begin{pmatrix} a_1-1 & a_2-1 & \dots & a_r-1 \\ & b_1 & b_2 & \dots & b_r \end{pmatrix}.$$

So we get as the constant term

$$\sum_{n\geq 0} p(n)q^n = \prod_{n\geq 1}\frac{1}{1-q^n} = \frac{1}{(q;q)_\infty},$$

as desired.

7. The triple product by Ismail and Ramanujan

Ramanujan's identity is:

$$\sum_{n\in\mathbb{Z}}\frac{(a;q)_n t^n}{(b;q)_n} = \frac{(b/a;q)_\infty (at;q)_\infty (q/at;q)_\infty (q;q)_\infty}{(q/a;q)_\infty (b/at;q)_\infty (b;q)_\infty (t;q)_\infty}$$

Ismail argues like this: Consider $b = q^N$.

$$\frac{1}{(b;q)_n} = \frac{1}{(q^N;q)_n} = \frac{(q^{N+n};q)_\infty}{(q^N;q)_\infty}$$

For $N+n \leq 0$ this vanishes. Therefore the identity means in this instance

$$\sum_{n \geq -N+1} \frac{(a;q)_n t^n}{(q^N;q)_n} = \frac{(q^N/a;q)_\infty (at;q)_\infty (q/at;q)_\infty (q;q)_\infty}{(q/a;q)_\infty (q^N/at;q)_\infty (q^N;q)_\infty (t;q)_\infty}$$

or

$$\frac{(1-q/a)\dots(1-q^{N-1}/a)}{(1-q)\dots(1-q^{N-1})} \sum_{n \geq -N+1} \frac{(a;q)_n t^n}{(q^N;q)_n} = (1-q/at)\dots(1-q^{N-1}/at)\frac{(at;q)_\infty}{(t;q)_\infty}$$

or

$$\frac{(1-a/q)\dots(1-a/q^{N-1})}{(1-q)\dots(1-q^{N-1})} \sum_{n \geq -N+1} \frac{(a;q)_n t^n}{(q^N;q)_n} = t^{-N+1}(1-at/q)\dots(1-at/q^{N-1})\frac{(at;q)_\infty}{(t;q)_\infty}$$

or

$$\sum_{n \geq -N+1} \frac{(a/q^{N-1};q)_{n+N-1} t^{n+N-1}}{(q;q)_{n+N-1}} = \frac{(at/q^{N-1};q)_\infty}{(t;q)_\infty}$$

or

$$\sum_{n \geq 0} \frac{(a/q^{N-1};q)_n t^n}{(q;q)_n} = \frac{(at/q^{N-1};q)_\infty}{(t;q)_\infty}.$$

But this follows from Cauchy's formula. (Sigh!)

For $|b|<1$ both sides constitute analytic functions that agree on an infinite sequence of points which converge to a limit in the interior of $|b|<1$. Hence the formula is true for all b with $|b|<1$. —

Now we replace t by t/a:

$$\sum_{n \in \mathbb{Z}} \frac{(a;q)_n t^n}{a^n (b;q)_n} = \frac{(b/a;q)_\infty (t;q)_\infty (q/t;q)_\infty (q;q)_\infty}{(q/a;q)_\infty (b/t;q)_\infty (b;q)_\infty (t/a;q)_\infty}.$$

Now let $a \to \infty$:

$$\sum_{n \in \mathbb{Z}} \frac{(-1)^n q^{\binom{n}{2}} t^n}{(b;q)_n} = \frac{(t;q)_\infty (q/t;q)_\infty (q;q)_\infty}{(b/t;q)_\infty (b;q)_\infty}.$$

Now set $b=0$:

$$\sum_{n \in \mathbb{Z}} (-1)^n q^{\binom{n}{2}} t^n = (t;q)_\infty (q/t;q)_\infty (q;q)_\infty.$$

Finally we replace t by $-t$ and get Jacobi's triple product:

$$\sum_{n \in \mathbb{Z}} q^{\binom{n}{2}} t^n = (-t;q)_\infty (-q/t;q)_\infty (q;q)_\infty.$$

8. Jacobi's triple product — an Euler like proof

Write

$$F(z) = \prod_{n\geq 1}(1-q^{2n})(1+zq^{2n-1})(1+z^{-1}q^{2n-1}) = \sum_{i\in\mathbb{Z}} c(i)z^i.$$

Now

$$F(zq^2) = \prod_{n\geq 1}(1-q^{2n})(1+zq^{2n+1})(1+z^{-1}q^{2n-3}) = \frac{1+\frac{1}{zq}}{1+zq}F(z)$$

or

$$F(z) = zq\,F(zq^2).$$

Compare coefficients:

$$c(n+1) = c(n)q^{2n+1};$$

since $c(n) = c(-n)$ everything boils down to finding $c(0)$, since

$$c(n) = c(0)\prod_{i=0}^{n-1} q^{2i+1} = c(0)q^{n^2}.$$

But

$$\begin{aligned} c(0) = [z^0]F(z) &= \prod_{n\geq 1}(1-q^{2n})[z^0]\prod_{n\geq 1}(1+zq^{2n-1})(1+z^{-1}q^{2n-1}) \\ &= \prod_{n\geq 1}(1-q^{2n})\sum_{i\geq 0}\left([z^i]\prod_{n\geq 1}(1+zq^{2n-1})\right)^2. \end{aligned}$$

Now observe that

$$\begin{aligned} \prod_{n\geq 1}(1+zq^{2n-1}) &= \prod_{n\geq 0}(1+(zq)q^{2n}) \\ &= \sum_{n\geq 0}\frac{(zq)^n q^{n(n-1)}}{(q^2;q^2)_n} = \sum_{n\geq 0}\frac{z^n q^{n^2}}{(q^2;q^2)_n}. \end{aligned}$$

Therefore we can continue

$$\begin{aligned} c(0) &= \prod_{n\geq 1}(1-q^{2n})\sum_{i\geq 0}\left([z^i]\prod_{n\geq 1}(1+zq^{2n-1})\right)^2 \\ &= \prod_{n\geq 1}(1-q^{2n})\sum_{i\geq 0}\frac{q^{2i^2}}{\left((q^2;q^2)_i\right)^2} \\ &= \prod_{n\geq 1}(1-q^{2n})\prod_{n\geq 1}\frac{1}{(1-q^{2n})} = 1. \end{aligned}$$

We have used "Euler's corollary"

$$\sum_{n\geq 0}\frac{q^{n^2}}{\left((q;q)_n\right)^2}=\prod_{m\geq 1}\frac{1}{1-q^m}=\frac{1}{(q;q)_\infty}.$$

9. Another example by this technique

Let us prove

$$\frac{1}{(z;q)_n}=\sum_{k\geq 0}z^k\frac{(q^n;q)_k}{(q;q)_k}. \tag{2}$$

Set

$$F_n(z)=\frac{1}{(z;q)_n}=\sum_{k\geq 0}c_n(k)z^k.$$

Then

$$F_n(zq)=\frac{1-z}{1-zq^n}F_n(z),$$

or

$$(1-zq^n)F_n(zq)=(1-z)F_n(z);$$

compare coefficients:

$$q^m c_n(m)-q^{n+m-1}c_n(m-1)=c_n(m)-c_n(m-1),$$

or

$$(q^m-1)c_n(m)-(q^{n+m-1}-1)c_n(m-1)=0.$$

This looks better as

$$c_n(m)=\frac{1-q^{n+m-1}}{1-q^m}c_n(m-1),$$

which holds for $m\geq 1$; $c_n(0)=1$. Thus

$$c_n(m)=\prod_{i=1}^{m}\frac{1-q^{n+i-1}}{1-q^i},$$

as desired.

10. A corollary of the triple product

Consider again

$$\begin{aligned}\prod_{n\geq 1}(1-xq^{2n+1})(1-x^{-1}q^{2n-1})(1-q^{2n}) &= \frac{1}{1-xq}\sum_{m\in\mathbb{Z}}(-1)^m x^m q^{m^2}\\ &= \frac{1}{1-xq}\left(1+\sum_{m\geq 1}q^{m^2}\left((-x)^m+(-x)^{-m}\right)\right)\\ &= 1+\sum_{m\geq 1}q^m\left(\sum_{i=1}^{M}\left((-x)^i+(-x)^{-i}\right)x^{m-i^2}+x^m\right),\end{aligned}$$

with $M=\lfloor\sqrt{m}\rfloor$. Now let $x=q^{-1}$, then

$$\begin{aligned}\prod_{m\geq 1}(1-q^{2m})^3 &= 1+\sum_{m\geq 1}\left(1+\sum_{i=1}^{M}(-1)^i\left(q^i+q^{-i}\right)q^{i^2}\right)\\ &= 1+\sum_{m\geq 1}\left(1+\sum_{i=1}^{M}(-1)^i\left(q^{i(i+1)}+q^{i(i-1)}\right)\right)\\ &= 1+\sum_{m\geq 1}(-1)^M q^{M(M+1)}.\end{aligned}$$

Now for a fixed value M there are several m's leading to that value: $m=M^2, M^2+1,\ldots,M^2+2M$. Hence

$$\prod_{m\geq 1}(1-q^{2m})^3=\sum_{M\geq 0}(-1)^M(2M+1)q^{M(M+1)}.$$

It looks better when we replace q^2 by q:

$$\prod_{m\geq 1}(1-q^{m})^3=\sum_{M\geq 0}(-1)^M(2M+1)q^{\binom{M+1}{2}}.$$

Another one:

$$\begin{aligned}F(q) &= q\prod_{m\geq 1}(1-q^m)^4\\ &= q\prod_{m\geq 1}(1-q^m)\prod_{m\geq 1}(1-q^m)^3\\ &= q\sum_{m\in\mathbb{Z}}(-1)^m q^{m(3m-1)/2}\sum_{k\geq 0}(-1)^k(2k+1)q^{\binom{k+1}{2}}.\end{aligned}$$

Assume $n \equiv 0 \mod 5$ and consider $[q^n]F(q)$. A contribution is possible for

$$1 + \frac{m(3m-1)}{2} + \frac{k(k+1)}{2} \equiv 0 \mod 5.$$

Or,

$$2 + 3m^2 - m + k^2 + k \equiv 0 \mod 5,$$

or (multiplying by $4 \equiv -1$)

$$3 + 2m^2 + m + 4k^2 + 4k \equiv 0 \mod 5,$$

or

$$2 + 2m^2 + m + (2k+1)^2 \equiv 0 \mod 5,$$

or

$$2 + 2m^2 - 4m + (2k+1)^2 \equiv 0 \mod 5,$$

or finally

$$2(m-1)^2 + (2k+1)^2 \equiv 0 \mod 5.$$

But we see almost immediately that this requires

$$m - 1 \equiv 0 \mod 5 \qquad \text{and} \qquad 2k + 1 \equiv 0 \mod 5.$$

And then $(-1)^{k+m}(2k+1)$ is a multiple of 5, and so $[q^n]F(q)$ is a multiple of 5 for n a multiple of 5. Now write

$$q \prod_{i \geq 1} \frac{1}{(1-q^i)} = F(q) \prod_{k \geq 1} \frac{1}{(1-q^k)^5}.$$

Observe that

$$\frac{1}{(1-z)^5} = \sum_{j \geq 0} \frac{(j+4)(j+3)(j+2)(j+1)}{24} z^j;$$

there is always a factor 5 in it except when it is like $4 \times 3 \times 2 \times 1$, in other words when j is a multiple of 5.

But we have

$$\frac{1}{(1-q)^5} \cdot \frac{1}{(1-q^2)^5} \cdots;$$

how can we have a coefficient that is not divisible by 5? Only if we can pick such a coefficient from each factor. In other words, we pick $q^{5j_1+10j_2+\cdots}$, but that is again of the form q^{5j}. Hence

$$[q^{5j}]q \prod_{i \geq 1} \frac{1}{(1-q^i)} \equiv 0 \mod 5,$$

which means

$$p(5j-1) \equiv 0 \mod 5,$$

or, nicer,

$$p(5n+4) \equiv 0 \mod 5.$$

Observe that we have a convolution of two series and look at the coefficient of a multiple of 5. There are contributions 0–0, 1–4, 2–3, 3–2, 4–1. In the last 4 instances, we get a multiple of 5 for the trivial reason; in the remaining case also, because of the first term.

The last one: Consider

$$\begin{aligned} G(q) &= q^2 \prod_{i\geq 1}(1-q^i)^6 = q^2\left(\sum_{i\geq 0}(-1)^i(2i+1)q^{\binom{i+1}{2}}\right)^2 \\ &= q^2 \sum_{j,k\geq 0}(-1)^{j+k}(2j+1)(2k+1)q^{\binom{j+1}{2}+\binom{k+1}{2}}. \end{aligned}$$

We have a contribution to $[q^{7j}]G(q)$ whenever

$$2+\frac{j(j+1)}{2}+\frac{k(k+1)}{2} \equiv 0 \mod 7,$$

or

$$4+j^2+j+k^2+k \equiv 0 \mod 7,$$

or

$$2+4j^2+4j+4k^2+4k \equiv 0 \mod 7,$$

or

$$(2j+1)^2+(2k+1)^2 \equiv 0 \mod 7.$$

From this we conclude

$$2j+1 \equiv 0 \mod 7 \qquad \text{and} \qquad 2k+1 \equiv 0 \mod 7.$$

But the contribution of such a term is

$$(-1)^{j+k}(2j+1)(2k+1)$$

which must be divisible by 7, by the above. Therefore 7 divides $[q^{7j}]G(q)$ for all j. But

$$q^2\prod_{i\geq 1}\frac{1}{(1-q^i)} = G(q)\prod_{k\geq 1}\frac{1}{(1-q^k)^7}.$$

Thus 7 divides

$$[q^{7j}]q^2\prod_{i\geq 1}\frac{1}{(1-q^i)} = p(7j-2).$$

Consequently

$$p(7n+5) \equiv 0 \mod 7.$$

11. Gaussian polynomials

Consider lattice paths on the first quadrant, starting at $(0,0)$ and moving to $(n-k,k)$ by n steps, each either one unit to the right of one up. We will split the $\binom{n}{k}$ paths according to the area under the curve. Since

$$(x+y)^2 = xx + xy + yx + yy,$$

we can keep track of the area by rewriting each term with the x's first and adding a unit area whenever yx is changed to xy. We will use a parameter q to do the counting, requiring that

$$yx = qxy.$$

Since we wish to collect the q's together, also assume that

$$yq = qy,$$
$$xq = qx.$$

Define the q–binomial coefficients (Gaussian polynomials) by

$$(x+y)^n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q x^{n-k} y^k,$$

when

$$yx = qxy, \qquad yq = qy, \qquad xq = qx.$$

First

$$(x+y)^{n+1} = (x+y)^n (x+y)$$

gives

$$\sum_{k=0}^{n} \begin{bmatrix} n+1 \\ k \end{bmatrix}_q x^{n+1-k} y^k = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q x^{n-k} y^k (x+y).$$

Since

$$y^k x = q^k x y^k$$

we have

$$\begin{bmatrix} n+1 \\ k \end{bmatrix}_q = \begin{bmatrix} n \\ k \end{bmatrix}_q q^k + \begin{bmatrix} n \\ k-1 \end{bmatrix}_q.$$

Similarly,

$$(x+y)^{n+1} = (x+y)(x+y)^n$$

gives

$$\begin{bmatrix} n+1 \\ k \end{bmatrix}_q = \begin{bmatrix} n \\ k \end{bmatrix}_q + q^{n+1-k} \begin{bmatrix} n \\ k-1 \end{bmatrix}_q.$$

We might take differences to get

$$0 = \begin{bmatrix} n \\ k \end{bmatrix}_q (1-q^k) - (1-q^{n+1-k}) \begin{bmatrix} n \\ k-1 \end{bmatrix}_q,$$

or

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{1-q^{n+1-k}}{1-q^k} \begin{bmatrix} n \\ k-1 \end{bmatrix}_q,$$

which can be iterated:

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(1-q^{n+1-k})\dots(1-q^n)}{(1-q^k)\dots(1-q^1)} \begin{bmatrix} n \\ 0 \end{bmatrix}_q.$$

But $\begin{bmatrix} n \\ 0 \end{bmatrix}_q = 1$, and so

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(1-q^{n+1-k})\dots(1-q^n)}{(1-q^k)\dots(1-q^1)} = \frac{(q;q)_n}{(q;q)_k (q;q)_{n-k}}.$$

Observe that (2) is now

$$\frac{1}{(z;q)_n} = \sum_k \begin{bmatrix} n+k-1 \\ k \end{bmatrix}_q z^k.$$

It can also be obtained by replacing a by q^n in Cauchy's theorem (check!)
Likewise, a replaced by q^{-n} leads to

$$\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q (-1)^k q^{\binom{k}{2}} x^k = (x;q)_n.$$

Defining

$$[n]_q = 1 + q + \dots + q^{n-1}$$

and

$$[n]_q! = [1]_q [2]_q \dots [n]_q,$$

we can also write

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q! [n-k]_q!}.$$

Now

$$(x+xy)^n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q x^{n-k} (xy)^k.$$

Observe

$$(xy)^k = xyxy\dots xy = x^k y^k q^{\binom{k}{2}}.$$

Also

$$\begin{aligned}(x+xy)\dots(x+xy)&\\ &= x(1+y)\dots x(1+y)x(1+y)\\ &= x(1+y)\dots x(1+y)x^2(1+qy)(1+y)\\ &= \dots = x^n(1+y)(1+qy)\dots(1+q^{n-1}y).\end{aligned}$$

Therefore

$$(1+y)(1+qy)\dots(1+q^{n-1}y) = \sum_{k=0}^{n} q^{\binom{k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q y^k.$$

Replace y with y/x to obtain

$$(x+y)(x+qy)\dots(x+q^{n-1}y) = \sum_{k=0}^{n} q^{\binom{k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q x^{n-k}y^k.$$

Let us prove a q–version of the Vandermonde formula:

We start from

$$\prod_{k=0}^{a+b-1}(1+xq^i) = \prod_{k=0}^{a-1}(1+xq^i)\prod_{k=a}^{a+b-1}(1+xq^i) = \prod_{k=0}^{a-1}(1+xq^i)\prod_{k=0}^{b-1}(1+(xq^a)q^i).$$

Now we read off coefficients on both sides:

$$\begin{bmatrix} a+b \\ n \end{bmatrix}_q q^{\binom{n}{2}} = \sum_{k=0}^{n} \begin{bmatrix} a \\ k \end{bmatrix}_q q^{\binom{k}{2}} \begin{bmatrix} b \\ n-k \end{bmatrix}_q q^{\binom{n-k}{2}} q^{(n-k)a}.$$

A rearrangement leads to

$$\begin{bmatrix} a+b \\ n \end{bmatrix}_q = \sum_{k=0}^{n} q^{(a-k)(n-k)} \begin{bmatrix} a \\ k \end{bmatrix}_q \begin{bmatrix} b \\ n-k \end{bmatrix}_q.$$

12. Gaussian polynomials and inversions

An inversion is here a pair $i<j$ with $a_i = 1$ and $a_j = 0$.

Denote

$$a_{n,k} = \sum_i [\text{number of words of length } n \text{ and } k \text{ zeros and } i \text{ inversions}]\, q^i.$$

Distinguishing the first letter, we find

$$a_{n,k} = a_{n-1,k-1} + q^k a_{n-1,k}.$$

Furthermore,

$$a_{n,0} = 1.$$

This shows that $a_{n,k} = \begin{bmatrix} n \\ k \end{bmatrix}_q$. And now count the inversions plus the number of pairs $(0,0)$.

For fixed k, the counting polynomial is

$$q^{\binom{k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q .$$

Summing on k, we consider the letter in position j and its contribution as the right letter of a pair $(0,0)$ or $(1,0)$. If it is 1, there is a contribution q^0, but if it is 0, there is a contribution q^{j-1}. This gives a combinatorial proof of the identity

$$\sum_{k=0}^{n} q^{\binom{k}{2}} x^k \begin{bmatrix} n \\ k \end{bmatrix}_q = \prod_{k=0}^{n-1} (1 + xq^k).$$

13. A finite version of the triple product

$$\prod_{i=1}^{n} (1 + xq^{2i-1})(1 + x^{-1}q^{2i-1}) = \sum_{k=-n}^{n} q^{k^2} x^k \begin{bmatrix} 2n \\ n+k \end{bmatrix}_{q^2}.$$

For the proof we use the finite expansions:

$$\begin{aligned} &\prod_{i=1}^{n} (1 + xq^i) \prod_{j=0}^{n-1} (1 + x^{-1}q^j) \\ &\quad = \sum_{m=0}^{n} (qx)^m q^{\binom{m}{2}} \begin{bmatrix} n \\ m \end{bmatrix}_q \sum_{l=0}^{n} x^{-l} q^{\binom{l}{2}} \begin{bmatrix} n \\ l \end{bmatrix}_q \\ &\quad = \dots \\ &\quad = \sum_{k=-n}^{n} x^k q^{\binom{k+1}{2}} \begin{bmatrix} 2n \\ n+k \end{bmatrix}_q . \end{aligned}$$

Now replace q by q^2 and then x by x/q:

$$\prod_{i=1}^{n} (1 + xq^{2i-1}) \prod_{j=0}^{n-1} (1 + x^{-1}q^{2j+1}) = \sum_{k=-n}^{n} x^k q^{k^2} \begin{bmatrix} 2n \\ n+k \end{bmatrix}_{q^2},$$

as desired.

Now take the limit $n \to \infty$:

$$\prod_{i \geq 1} (1 + xq^{2i-1}) \prod_{j \geq 0} (1 + x^{-1}q^{2j+1}) = \sum_{k=-\infty}^{\infty} x^k q^{k^2} \frac{1}{(q^2;q^2)_\infty}.$$

Observe that following limit relation: If n, k, and $n-k$ all go to infinity, then

$$\begin{bmatrix} n \\ k \end{bmatrix}_q \to \frac{(q;q)_\infty}{(q;q)_\infty (q;q)_\infty} = \frac{1}{(q;q)_\infty}.$$

If k is fixed, the limit is

$$\frac{1}{(q;q)_k}.$$

14. Watson's quintuple product

$$\prod_{n\geq 1}(1-\frac{1}{x}q^{n-1})(1-xq^n)(1-\frac{1}{x^2}q^{2n-1})(1-x^2q^{2n-1})(1-q^n)=\sum_{k\in\mathbb{Z}}q^{\frac{k(3k+1)}{2}}(x^{3k}-x^{-3k-1}).$$

We rewrite it as

$$\prod_{n\geq 1}(1-\frac{1}{x}q^{n-1})(1-xq^n)(1-\frac{1}{x^2}q^{2n-1})(1-x^2q^{2n-1})=\prod_{i\geq 1}\frac{1}{1-q^i}\sum_{k\in\mathbb{Z}}q^{\frac{k(3k+1)}{2}}(x^{3k}-x^{-3k-1}).$$

Recall the triple product:

$$\prod_{n\geq 1}(1-\frac{1}{x}q^{n-1})(1-xq^n)=\prod_{i\geq 1}\frac{1}{1-q^i}\sum_{k\in\mathbb{Z}}(-1)^kq^{\frac{k(k+1)}{2}}x^k.$$

We use that twice in the Watson product:

$$\prod_{i\geq 1}\frac{1}{1-q^i}\sum_{k\in\mathbb{Z}}(-1)^kq^{\frac{k(k+1)}{2}}x^k\times\prod_{i\geq 1}\frac{1}{1-q^{2i}}\sum_{k\in\mathbb{Z}}(-1)^kq^{k^2}x^{2k}$$
$$=\prod_{i\geq 1}\frac{1}{1-q^i}\sum_{k\in\mathbb{Z}}q^{\frac{k(3k+1)}{2}}(x^{3k}-x^{-3k-1}).$$

This is what we have to prove.

Rewrite it:

$$\sum_{k\in\mathbb{Z}}(-1)^kq^{\frac{k(k+1)}{2}}x^k\times\sum_{k\in\mathbb{Z}}(-1)^kq^{k^2}x^{2k}=\prod_{i\geq 1}(1-q^{2i})\sum_{k\in\mathbb{Z}}q^{\frac{k(3k+1)}{2}}(x^{3k}-x^{-3k-1}).$$

Now use Euler's pentagonal theorem in the form

$$\prod_{i\geq 1}(1-q^{2i})=\sum_{k\in\mathbb{Z}}(-1)^kq^{k(3k-1)}.$$

Thus we must prove

$$\sum_{k\in\mathbb{Z}}(-1)^kq^{\frac{k(k+1)}{2}}x^k\times\sum_{l\in\mathbb{Z}}(-1)^lq^{l^2}x^{2l}=\sum_{n\in\mathbb{Z}}(-1)^nq^{n(3n-1)}\sum_{m\in\mathbb{Z}}q^{\frac{m(3m+1)}{2}}(x^{3m}-x^{-3m-1}).$$

We work on the LHS:

$$\begin{aligned}
&\sum_{k,l}(-1)^{k+l}x^{k+2l}q^{\binom{k+1}{2}+l^2}\\
&\quad=\sum_m x^{3m}\sum_{k+2l=3m}(-1)^{k+l}q^{\binom{k+1}{2}+l^2}\\
&\quad+\sum_m x^{3m-1}\sum_{k+2l=3m-1}(-1)^{k+l}q^{\binom{k+1}{2}+l^2}\\
&\quad+\sum_m x^{3m-2}\sum_{k+2l=3m-2}(-1)^{k+l}q^{\binom{k+1}{2}+l^2}\\
&\quad=S_0+S_1+S_2.
\end{aligned}$$

For S_0 we make the change of variables $k+2l=3m$ and $l-m=n$. Hence $k+l=2m-n$ and $\binom{k+1}{2}+l^2=\frac{m(3m+1)}{2}+n(3n-1)$, and so

$$S_0=\sum_m q^{\frac{m(3m+1)}{2}}x^{3m}\sum_n(-1)^n q^{n(3n-1)}.$$

For S_1 we make the change of variables $k+2l=3m-1$ and $l-m=n$. Hence $k+l=2m-n-1$ and $\binom{k+1}{2}+l^2=\frac{m(3m-1)}{2}+n(3n+1)$, and so

$$\begin{aligned}
S_1&=-\sum_m q^{\frac{m(3m-1)}{2}}x^{3m-1}\sum_n(-1)^n q^{n(3n+1)}\\
&=-\sum_m q^{\frac{m(3m+1)}{2}}x^{-3m-1}\sum_n(-1)^n q^{n(3n-1)}.
\end{aligned}$$

For S_2 we make the change of variables $k+2l=3m-2$ and $l-m=n$. Hence $k+l=2m-n-2$ and $\binom{k+1}{2}+l^2=\frac{3m^2-3m+2}{2}+3n(n+1)$, and so

$$S_2=\sum_m q^{\frac{3m^2-3m+2}{2}}x^{3m-2}\sum_n(-1)^n q^{3n(n+1)}.$$

But

$$\sum_n(-1)^n(q^3)^{n(n+1)}=\sum_{n\geq 0}(-1)^n(q^3)^{n(n+1)}+\sum_{n\leq -1}(-1)^n(q^3)^{n(n+1)}$$

and also

$$\sum_n(-1)^n(q^3)^{n(n+1)}=\sum_{n\geq 0}(-1)^n(q^3)^{n(n+1)}+\sum_{n\geq 1}(-1)^n(q^3)^{n(n-1)}$$

and

$$\sum_n(-1)^n(q^3)^{n(n+1)}=\sum_{n\geq 0}(-1)^n(q^3)^{n(n+1)}-\sum_{n\geq 0}(-1)^n(q^3)^{(n+1)n}=0.$$

Hence $S_2=0$, and S_0+S_1 is exactly the righthandside of the quintuple product.

Write the formula like this:

$$\prod_{n\geq 1}(1-\frac{1}{x}q^n)(1-xq^n)(1-\frac{1}{x^2}q^{2n-1})(1-x^2q^{2n-1})(1-q^n)$$
$$= \frac{1}{1-\frac{1}{x}}\sum_{k\in\mathbb{Z}} q^{\frac{k(3k+1)}{2}}(x^{3k}-x^{-3k-1})$$

or

$$\prod_{n\geq 1}(1-\frac{1}{x}q^n)(1-xq^n)(1-\frac{1}{x^2}q^{2n-1})(1-x^2q^{2n-1})(1-q^n)$$
$$= \frac{x}{x-1}\sum_{k\in\mathbb{Z}} q^{\frac{k(3k+1)}{2}}(x^{3k}-x^{-3k-1}).$$

Now plug in $x=1$, which amounts to take a limit on the right hand side. The result is

$$\prod_{n\geq 1}(1-q^n)^3(1-q^{2n-1})^2 = \sum_{k\in\mathbb{Z}}(6k+1)q^{\frac{k(3k+1)}{2}}.$$

15. Septuple product of Farkas and Kra

For polynomials f and g in the variable n with integer coefficients let us define

$$\Omega(f) = \sum_{n\in\mathbb{Z}}(-1)^n q^{f(n)}$$

and

$$\Omega(f,g) = \sum_{n\in\mathbb{Z}}(-1)^n q^{f(n)}x^{g(n)}.$$

Theorem 3.

$$\prod_{n\geq 1}(1-q^{2n})^2(1-xq^{2n-2})(1-x^{-1}q^{2n})(1-x^2q^{4n-2})$$
$$\times(1-x^{-2}q^{4n-2})(1-x^2q^{4n-4})(1-x^{-2}q^{4n})$$
$$= \Omega(5n^2+n)\Big(\Omega(5n^2+3n,5n+3)+\Omega(5n^2-3n,5n)\Big)$$
$$-\Omega(5n^2+3n)\Big(\Omega(5n^2+n,5n+2)+\Omega(5n^2-n,5n+1)\Big).$$

Here we give an elementary derivation just from Jacobi's triple product formula, following Robin Chapman.

Let P denote the product on the LHS of the identity in Theorem 3. First observe that we can rewrite P as

$$P = \prod_{n\geq 1}(1-q^{2n})^2(1-xq^{2n-2})(1-x^{-1}q^{2n})(1-x^2q^{2n-2})(1-x^{-2}q^{2n}). \qquad (1)$$

We now use Jacobi's triple product formula in the form

$$\prod_{n\geq 1}(1-q^{2n})(1-xq^{2n-2})(1-x^{-1}q^{2n}) = \sum_{k\in\mathbb{Z}}(-1)^k x^k q^{k(k-1)}. \qquad (2)$$

Replacing x by x^2 in (2) gives

$$\prod_{n\geq 1}(1-q^{2n})(1-x^2q^{2n-2})(1-x^{-2}q^{2n}) = \sum_{k\in\mathbb{Z}}(-1)^k x^{2k}q^{k(k-1)}. \tag{3}$$

Multiplying (2) by (3) and using (1) gives

$$P = \sum_{k,l\in\mathbb{Z}}(-1)^{k+l}x^{k+2l}q^{k^2-k+l^2-l}.$$

Thus

$$P = \sum_{r\in\mathbb{Z}} a_r(q)x^r$$

where

$$a_r(q) = \sum_{k+2l=r}(-1)^{k+l}q^{k^2-k+l^2-l} = \sum_{l\in\mathbb{Z}}(-1)^{r-l}q^{5l^2-(4r-1)l+r^2-r}.$$

We now evaluate $a_r(q)$ by dividing into cases according to the residue of r modulo 5. If $r = 5m$ then

$$5l^2-(4r-1)l+r^2-r = 5(l-2m)^2+(l-2m)+5m^2-3m$$

and so

$$a_{5m}(q) = (-1)^m q^{5m^2-3m}\sum_{n\in\mathbb{Z}}(-1)^n q^{5n^2+n} = (-1)^m q^{5m^2-3m}\Omega(5n^2+n).$$

If $r = 5m+1$ then

$$5l^2-(4r-1)l+r^2-r = 5(2m-l)^2+3(2m-l)+5m^2-m$$

and so

$$a_{5m+1}(q) = -(-1)^m q^{5m^2-m}\sum_{n\in\mathbb{Z}}(-1)^n q^{5n^2+3n} = -(-1)^m q^{5m^2-m}\Omega(5n^2+3n).$$

If $r = 5m+2$ then

$$5l^2-(4r-1)l+r^2-r = 5(l-2m-1)^2+3(l-2m-1)+5m^2+m$$

and so

$$a_{5m+2}(q) = -(-1)^m q^{5m^2+m}\sum_{n\in\mathbb{Z}}(-1)^n q^{5n^2+3n} = -(-1)^m q^{5m^2+m}\Omega(5n^2+3n).$$

If $r = 5m+3$ then

$$5l^2-(4r-1)l+r^2-r = 5(2m+1-l)^2+(2m+1-l)+5m^2+3m$$

and so

$$a_{5m+3}(q) = (-1)^m q^{5m^2+3m}\sum_{n\in\mathbb{Z}}(-1)^n q^{5n^2+n} = (-1)^m q^{5m^2+3m}\Omega(5n^2+n).$$

If $r = 5m+4$ then

$$5l^2-(4r-1)l+r^2-r = 5(l-2m-2)^2+5(l-2m-2)+5m^2+5m+2$$

and so

$$a_{5m+4}(q) = (-1)^m q^{5m^2+5m+2}\sum_{n\in\mathbb{Z}}(-1)^n q^{5n^2+5n}.$$

But in this sum the terms for $n = j$ and $n = -j-1$ cancel and so $a_{5m+4}(q) = 0$. It follows that

$$\begin{aligned}
P &= \Omega(5n^2+n)\sum_{m\in\mathbb{Z}}(-1)^m\Big(q^{5m^2-3m}x^{5m}+q^{5m^2+3m}x^{5m+3}\Big)\\
&\quad-\Omega(5n^2+3n)\sum_{m\in\mathbb{Z}}(-1)^m\Big(q^{5m^2-m}x^{5m+1}+q^{5m^2+m}x^{5m+2}\Big)\\
&=\Omega(5n^2+n)\Big(\Omega(5n^2-3n,5n)+\Omega(5n^2+3n,5n+3)\Big)\\
&\quad-\Omega(5n^2+3n)\Big(\Omega(5n^2-n,5n+1)+\Omega(5n^2+n,5n+2)\Big)
\end{aligned}$$

which establishes Theorem 3.

16. Schur

Define

$$\begin{aligned}
F^{(0)}(n,k) &= \begin{bmatrix} n \\ k \end{bmatrix}_q - q^{n-2k+2}\begin{bmatrix} n \\ k-2 \end{bmatrix}_q\\
F^{(1)}(n,k) &= \begin{bmatrix} n \\ k \end{bmatrix}_q - q^{n-2k+3}\begin{bmatrix} n \\ k-3 \end{bmatrix}_q.
\end{aligned}$$

Then (almost provable by Maple)

$$F^{(0)}(n,k) = F^{(1)}(n-1,k) + q^{n-1}F^{(0)}(n-2,k-1).$$

Set

$$F_n = F^{\mathrm{odd}(n)}(n+1, \lfloor \tfrac{n+1}{2} \rfloor - j), \qquad G_n = F^{\mathrm{even}(n)}(n+1, \lfloor \tfrac{n}{2} \rfloor - j).$$

Then

$$F_{n-1} = F^{\mathrm{even}(n)}(n, \lfloor \tfrac{n}{2} \rfloor - j), \qquad G_{n-1} = F^{\mathrm{odd}(n)}(n, \lfloor \tfrac{n-1}{2} \rfloor - j),$$

and

$$F_{n-2} = F^{\mathrm{odd}(n)}(n-1, \lfloor \tfrac{n-1}{2} \rfloor - j), \qquad G_{n-2} = F^{\mathrm{even}(n)}(n-1, \lfloor \tfrac{n-2}{2} \rfloor - j).$$

Assume n even:

$$F_n = F^{(0)}(n+1, \tfrac{n}{2} - j), \qquad G_n = F^{(1)}(n+1, \tfrac{n}{2} - j),$$

$$F_{n-1} = F^{(1)}(n, \tfrac{n}{2} - j), \qquad G_{n-1} = F^{(0)}(n, \tfrac{n}{2} - j - 1),$$

$$F_{n-2} = F^{(0)}(n-1, \tfrac{n}{2} - j - 1), \qquad G_{n-2} = F^{(1)}(n-1, \tfrac{n}{2} - j - 1).$$

Hence

$$F_n = F_{n-1} + q^n F_{n-2}.$$

Assume n odd. Then

$$F_n = F^{(1)}(n+1, \tfrac{n+1}{2} - j), \qquad G_n = F^{(0)}(n+1, \tfrac{n-1}{2} - j),$$

$$F_{n-1} = F^{(0)}(n, \tfrac{n-1}{2} - j), \qquad G_{n-1} = F^{(1)}(n, \tfrac{n-1}{2} - j),$$

$$F_{n-2} = F^{(1)}(n-1, \tfrac{n-1}{2} - j), \qquad G_{n-2} = F^{(0)}(n-1, \tfrac{n-1}{2} - j - 1).$$

Hence

$$G_n = G_{n-1} + q^n G_{n-2}.$$

We can take suitable linear combinations, and will get these recursions for all n, but the coefficients must "fit." What does that actually mean?

$$R_n = \sum_{j\geq 0} f_j F^{\mathrm{odd}(n)}(n+1, \lfloor \tfrac{n+1}{2} \rfloor - \alpha_j) = \sum_{j\geq 0} g_j F^{\mathrm{even}(n)}(n+1, \lfloor \tfrac{n}{2} \rfloor - \beta_j).$$

Write that again as

$$\begin{aligned} R_n &= \sum_{j\geq 0} \left\{ f_j \begin{bmatrix} n+1 \\ \lfloor \frac{n+1}{2} \rfloor - \alpha_j \end{bmatrix}_q - f_j q^{2\alpha_j+3} \begin{bmatrix} n+1 \\ \lfloor \frac{n+1}{2} \rfloor - 2 - \alpha_j \end{bmatrix}_q \right\} \\ &= \sum_{j\geq 0} \left\{ g_j \begin{bmatrix} n+1 \\ \lfloor \frac{n}{2} \rfloor - \beta_j \end{bmatrix}_q - g_j q^{2\beta_j+4} \begin{bmatrix} n+1 \\ \lfloor \frac{n+1}{2} \rfloor - 3 - \beta_j \end{bmatrix}_q \right\}. \end{aligned}$$

Sufficient conditions are:

$$f_0 \begin{bmatrix} n+1 \\ \lfloor \frac{n+1}{2} \rfloor - \alpha_0 \end{bmatrix}_q = g_0 \begin{bmatrix} n+1 \\ \lfloor \frac{n}{2} \rfloor - \beta_0 \end{bmatrix}_q,$$

$$\begin{aligned} f_{j+1} \begin{bmatrix} n+1 \\ \lfloor \frac{n+1}{2} \rfloor - \alpha_{j+1} \end{bmatrix}_q &= -g_j q^{2\beta_j+4} \begin{bmatrix} n+1 \\ \lfloor \frac{n+1}{2} \rfloor - 3 - \beta_j \end{bmatrix}_q, \\ g_{j+1} \begin{bmatrix} n+1 \\ \lfloor \frac{n}{2} \rfloor - \beta_{j+1} \end{bmatrix}_q &= -f_j q^{2\alpha_j+3} \begin{bmatrix} n+1 \\ \lfloor \frac{n}{2} \rfloor - 2 - \alpha_j \end{bmatrix}_q. \end{aligned}$$

Observe that

$$\begin{bmatrix} n+1 \\ \lfloor \frac{n+1}{2} \rfloor - \alpha_0 \end{bmatrix}_q = \begin{bmatrix} n+1 \\ n+1 - \lfloor \frac{n+1}{2} \rfloor + \alpha_0 \end{bmatrix}_q = \begin{bmatrix} n+1 \\ \lfloor \frac{n}{2} \rfloor + 1 + \alpha_0 \end{bmatrix}_q.$$

Again, sufficient for that is

$$\begin{aligned} \alpha_0 + \beta_0 + 1 &= 0, \\ \alpha_{j+1} &= \beta_j + 3, \\ \beta_{j+1} &= \alpha_j + 2, \end{aligned}$$

and

$$\begin{aligned} f_0 &= g_0 = 1, \\ f_{j+1} &= -q^{2\beta_j+4} g_j, \\ g_{j+1} &= -q^{2\alpha_j+3} f_j. \end{aligned}$$

This, however, is easy to solve:

$$\begin{aligned} \alpha_{2j} &= 5j + \alpha_0, \\ \alpha_{2j+1} &= 5j + 2 - \alpha_0. \end{aligned}$$

One finds

$$f_j = (-1)^j q^{\frac{5j^2}{2}+(-1)^j\frac{j}{2}+(-1)^j 2j\alpha_0}$$

and

$$-g_{j+1} = q^{2\alpha_j+3} f_j = (-1)^j q^{\frac{5(j+1)^2}{2}+(-1)^j\frac{j+1}{2}+(-1)^j 2(j+1)\alpha_0}.$$

So

$$\begin{aligned} R_n = & \begin{bmatrix} n+1 \\ \lfloor \frac{n+1}{2} \rfloor - \alpha_0 \end{bmatrix}_q \\ & + \sum_{j\geq 1} (-1)^j q^{\frac{5j^2}{2}+(-1)^j\frac{j}{2}+(-1)^j 2j\alpha_0} \begin{bmatrix} n+1 \\ \lfloor \frac{n+1}{2} \rfloor - \alpha_j \end{bmatrix}_q \\ & - \sum_{j\geq 1} (-1)^j q^{\frac{5(j+1)^2}{2}+(-1)^j\frac{j+1}{2}+(-1)^j 2(j+1)\alpha_0} \begin{bmatrix} n+1 \\ \lfloor \frac{n}{2} \rfloor - 2 - \alpha_j \end{bmatrix}_q . \end{aligned}$$

Setting $\alpha_0 = 0$ we get

$$D_{n-1} = \sum_{j\in\mathbb{Z}} (-1)^j q^{\frac{j(5j-1)}{2}} \begin{bmatrix} n \\ \lfloor \frac{n+5j}{2} \rfloor \end{bmatrix}_q \tag{3}$$

or $(j \leftrightarrow -j)$

$$D_{n-1} = \sum_{j\in\mathbb{Z}} (-1)^j q^{\frac{j(5j+1)}{2}} \begin{bmatrix} n \\ \lfloor \frac{n-5j}{2} \rfloor \end{bmatrix}_q .$$

The initial conditions are $D_0 = 1$, $D_1 = 1 + q$.

Setting $\alpha_0 = -1$ we get

$$E_{n-1} = \sum_{j\in\mathbb{Z}} (-1)^j q^{\frac{j(5j-3)}{2}} \begin{bmatrix} n \\ \lfloor \frac{n-1+5j}{2} \rfloor \end{bmatrix}_q \tag{4}$$

or, using the symmetry of the Gaussian polynomials,

$$E_{n-1} = \sum_{j\in\mathbb{Z}} (-1)^j q^{\frac{j(5j-3)}{2}} \left[\begin{matrix} n \\ \lfloor \frac{n-5j+2}{2} \rfloor \end{matrix} \right]_q .$$

The initial conditions are $E_0 = 1$, $E_1 = 1$.

(I spotted small errors in Schur's paper.)

Schur considers the limit for $n \to \infty$.

From the combinatorial meaning of the Gaussian polynomials it follows, that, if both parameters go to infinity, it approaches $\frac{1}{(q;q)_\infty}$.

Since we have the factors $q^{\frac{j(5j+1)}{2}}$, we can fix j and show that the coefficients up to this value (which grows more than linearly in j) stabilise. In other words, consider

$$\left[\begin{matrix} n \\ \lfloor \frac{n}{2} \rfloor + \text{constant} \end{matrix} \right]_q ,$$

which has the limit as described, and we thus find

$$D_\infty = \frac{1}{(q;q)_\infty} \sum_{j\in\mathbb{Z}} (-1)^j q^{\frac{j(5j-1)}{2}},$$
$$E_\infty = \frac{1}{(q;q)_\infty} \sum_{j\in\mathbb{Z}} (-1)^j q^{\frac{j(5j-3)}{2}}.$$

Now take the famous triple product

$$\sum_{n\in\mathbb{Z}} z^n q^{n^2} = \prod_{n\geq 0} (1-q^{2n+2})(1+zq^{2n+1})(1+z^{-1}q^{2n+1})$$

replace q by $q^{5/2}$ and then z by $-q^{1/2}$. The result is

$$\sum_{n\in\mathbb{Z}} (-1)^n q^{\frac{n}{2}+\frac{5n^2}{2}} = \prod_{n\geq 0} (1-q^{5n+5})(1-q^{5n+3})(1-q^{5n+2}).$$

Therefore

$$D_\infty = \frac{1}{(q;q)_\infty} \prod_{n\geq 0} (1-q^{5n+5})(1-q^{5n+3})(1-q^{5n+2}).$$

But now we can cancel and get

$$D_\infty = \prod_{n\geq 0} \frac{1}{(1-q^{5n+1})(1-q^{5n+4})}.$$

Now let us also do that for E_∞:

$$E_\infty = \frac{1}{(q;q)_\infty} \sum_{j\in\mathbb{Z}} (-1)^j q^{\frac{j(5j-3)}{2}}.$$

Replace q by $q^{5/2}$ and then z by $-q^{-3/2}$ in the triple product. Result:

$$\sum_{n\in\mathbb{Z}} (-1)^n q^{\frac{-3n}{2}+\frac{5n^2}{2}} = \prod_{n\geq 0} (1-q^{5n+5})(1-q^{5n+1})(1-q^{5n+4}).$$

And therefore

$$E_\infty = \prod_{n\geq 0} \frac{1}{(1-q^{5n+2})(1-q^{5n+3})}.$$

Consider now the determinant of the infinite matrix

$$F(z) := \begin{vmatrix} 1 & zq & & & & \dots \\ -1 & 1 & zq^2 & & & \dots \\ & -1 & 1 & zq^3 & & \dots \\ & & -1 & 1 & zq^4 & \dots \\ & & & \ddots & \ddots & \ddots \end{vmatrix}.$$

Expanding the determinant with respect to the first row, we derive the following recursion (aka functional equation)

$$F(z) = F(zq) + zqF(zq^2);$$

setting

$$F(z) = \sum_{n\geq 0} a_n z^n$$

this means $a_0 = 1$ and

$$a_n = q^n a_n + q a_{n-1} q^{2n-2}$$

or

$$a_n = \frac{q^{2n-1}}{1-q^n} a_{n-1}.$$

Iteration leads to

$$a_n = \frac{q^{n^2}}{(q;q)_n}.$$

Therefore

$$F(z) = \sum_{n\geq 0} \frac{q^{n^2} z^n}{(q;q)_n};$$

in particular

$$F(1) = \sum_{n\geq 0} \frac{q^{n^2}}{(q;q)_n}.$$

On the other hand, $F(1)$ is the limit of the "finite" determinants

$$f_n := \begin{vmatrix} 1 & q & & & & \dots \\ -1 & 1 & q^2 & & & \dots \\ & -1 & 1 & q^3 & & \dots \\ & & -1 & 1 & q^4 & \dots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots \\ & & & -1 & 1 & q^n \\ & & & & -1 & 1 \end{vmatrix}.$$

Expanding with respect to the last row we get

$$f_n = f_{n-1} + q^n f_{n-2}.$$

$$f_0 = |1| = 1, \qquad f_1 = \begin{vmatrix} 1 & q \\ -1 & 1 \end{vmatrix} = 1 + q.$$

Therefore $f_n = D_n$, and

$$F(1) = \lim_{n\to\infty} f_n = D_\infty,$$

or

$$\sum_{n\geq 0} \frac{q^{n^2}}{(q;q)_n} = \prod_{n\geq 0} \frac{1}{(1-q^{5n+1})(1-q^{5n+4})}.$$

This is the first Rogers–Ramanujan identity.

For the second we do this.

$$F(q) = \sum_{n\geq 0} \frac{q^{n(n+1)}}{(q;q)_n}$$

and it is the limit of

$$g_n := \begin{vmatrix} 1 & q^2 & & & & \dots \\ -1 & 1 & q^3 & & & \dots \\ & -1 & 1 & q^4 & & \dots \\ & & -1 & 1 & q^5 & \dots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots \\ & & & -1 & 1 & q^n \\ & & & & -1 & 1 \end{vmatrix}.$$

Expanding with respect to the last row we get

$$g_n = g_{n-1} + q^n g_{n-2}.$$

$$g_1 = |1| = 1, \qquad g_2 = \begin{vmatrix} 1 & q^2 \\ -1 & 1 \end{vmatrix} = 1 + q^2.$$

We might add $g_0 = 1$; therefore $g_n = E_n$, and

$$F(q) = \lim_{n\to\infty} g_n = E_\infty,$$

or

$$\sum_{n\geq 0}\frac{q^{n(n+1)}}{(q;q)_n}=\prod_{n\geq 0}\frac{1}{(1-q^{5n+2})(1-q^{5n+3})}.$$

This is the second Rogers–Ramanujan identity.

Remark. We have the formula

$$\sum_{j\geq 0}q^{j^2}\begin{bmatrix}n+1-j\\ j\end{bmatrix}_q=\sum_{h\in\mathbb{Z}}(-1)^h q^{\frac{h(5h+1)}{2}}\begin{bmatrix}n+1\\ \lfloor\frac{n+1-5h}{2}\rfloor\end{bmatrix}_q=D_n.$$

Call the lefthandside f_n. We prove the recursion $f_n=f_{n-1}+q^n f_{n-2}$: Define

$$F(n,j):=q^{j^2}\begin{bmatrix}n+1-j\\ j\end{bmatrix}_q,$$

then

$$F(n,j)=F(n-1,j)+q^n F(n-2,j-1).$$

This can be verified by a computer or otherwise. Summing that over $j\geq 1$ and adding 1 (for $j=0$) on both sides, we have the claim. We find likewise that $f_0=1$ and $f_1=1+q$. This finishes the proof.

There is also such a formula for the other quantities; it might be an exercise:

$$\sum_{j\geq 0}q^{j(j+1)}\begin{bmatrix}n-j\\ j\end{bmatrix}_q=\sum_{h\in\mathbb{Z}}(-1)^h q^{\frac{h(5h-3)}{2}}\begin{bmatrix}n+1\\ \lfloor\frac{n+3-5h}{2}\rfloor\end{bmatrix}_q=E_n.$$

Schur uses his method to derive the (easier) formula

$$1=\sum_{k\in\mathbb{Z}}(-1)^k q^{\frac{k(3k-1)}{2}}\begin{bmatrix}n\\ \lfloor\frac{n+3k}{2}\rfloor\end{bmatrix}_q.$$

In the limit this means

$$1=\sum_{k\in\mathbb{Z}}(-1)^k q^{\frac{k(3k-1)}{2}}\frac{1}{(q;q)_\infty},$$

which is again Euler's pentagonal theorem.

17. Schur streamlined by R. Chapman

Define

$$A(n,t,b)=\sum_{2r\equiv n+t\,(5)}q^{(2r-n+b)^2/10}\begin{bmatrix}n\\ r\end{bmatrix}_q.$$

Here n is a nonnegative integer, t any integer and b any rational. Note that $A(n,t,b)$ depends only on the congruence class of t modulo 5. Replacing r by $n-r$ in the definition gives

$$A(n,t,b)=A(n,-t,-b).$$

If we use the formula

$$\begin{bmatrix} n+1 \\ r \end{bmatrix}_q = \begin{bmatrix} n \\ r-1 \end{bmatrix}_q + q^r \begin{bmatrix} n \\ r \end{bmatrix}_q$$

we get $A(n+1,b,t)$ where

$$\begin{aligned} S_1 &= \sum_{2r \equiv n+t+1\,(5)} q^{(2r-n-1+b)^2/10} \begin{bmatrix} n \\ r-1 \end{bmatrix}_q \\ &= \sum_{2s \equiv n+t-1\,(5)} q^{(2s-n+1+b)^2/10} \begin{bmatrix} n \\ s \end{bmatrix}_q \\ &= A(n,t-1,b+1) \end{aligned}$$

and

$$S_2 = \sum_{2r \equiv n+t+1\,(5)} q^{r+(2r-n-1+b)^2/10} \begin{bmatrix} n \\ r \end{bmatrix}_q .$$

Now

$$\begin{aligned} r + \frac{(2r-n-1+b)^2}{10} &= \frac{4r^2 - 4r(n+1-b) + 10r + (n+1-b)^2}{10} \\ &= \frac{4r^2 - 4r(n-\frac{3}{2}-b) + (n+1-b)^2}{10} \\ &= \frac{(2r-n+\frac{3}{2}+b)^2 + (n+1-b)^2 - (n-\frac{3}{2}-b)^2}{10} \\ &= \frac{(2r-n+\frac{3}{2}+b)^2 + \frac{5}{2}(2n-\frac{1}{2}-2b)}{10}. \end{aligned}$$

Hence

$$A(n+1,t,b) = A(n,t-1,b+1) + q^{(4n-4b+1)/8} A(n,t+1,b+\tfrac{3}{2}).$$

Consequently also

$$\begin{aligned} A(n+1,t,b) &= A(n+1,-t,-b) \\ &= A(n,-t-1,-b+1) + q^{(4n+4b+1)/8} A(n,-t+1,-b+\tfrac{3}{2}) \\ &= A(n,t+1,b-1) + q^{(4n+4b+1)/8} A(n,t-1,b-\tfrac{3}{2}). \end{aligned}$$

For convenience we record these formulas with n replaced by $n-1$:

$$A(n,t,b) = A(n-1,t-1,b+1) + q^{(4n-4b-5)/8} A(n-1,t+1,b-\tfrac{3}{2})$$

and

$$A(n,t,b) = A(n-1,t+1,b-1) + q^{(4n+4b-5)/8} A(n-1,t-1,b+\tfrac{3}{2})$$

for $n \geq 1$.

For $n \geq 1$ we get

$$\begin{aligned} &A(n+1,t,\tfrac{1}{2}) \\ &= A(n,t+1,-\tfrac{1}{2}) + q^{(4n+1)/8} A(n,t-1,-1) \\ &= A(n,t+1,-\tfrac{1}{2}) + q^{(4n+1)/8} [A(n-1,t-2,0) + q^{(4n-1)/8} A(n-1,t,\tfrac{1}{2})] \\ &= A(n,-t-1,\tfrac{1}{2}) + q^{(4n+1)/8} A(n-1,t-2,0) + q^n A(n-1,t,\tfrac{1}{2}). \end{aligned}$$

Set $B(n,t) = A(n,t,\frac{1}{2}) - A(n,-t-1,\frac{1}{2})$. Then

$$B(n+1,t) = -B(n,t) + q^n B(n-1,t) + q^{(4n+1)/8}[A(n,t-2,0) - A(n,-3-t,0)].$$

But

$$A(n,-3-t,0) = A(n,t+3,0) = A(n,t-2,0)$$

and so

$$B(n+1,t) = -B(n,t) + q^n B(n-1,t).$$

Setting $C(n,t) = (-1)^n B(n,t)$ gives

$$C(n+1,t) = C(n,t) + q^n C(n,t)$$

which is almost the same recurrence satisfied by D_{n-1} and E_{n-1}. We calculate

$$C(0,0) = A(0,0,\tfrac{1}{2}) - A(0,-1,\tfrac{1}{2}) = q^{1/40}\begin{bmatrix}0\\0\end{bmatrix}_q = q^{1/40}$$

and

$$C(1,0) = A(1,-1,\tfrac{1}{2}) - A(1,0,\tfrac{1}{2}) = q^{1/40}\begin{bmatrix}1\\0\end{bmatrix}_q = q^{1/40}.$$

Thus

$$C(2,0) = C(1,0) + qC(0,0) = q^{1/40}(1+q)$$

so that

$$C(n,0) = q^{1/40} D_{n-1}$$

by induction. Also

$$C(0,-2) = A(0,-2,\tfrac{1}{2}) - A(0,1,\tfrac{1}{2}) = 0$$

and

$$C(1,-2) = A(1,1,\tfrac{1}{2}) - A(1,-2,\tfrac{1}{2}) = q^{9/40}\begin{bmatrix}1\\1\end{bmatrix}_q = q^{9/40}.$$

Thus

$$C(2,-2) = C(1,-2) + qC(0,-2) = q^{9/40}$$

so that

$$C(n,-2) = q^{9/40} E_{n-1}$$

by induction.

We now look in more detail at the sums $C(n,0)$ and $C(n,-2)$. First of all

$$A(n,0,\tfrac{1}{2}) = q^{1/40} \sum_{2r \equiv n\,(5)} q^{(2r-n)(2r-n+1)/10} \begin{bmatrix}n\\r\end{bmatrix}_q.$$

Let us write $2r - n = 5k$. Then $k \equiv n \pmod 2$ and this sum becomes

$$A(n,0,\tfrac{1}{2}) = q^{1/40} \sum_{k \equiv n\,(2)} q^{k(5k+1)/2} \begin{bmatrix}n\\ \frac{n+5k}{2}\end{bmatrix}_q = q^{1/40} \sum_{k \equiv n\,(2)} q^{k(5k+1)/2} \begin{bmatrix}n\\ \frac{n-5k}{2}\end{bmatrix}_q.$$

Similarly

$$A(n,-1,\tfrac{1}{2}) = q^{1/40} \sum_{2r \equiv n+1\,(5)} q^{(2r-n)(2r-n+1)/10} \begin{bmatrix}n\\r\end{bmatrix}_q$$

and putting $2r-n+1=-5k$ gives

$$A(n,-1,\tfrac{1}{2}) = q^{1/40}\sum_{k\equiv n+1\,(2)} q^{k(5k+1)/2}\begin{bmatrix} n \\ \frac{n-5k-1}{2}\end{bmatrix}_q.$$

Thus

$$D_{n-1} = q^{-1/40}C(n,0) = \sum_k (-1)^k q^{k(5k+1)/2}\begin{bmatrix} n \\ \lfloor\frac{n-5k}{2}\rfloor\end{bmatrix}_q$$

in accordance with Schur's formula (3).

We obtain (4) in a similar fashion. We have

$$A(n,-2,\tfrac{1}{2}) = q^{9/40}\sum_{2r\equiv n-2\,(5)} q^{(2r-n-1)(2r-n+2)/10}\begin{bmatrix} n \\ r\end{bmatrix}_q$$

and putting $2r-n+2=5k$ gives

$$\begin{aligned} A(n,-2,\tfrac{1}{2}) &= q^{9/40}\sum_{k\equiv n\,(2)} q^{k(5k-3)/2}\begin{bmatrix} n \\ \frac{n+5k-2}{2}\end{bmatrix}_q \\ &= q^{9/40}\sum_{k\equiv n\,(2)} q^{k(5k-3)/2}\begin{bmatrix} n \\ \frac{n-5k+2}{2}\end{bmatrix}_q. \end{aligned}$$

Also

$$A(n,1,\tfrac{1}{2}) = q^{9/40}\sum_{2r\equiv n+1\,(5)} q^{(2r-n-1)(2r-n+2)/10}\begin{bmatrix} n \\ r\end{bmatrix}_q$$

and putting $2r-n-2=-5k$ gives

$$A(n,1,\tfrac{1}{2}) = q^{9/40}\sum_{k\equiv n+1\,(2)} q^{k(5k-3)/2}\begin{bmatrix} n \\ \frac{n-5k+1}{2}\end{bmatrix}_q.$$

Hence

$$E_{n-1} = q^{-9/40}C(n,-2) = \sum_k (-1)^k q^{k(5k-3)/2}\begin{bmatrix} n \\ \lfloor\frac{n-5k+2}{2}\rfloor\end{bmatrix}_q$$

which is (4).

18. Schur's old determinant proves a brandnew theorem of Garrett–Ismail–Stanton

In a recent difficult paper [11] Garrett, Ismail, and Stanton prove, amongst many other things, the following generalization of the celebrated Rogers–Ramanujan identities:

$$\begin{aligned} \sum_{n\ge 0}\frac{q^{n^2+mn}}{(q;q)_n} &= (-1)^m q^{-\binom{m}{2}}E_{m-2}\prod_{n\ge 0}\frac{1}{(1-q^{5n+1})(1-q^{5n+4})} \\ &\quad - (-1)^m q^{-\binom{m}{2}}D_{m-2}\prod_{n\ge 0}\frac{1}{(1-q^{5n+2})(1-q^{5n+3})}, \end{aligned} \tag{5}$$

with the *Schur* polynomials, defined by

$$D_m = D_{m-1}+q^m D_{m-2}, \qquad D_0=1,\ D_1=1+q,$$

$$E_m = E_{m-1} + q^m E_{m-2}, \qquad E_0 = 1,\ E_1 = 1.$$

(Another proof, based on generalized Engel expansions, can be found in [2].)

Schur [13] has computed the limits

$$D_\infty = \prod_{n\geq 0} \frac{1}{(1-q^{5n+1})(1-q^{5n+4})}, \qquad E_\infty = \prod_{n\geq 0} \frac{1}{(1-q^{5n+2})(1-q^{5n+3})}.$$

In this section we will give a simple proof of this result, based on the following determinant of Schur:

$$\mathsf{Schur}(x) := \begin{vmatrix} 1 & xq^{1+m} & & & & \dots \\ -1 & 1 & xq^{2+m} & & & \dots \\ & -1 & 1 & xq^{3+m} & & \dots \\ & & -1 & 1 & xq^{4+m} & \dots \\ & & & \ddots & \ddots & \ddots \end{vmatrix}.$$

Expanding the determinant with respect to the first column ("top–recursion") we get

$$\mathsf{Schur}(x) = \mathsf{Schur}(xq) + xq^{1+m}\,\mathsf{Schur}(xq^2).$$

Setting

$$\mathsf{Schur}(x) = \sum_{n\geq 0} a_n x^n,$$

we get, upon comparing coefficients,

$$a_n = q^n a_n + q^{1+m} q^{2n-2} a_{n-1},$$

or

$$a_n = \frac{q^{2n-1+m}}{1-q^n} a_{n-1}.$$

Since $a_0 = 1$, iteration leads to

$$a_n = \frac{q^{n^2+mn}}{(1-q)(1-q^2)\dots(1-q^n)},$$

and thus the left hand side of (5) can be expressed by $\mathsf{Schur}(1)$.

On the other hand, $\mathsf{Schur}(1)$ is the limit of the *finite* determinants

$$\mathsf{Schur}_n := \begin{vmatrix} 1 & q^{1+m} & & & & \dots \\ -1 & 1 & q^{2+m} & & & \dots \\ & -1 & 1 & q^{3+m} & & \dots \\ & & -1 & 1 & q^{4+m} & \dots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots \\ & & & -1 & 1 & q^{n+m} \\ & & & & -1 & 1 \end{vmatrix}.$$

Expanding this determinant with respect to the last row ("bottom–recursion") we get

$$\mathsf{Schur}_n = \mathsf{Schur}_{n-1} + q^{n+m}\,\mathsf{Schur}_{n-2}\,.$$

We see that the sequences $\langle D_{n+m}\rangle_n$ and $\langle E_{n+m}\rangle_n$ satisfy this recursion, and thus any linear combination; set

$$\mathsf{Schur}_n = \lambda D_{n+m} + \mu E_{n+m}.$$

We can determine the parameters λ and μ using the initial conditions $\mathsf{Schur}_0 = 1$, $\mathsf{Schur}_1 = 1 + q^{1+m}$, which leads to

$$\lambda = \frac{E_m - E_{m-1}}{D_{m-1}E_m - D_m E_{m-1}} = \frac{q^m E_{m-2}}{D_{m-1}E_m - D_m E_{m-1}}$$

and

$$\mu = \frac{D_m - D_{m-1}}{D_m E_{m-1} - D_{m-1}E_m} = \frac{q^m D_{m-2}}{D_m E_{m-1} - D_{m-1}E_m}.$$

We can find the nicer form

$$\lambda = \frac{q^m E_{m-2}}{D_{m-1}E_m - D_m E_{m-1}} = (-1)^m q^{-\binom{m}{2}} E_{m-2},$$

which amounts to prove that

$$D_{m-1}E_m - D_m E_{m-1} = (-1)^m q^{\binom{m+1}{2}}.$$

This however is an easy induction on m, the beginning $m = 0$ being trivial. The induction step goes like this:

$$\begin{aligned} D_m E_{m+1} - D_{m+1}E_m &= D_m(E_m + q^{m+1}E_{m-1}) - (D_m + q^{m+1}D_{m-1})E_m \\ &= q^{m+1}(D_m E_{m-1} - D_{m-1}E_m) \\ &= -q^{m+1}(-1)^m q^{\binom{m+1}{2}} = (-1)^{m+1} q^{\binom{m+2}{2}}. \end{aligned}$$

Analogously we obtain

$$\mu = \frac{q^m D_{m-2}}{D_m E_{m-1} - D_{m-1}E_m} = (-1)^{m+1} q^{-\binom{m}{2}} D_{m-2}.$$

Hence

$$\mathsf{Schur}_n = (-1)^m q^{-\binom{m}{2}} E_{m-2} D_{n+m} - (-1)^m q^{-\binom{m}{2}} D_{m-2} E_{n+m}.$$

Performing the limit $n \to \infty$ this turns into

$$\mathsf{Schur}(1) = (-1)^m q^{-\binom{m}{2}} E_{m-2} D_\infty - (-1)^m q^{-\binom{m}{2}} D_{m-2} E_\infty,$$

which is (5).

19. Engel

We consider Laurent series

$$A = \sum_{n \geq \nu} C_n q^n,$$

and denote the integral part by

$$\lfloor A \rfloor = \sum_{\nu \leq n \leq 0} C_n q^n,$$

which is a polynomial in $\frac{1}{q}$.

There is the unique expansion

$$A = a_0 + \sum_{n\geq 1} \frac{1}{a_1 a_2 \dots a_n},$$

where the a_i's are polynomials (in $\frac{1}{q}$) of strictly increasing degrees. It helps to imagine a term like $a = \frac{1}{q^{10}} - 1$ as $a = \frac{1}{q^{10}}(1 - q^{10})$, so that

$$\frac{1}{a} = \frac{q^{10}}{1 - q^{10}} = q^{10} + \mathcal{O}(q^{11}),$$

so that terms of small degree are no longer affected after a while.

The quantities in question may be computed recursively viz.

$$a_n := \left\lfloor \frac{1}{A_n} \right\rfloor, \ A_{n+1} := a_n A_n - 1, \qquad A_0 := A, \ a_0 := \lfloor A \rfloor.$$

Example: If $A = 1 + q + \dots + q^{10}$, then we compute

$$a_0 = 1, \ \frac{1}{a_1} = \frac{q}{1-q}, \ \frac{1}{a_2} = -q^{10},$$

and thus

$$A = 1 + \frac{q}{1-q} - \frac{q}{1-q} q^{10}.$$

Let us prove one of Euler's partition identities with this device:

$$\sum_{n\geq 0} \frac{t^n q^{\binom{n}{2}}}{(q;q)_n} = (-t;q)_\infty.$$

Replace t by zq :

$$\sum_{n\geq 0} \frac{z^n q^{\binom{n+1}{2}}}{(q;q)_n} = \prod_{n\geq 1} (1 + zq^n).$$

Let us prove this form; z is just a parameter.

We have

$$A = A_0 = \prod_{n\geq 1} (1 + zq^n)$$

and so $a_0 = 1$.

Claim:

$$a_n = z^{-1}(q^{-n} - 1),$$
$$A_n = \sum_{m\geq 1} (1+zq)(1+zq^2)\dots(1+zq^{m-1})zq^{mn}, \qquad n \geq 1.$$

Proof (inductive):

$$a_n A_n - 1 = z^{-1}(q^{-n} - 1) \sum_{m\geq 1} (1+zq)(1+zq^2)\dots(1+zq^{m-1})zq^{mn} - 1$$

$$
\begin{aligned}
&= \sum_{m\geq 1}(1+zq)(1+zq^2)\dots(1+zq^{m-1})q^{n(m-1)} \\
&\quad - \sum_{m\geq 1}(1+zq)(1+zq^2)\dots(1+zq^{m-1})q^{mn} - 1 \\
&= \sum_{m\geq 2}(1+zq)(1+zq^2)\dots(1+zq^{m-1})q^{n(m-1)} \\
&\quad - \sum_{m\geq 1}(1+zq)(1+zq^2)\dots(1+zq^{m-1})q^{mn} \\
&= \sum_{m\geq 1}(1+zq)(1+zq^2)\dots(1+zq^{m})q^{nm} \\
&\quad - \sum_{m\geq 1}(1+zq)(1+zq^2)\dots(1+zq^{m-1})q^{mn} \\
&= \sum_{m\geq 1}(1+zq)(1+zq^2)\dots(1+zq^{m-1})q^{mn}\Big((1+zq^m)-1\Big) \\
&= \sum_{m\geq 1}(1+zq)(1+zq^2)\dots(1+zq^{m-1})zq^{m(n+1)} = A_{n+1}.
\end{aligned}
$$

Note that

$$A_n = zq^n + (1+zq)zq^{2n} + \mathcal{O}(q^{3n}),$$

and so

$$
\begin{aligned}
a_n &= \left\lfloor \frac{1}{A_n} \right\rfloor \\
&= \left\lfloor \frac{1}{zq^n + (1+zq)zq^{2n} + \mathcal{O}(q^{3n})} \right\rfloor \\
&= \left\lfloor \frac{1}{zq^n} \frac{1}{1 + (1+zq)q^n + \mathcal{O}(q^{2n})} \right\rfloor \\
&= \left\lfloor \frac{1}{zq^n}\Big(1 - (1+zq)q^n + \mathcal{O}(q^{2n})\Big) \right\rfloor \\
&= \left\lfloor z^{-1}q^{-n} - z^{-1}(1+zq) + \mathcal{O}(q^n) \right\rfloor \\
&= z^{-1}q^{-n} - z^{-1}.
\end{aligned}
$$

The beginning is also easy;

$$A - 1 = \sum_{m\geq 1}(1+zq)\dots(1+zq^{m-1})zq^m,$$

by selecting the highest power chosen.

Hence

$$\frac{1}{a_n} = \frac{zq^n}{1-q^n},$$

$$\frac{1}{a_1 \dots a_n} = \frac{z^n q^{\binom{n+1}{2}}}{(1-q)\dots(1-q^n)},$$

and

$$A = \prod_{n\geq 1}(1+zq^n) = 1 + \sum_{n\geq 1} \frac{z^n q^{\binom{n+1}{2}}}{(1-q)\dots(1-q^n)},$$

as desired.

Now we derive the first Rogers–Ramanujan Identity:

Set

$$A := A_0 = \prod_{n\geq 0} \frac{1}{(1-q^{5n+1})(1-q^{5n+4})}.$$

We use the Schur polynomials:

$$D_n = D_{n-1} + q^n D_{n-2}, \qquad D_{-1} = 1,\ D_{-2} = 0.$$

We have seen already that $D_\infty = A$.

We sum the recursion:

$$\sum_{j=0}^{N} q^j D_{j-2} = \sum_{j=0}^{N} (D_j - D_{j-1}) = D_N - 1.$$

Taking the limit,

$$\sum_{j=0}^{\infty} q^j D_{j-2} = A - 1 =: A_1.$$

Claim:

$$a_n = \begin{cases} 1 & \text{if } n = 0 \\ q^{-2n+1} - q^{-n+1} & \text{if } n \geq 1 \end{cases}$$
$$A_n = \sum_{j\geq 0} q^{nj+(n-1)} D_{j-2} \qquad \text{for } n \geq 1\ .$$

The induction step:

$$\begin{aligned}
&(q^{-2n+1} - q^{-n+1}) \sum_{j\geq 0} q^{nj+(n-1)} D_{j-2} - 1 \\
&= \sum_{j\geq 1} q^{nj-n} D_{j-2} - \sum_{j\geq 1} q^{nj} D_{j-2} - 1 \\
&= \sum_{j\geq 0} q^{nj} D_{j-1} - \sum_{j\geq 1} q^{nj} D_{j-2} - 1 \\
&= \sum_{j\geq 1} q^{nj} D_{j-1} - \sum_{j\geq 1} q^{nj} D_{j-2} \\
&= \sum_{j\geq 1} q^{nj} (D_{j-1} - D_{j-2})
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{j\geq 1} q^{nj} q^{j-1} D_{j-3} \\
&= \sum_{j\geq 2} q^{(n+1)j-1} D_{j-3} \\
&= \sum_{j\geq 1} q^{(n+1)(j+1)-1} D_{j-2} \\
&= \sum_{j\geq 0} q^{(n+1)j+n} D_{j-2}.
\end{aligned}
$$

Great; finally

$$A_n = q^{2n-1} + q^{3n-1} + \mathcal{O}(q^{4n-1})$$

and so

$$
\begin{aligned}
\left\lfloor \frac{1}{A_n} \right\rfloor &= \left\lfloor \frac{1}{q^{2n-1} + q^{3n-1} + \mathcal{O}(q^{4n-1})} \right\rfloor \\
&= \left\lfloor q^{1-2n} \frac{1}{1 + q^n + \mathcal{O}(q^{2n})} \right\rfloor \\
&= \left\lfloor q^{1-2n} (1 - q^n + \mathcal{O}(q^{2n})) \right\rfloor \\
&= q^{1-2n} - q^{1-n}.
\end{aligned}
$$

Thus

$$\frac{1}{a_n} = \frac{1}{q^{1-2n} - q^{1-n}} = \frac{q^{2n-1}}{1-q^n}.$$

and

$$\frac{1}{a_1 \dots a_n} = \frac{q^{n^2}}{(1-q)\dots(1-q^n)}.$$

So we have proved the first Rogers–Ramanujan identity:

$$\prod_{n\geq 0} \frac{1}{(1-q^{5n+1})(1-q^{5n+4})} = \sum_{n\geq 0} \frac{q^{n^2}}{(1-q)\dots(1-q^n)}.$$

This approach has avoided Schur's determinants.

The second one is completely analogous and is left as an exercise.—

Next we will prove the so-called q–analogue of Kummer's theorem:

$$\sum_{n\geq 0} \frac{q^{n(n-1)} z^n}{(q;q)_n (z;q)_n} = \frac{1}{(z;q)_\infty}.$$

We will prove it in the form (z replaced by zq)

$$\sum_{n\geq 0} \frac{q^{n^2} z^n}{(q;q)_n (zq;q)_n} = \frac{1}{(zq;q)_\infty}.$$

We need a slight variation of the Engel expansion: Not only the nonpositive terms are taken out as digits, but also the term with 'q'. The relevant formulæ are these:

$$A_{n+1} = q(a_n A_n - 1); \qquad a_n = \left\lfloor \frac{1}{A_n} \right\rfloor;$$

as digits we have to take qa_n, so that the formula will look like

$$\frac{1}{(zq;q)_\infty} = \sum_{n\geq 0} \frac{q^{-n}}{a_1 \dots a_n}.$$

Claim (for $n \geq 1$):

$$A_n = \sum_{j\geq 1} \frac{zq^{n(j+1)}}{(zq^n;q)_j}, \qquad a_n = q^{-2n} z^{-1}(1-q^n)(1-zq^n).$$

First note that if

$$l_N = \prod_{k=1}^{N} \frac{1}{1-zq^k},$$

then

$$l_N - l_{N-1} = \frac{zq^N}{(zq;q)_N},$$

and so

$$l_N = 1 + \sum_{j=1}^{N} \frac{zq^j}{(zq;q)_j}.$$

Therefore

$$l_\infty = \frac{1}{(zq;q)_\infty} = 1 + \sum_{j\geq 1} \frac{zq^j}{(zq;q)_j}.$$

We start the algorithm with

$$A_0 = \sum_{j\geq 1} \frac{zq^j}{(zq;q)_j}.$$

Let us just do the induction step; the beginning is similar.

$$\begin{aligned}
A_{n+1} &= q(a_n A_n - 1) \\
&= q\left(\frac{(1-q^n)(1-zq^n)}{zq^{2n}} \sum_{j\geq 1} \frac{zq^{n(j+1)}}{(zq^n;q)_j} - 1 \right) \\
&= q\left((1-q^n) \sum_{j\geq 1} \frac{q^{n(j-1)}}{(zq^{n+1};q)_{j-1}} - 1 \right) \\
&= q\left(\sum_{j\geq 1} \frac{q^{nj}}{(zq^{n+1};q)_j} - \sum_{j\geq 1} \frac{q^{nj}}{(zq^{n+1};q)_{j-1}} \right)
\end{aligned}$$

$$= q \sum_{j \geq 1} \frac{q^{nj}}{(zq^{n+1};q)_j}(1 - 1 + zq^{n+j})$$
$$= \sum_{j \geq 1} \frac{zq^{(n+1)(j+1)}}{(zq^{n+1};q)_j}.$$

And we note that

$$\begin{aligned} A_n &= \frac{zq^{2n}}{1-zq^n} + \frac{zq^{3n}}{(1-zq^n)(1-zq^{n+1})} + \frac{zq^{4n}}{(1-zq^n)(1-zq^{n+1})(1-zq^{n+2})} + \dots \\ &= zq^{2n}(1 + zq^n + z^2q^{2n} + \dots) + zq^{3n}(1 + zq^n + \dots) + zq^{4n}(1 + \dots) \\ &= zq^{2n} + z(1+z)q^{3n} + z(1+z+z^2)q^{4n} + \dots, \end{aligned}$$

or

$$\frac{1}{A_n} = \frac{1}{zq^{2n}}(1 - (1+z)q^n + zq^{2n}) + \dots,$$

thus

$$a_n = \left\lfloor \frac{1}{A_n} \right\rfloor = \frac{1}{zq^{2n}}(1 - (1+z)q^n + zq^{2n}) = \frac{(1-q^n)(1-zq^n)}{zq^{2n}},$$

as desired.

So we obtain

$$1 + \sum_{n \geq 1} \frac{q^{-n}q^{2+4+\dots+2n}z^n}{(q;q)_n(zq;q)_n} = 1 + \sum_{n \geq 1} \frac{q^{n^2}z^n}{(q;q)_n(zq;q)_n},$$

as claimed.

The instance $z = 1$ was known to Euler and appears in the Andrews/Knopfmacher/-Knopfmacher paper:

$$\sum_{n \geq 0} \frac{q^{n^2}}{(q;q)_n^2} = \prod_{n \geq 1} \frac{1}{1-q^n}.$$

20. An inversion formula

$$\left(\sum_{k \geq 0} \frac{t^k}{[k]_q!} \right)^{-1} = \sum_{k \geq 0} (-1)^k q^{\binom{k}{2}} \frac{t^k}{[k]_q!}.$$

Well, consider

$$G(t) = \prod_{i \geq 0} \left(1 - t(1-q)q^i\right);$$

now use one of Euler's partition identities to expand that:

$$G(t) = \sum_{k\geq 0} (-1)^k q^{\binom{k}{2}} \frac{t^k (1-q)^k}{(q;q)_k}.$$

But also

$$\frac{1}{G(t)} = \prod_{i\geq 0} \frac{1}{1-t(1-q)q^i};$$

now use the other Euler formula to expand that:

$$\sum_{k\geq 0} \frac{t^k (1-q)^k}{(q;q)_k},$$

and that's it.

Now consider 2 Eulerian generating functions

$$A(t) = \sum_{k\geq 0} \frac{a_k t^k}{[k]_q!}$$

and

$$B(t) = \sum_{k\geq 0} \frac{b_k t^k}{[k]_q!}.$$

If

$$A(t) = B(t) \sum_{k\geq 0} \frac{t^k}{[k]_q!},$$

then because of the above also

$$B(t) = A(t) \sum_{k\geq 0} \frac{(-1)^k q^{\binom{k}{2}} t^k}{[k]_q!}.$$

Now consider $[n]_q![t^n]\ldots$:

$$a_n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q b_k$$

$$b_n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q (-1)^{n-k} q^{\binom{n-k}{2}} a_k.$$

21. Rogers–Ramanujan by Bressoud

We follow Goulden and Jackson:

Set

$$G(x) = \frac{1}{(xq;q)_\infty} \sum_{n\geq 0} (-1)^n x^{2n} q^{\frac{n(5n-1)}{2}} (1-xq^{2n}) \frac{(xq;q)_{n-1}}{(q;q)_n}. \tag{6}$$

(For $n=0$, the term of the sum is 1.)

Now write

$$1-xq^{2n}=(1-q^n)+q^n(1-xq^n).$$

Then

$$\begin{aligned}(xq;q)_\infty G(x) &= 1+\sum_{n\geq 1}(-1)^n x^{2n}q^{\frac{n(5n-1)}{2}}\frac{(xq;q)_{n-1}}{(q;q)_{n-1}}+\sum_{n\geq 1}(-1)^n x^{2n}q^{\frac{n(5n+1)}{2}}\frac{(xq;q)_n}{(q;q)_n}\\ &= \sum_{n\geq 0}(-1)^n x^{2n}q^{\frac{n(5n+1)}{2}}\frac{(xq;q)_n}{(q;q)_n}-\sum_{n\geq 0}(-1)^n x^{2n+2}q^{\frac{(n+1)(5n+4)}{2}}\frac{(xq;q)_n}{(q;q)_n} \qquad (7)\\ &= \sum_{n\geq 0}(-1)^n x^{2n}q^{\frac{n(5n+1)}{2}}(1-x^2q^{4n+2})\frac{(xq;q)_n}{(q;q)_n}.\end{aligned}$$

Now set $D=(xq;q)_\infty(G(x)-G(xq))$. Please note that

$$(xq^2;q)_\infty=(xq;q)_\infty/(1-xq).$$

Thus

$$\begin{aligned}D &= \sum_{n\geq 0}(-1)^n x^{2n}q^{\frac{n(5n+1)}{2}}(1-x^2q^{4n+2})\frac{(xq;q)_n}{(q;q)_n}\\ &\quad -(1-xq)\sum_{n\geq 0}(-1)^n x^{2n}q^{2n+\frac{n(5n-1)}{2}}(1-xq^{2n+1})\frac{(xq^2;q)_{n-1}}{(q;q)_n};\end{aligned}$$

here we have used both versions, (6) and (7).

Hence

$$\begin{aligned}D &= \sum_{n\geq 0}(-1)^n x^{2n}q^{\frac{n(5n+1)}{2}}(1-x^2q^{4n+2})\frac{(xq;q)_n}{(q;q)_n}\\ &\quad -\sum_{n\geq 0}(-1)^n x^{2n}q^{n+\frac{n(5n+1)}{2}}(1-xq^{2n+1})\frac{(xq;q)_n}{(q;q)_n}\\ &= \sum_{n\geq 0}(-1)^n x^{2n}q^{\frac{n(5n+1)}{2}}\frac{(xq;q)_n}{(q;q)_n}\Big(1-x^2q^{4n+2}-q^n(1-xq^{2n+1})\Big)\\ &= \sum_{n\geq 0}(-1)^n x^{2n}q^{\frac{n(5n+1)}{2}}\frac{(xq;q)_n}{(q;q)_n}\Big((1-q^n)+xq^{3n+1}(1-xq^{n+1})\Big)\\ &= \sum_{n\geq 1}(-1)^n x^{2n}q^{\frac{n(5n+1)}{2}}\frac{(xq;q)_n}{(q;q)_{n-1}}+qx\sum_{n\geq 0}(-1)^n x^{2n}q^{\frac{n(5n+7)}{2}}\frac{(xq;q)_{n+1}}{(q;q)_n}\\ &= -\sum_{n\geq 0}(-1)^n x^{2n+2}q^{\frac{(n+1)(5n+6)}{2}}\frac{(xq;q)_{n+1}}{(q;q)_n}+qx\sum_{n\geq 0}(-1)^n x^{2n}q^{\frac{n(5n+7)}{2}}\frac{(xq;q)_{n+1}}{(q;q)_n}\\ &= qx\sum_{n\geq 0}(-1)^n x^{2n}q^{\frac{n(5n+7)}{2}}\frac{(xq;q)_{n+1}}{(q;q)_n}(1-xq^{2n+2})\\ &= qx(qx;q)_\infty G(xq^2).\end{aligned}$$

Let us confirm that:

$$\begin{aligned} qx(qx;q)_\infty G(xq^2) &= qx(1-qx)(1-q^2x)(q^3x;q)_\infty G(xq^2) \\ &= qx(1-qx)(1-q^2x)\sum_{n\geq 0}(-1)^n x^{2n} q^{4n+\frac{n(5n-1)}{2}}(1-xq^{2n+2})\frac{(xq^3;q)_{n-1}}{(q;q)_n} \\ &= qx\sum_{n\geq 0}(-1)^n x^{2n} q^{\frac{n(5n+7)}{2}}(1-xq^{2n+2})\frac{(xq;q)_{n+1}}{(q;q)_n}. \end{aligned}$$

Fine. Summarizing,

$$(xq;q)_\infty(G(x)-G(xq)) = qx(qx;q)_\infty G(xq^2),$$

or

$$G(x)-G(xq) = qxG(xq^2).$$

We have

$$G(x) = G(xq) + qxG(xq^2)$$

and $G(0)=1$. Setting

$$G(x) = \sum_{n\geq 0} c_n x^n$$

and comparing coefficients, we find

$$c_n = q^n c_n + q^{1+2(n-1)} c_{n-1},$$

or

$$c_n = q^n c_n + q^{2n-1} c_{n-1},$$

or

$$c_n = \frac{q^{2n-1}}{1-q^n} c_{n-1}.$$

Since $c_0 = 1$, iteration leads us to

$$c_n = \frac{q^{n^2}}{(q;q)_n}.$$

Again:

$$G(x) = \frac{1}{(xq;q)_\infty}\left[1+\sum_{n\geq 1}(-1)^n x^{2n} q^{\frac{n(5n-1)}{2}}(1-xq^{2n})\frac{(xq;q)_{n-1}}{(q;q)_n}\right] = \sum_{n\geq 1}\frac{q^{n^2}x^n}{(q;q)_n}.$$

Plug in $x=1$:

$$\begin{aligned} \sum_{n\geq 0}\frac{q^{n^2}}{(q;q)_n} &= \frac{1}{(q;q)_\infty}\left[1+\sum_{n\geq 1}(-1)^n q^{\frac{n(5n-1)}{2}}(1-q^{2n})\frac{(q;q)_{n-1}}{(q;q)_n}\right] \\ &= \frac{1}{(q;q)_\infty}\left[1+\sum_{n\geq 1}(-1)^n q^{\frac{n(5n-1)}{2}}(1+q^n)\frac{(q;q)_n}{(q;q)_n}\right] \end{aligned}$$

$$= \frac{1}{(q;q)_\infty} \sum_{n\in\mathbb{Z}} (-1)^n q^{\frac{n(5n+1)}{2}}$$
$$= \frac{1}{(q;q)_\infty} \prod_{n\geq 1} (1-q^{5n-3})(1-q^{5n-2})(1-q^{5n}).$$

This was by the Triple product; we have seen that already. Cancelling out we find

$$\prod_{n\geq 0} \frac{1}{(1-q^{5n+1})(1-q^{5n+4})} = \sum_{n\geq 0} \frac{q^{n^2}}{(q;q)_n}.$$

Wow! That was hard; now we can go for a beer.

Plugging in $x = q$ gives us the other one;

$$\prod_{n\geq 0} \frac{1}{(1-q^{5n+2})(1-q^{5n+3})} = \sum_{n\geq 0} \frac{q^{n(n+1)}}{(q;q)_n}.$$

22. Rogers' second proof

Lemma:

Set

$$S_{2l} = \sum_{k=-l}^{l(+1)} \frac{(-1)^k q^{\frac{k(3k-1)}{2}}}{(q;q)_{l+k}(q;q)_{l-k}}$$

and

$$S_{2l+1} = \sum_{k=-l}^{l+1} \frac{(-1)^k q^{\frac{k(3k-1)}{2}}}{(q;q)_{l+k}(q;q)_{l+1-k}}.$$

Then

$$S_{2l} = S_{2l+1} = \frac{1}{(q;q)_l}.$$

Proof. In S_{2l}, combine terms with indices $-k$ and $k+1$:

$$\frac{(-1)^k q^{\frac{k(3k+1)}{2}}}{(q;q)_{l+k}(q;q)_{l-k}} + \frac{(-1)^{k+1} q^{\frac{(k+1)(3k+2)}{2}}}{(q;q)_{l+k+1}(q;q)_{l-k-1}}$$
$$= \frac{(-1)^k q^{\frac{k(3k+1)}{2}}}{(q;q)_{l+k}(q;q)_{l-k}} \left[1 - \frac{q^{2k+1}(1-q^{l-k})}{1-q^{l+k+1}}\right]$$
$$= \frac{(-1)^k q^{\frac{k(3k+1)}{2}}}{(q;q)_{l+k+1}(q;q)_{l-k}} (1-q^{2k+1}).$$

If one combines terms $-k$ and $k+1$ in S_{2l+1}, we get the same thing (check!). Thus $S_{2l} = S_{2l+1}$. Now we prove that

$$S_{2l+1} = (1-q^{l+1}) S_{2l+2};$$

this will prove the lemma by induction.

Consider the terms in S_{2l+1} with indices k and $-k$: $(k \neq 0)$

$$\frac{(-1)^k q^{\frac{k(3k-1)}{2}}}{(q;q)_{l+1-k}(q;q)_{l+k}} + \frac{(-1)^k q^{\frac{k(3k+1)}{2}}}{(q;q)_{l+k+1}(q;q)_{l-k}}$$
$$= \frac{(-1)^k q^{\frac{k(3k-1)}{2}}}{(q;q)_{l+k}(q;q)_{l+1-k}}\left[1 + \frac{q^k(1-q^{l+1-k})}{1-q^{l+k+1}}\right]$$
$$= (1-q^{l+1})\frac{(-1)^k q^{\frac{k(3k-1)}{2}}}{(q;q)_{l+1+k}(q;q)_{l+1-k}}(1+q^k).$$

Consider the terms in S_{2l+2} with indices k and $-k$: $(k \neq 0)$

$$\frac{(-1)^k q^{\frac{k(3k-1)}{2}}}{(q;q)_{l+1+k}(q;q)_{l+1-k}}(1+q^k).$$

When $k = 0$, the respective terms are

$$\frac{1-q^{l+1}}{(q;q)_{l+1}(q;q)_{l+1}}$$

and

$$\frac{1}{(q;q)_{l+1}(q;q)_{l+1}}.$$

This proves the lemma.

Rewrite the result

$$\sum_{k=-l}^{l} \frac{(-1)^k q^{\frac{k(3k-1)}{2}}}{(q;q)_{l+k}(q;q)_{l-k}} = \frac{1}{(q;q)_l}$$

as

$$\sum_{k=-l}^{l} (-1)^k q^{\frac{k(3k-1)}{2}} \begin{bmatrix} 2l \\ l+k \end{bmatrix}_q = \frac{(q;q)_{2l}}{(q;q)_l}.$$

Now perform the limit $l \to \infty$:

$$\sum_{k\in\mathbb{Z}} (-1)^k q^{\frac{k(3k-1)}{2}} \frac{1}{(q;q)_\infty} = 1,$$

or

$$\sum_{k\in\mathbb{Z}} (-1)^k q^{\frac{k(3k-1)}{2}} = (q;q)_\infty,$$

which is Euler's pentagonal theorem once again.—

Idea: Expand

$$(-\sqrt{q}e^{i\theta};q)_\infty(-\sqrt{q}e^{-i\theta};q)_\infty$$

in 2 different ways as a Fourier series.

Since the Fourier expansion is unique, the coefficients must coincide. Hence we have identities and can replace $e^{in\theta}$ by anything we want. —

By Euler's partition identity:

$$(-xe^{i\theta}q;q)_\infty = \sum_{n\geq 0} \frac{q^{\binom{n+1}{2}}}{(q;q)_n} x^n e^{in\theta}$$

and

$$(-xe^{-i\theta}q;q)_\infty = \sum_{m\geq 0} \frac{q^{\binom{m+1}{2}}}{(q;q)_m} x^m e^{-im\theta}.$$

So the product is

$$\begin{aligned}(-xe^{i\theta}q;q)_\infty(-xe^{-i\theta}q;q)_\infty &= \sum_{n,m\geq 0} \frac{q^{\binom{n+1}{2}+\binom{m+1}{2}}}{(q;q)_n(q;q)_m} x^{n+m} e^{i(n-m)\theta} \\ &= \sum_{p\geq 0} x^p \sum_{n=0}^{p} \frac{q^{\binom{n+1}{2}+\binom{p-n+1}{2}}}{(q;q)_n(q;q)_{p-n}} e^{i(2n-p)\theta}.\end{aligned}$$

Break up the last sum into 2 parts for $p = 2l$ and $p = 2l+1$ and set $n = l+k$. Then

$$\begin{aligned}(-xe^{i\theta}q;q)_\infty(-xe^{-i\theta}q;q)_\infty &= \sum_{l\geq 0} q^{l(l+1)} x^{2l} \sum_{k=-l}^{l} \frac{q^{k^2} e^{2ki\theta}}{(q;q)_{l+k}(q;q)_{l-k}} \\ &\quad + \sum_{l\geq 0} q^{(l+1)^2} x^{2l+1} \sum_{k=-l}^{l+1} \frac{q^{k(k-1)} e^{(2k-1)i\theta}}{(q;q)_{l+k}(q;q)_{l+1-k}}.\end{aligned}$$

When $qx = q^{1/2}$ this relation becomes

$$\begin{aligned}&(-\sqrt{q}e^{i\theta};q)_\infty(-\sqrt{q}e^{-i\theta};q)_\infty \\ &= \sum_{l\geq 0} q^{l^2} \sum_{k=-l}^{l} \frac{q^{k^2} e^{2ki\theta}}{(q;q)_{l+k}(q;q)_{l-k}} + \sqrt{q} \sum_{l\geq 0} q^{l(l+1)} \sum_{k=-l}^{l+1} \frac{q^{k(k-1)} e^{(2k-1)i\theta}}{(q;q)_{l+k}(q;q)_{l+1-k}} \\ &= \sum_{k\in\mathbb{Z}} q^{k^2} e^{2ki\theta} \sum_{l\geq k} \frac{q^{l^2}}{(q;q)_{l+k}(q;q)_{l-k}} + \sqrt{q} \sum_{k\in\mathbb{Z}} q^{k(k-1)} e^{(2k-1)i\theta} \sum_{l\geq k-1} \frac{q^{l(l+1)}}{(q;q)_{l+k}(q;q)_{l+1-k}}\end{aligned}$$

This gives one Fourier expansion. To get the other one, start with the triple product

$$(q;q)_\infty(x;q)_\infty(q/x;q)_\infty = \sum_{k\in\mathbb{Z}} (-1)^k q^{k(k-1)/2} x^k.$$

Replace x by $-\sqrt{q}e^{i\theta}$:

$$(q;q)_\infty(-\sqrt{q}e^{i\theta};q)_\infty(-\sqrt{q}e^{-i\theta};q)_\infty = \sum_{k\in\mathbb{Z}} q^{k^2/2} e^{ik\theta}$$

or

$$(-\sqrt{q}e^{i\theta};q)_\infty(-\sqrt{q}e^{i\theta};q)_\infty = \frac{1}{(q;q)_\infty}\left[\sum_{k\in\mathbb{Z}} q^{2k^2} e^{2ki\theta} + \sqrt{q}\sum_{k\in\mathbb{Z}} q^{2k(k-1)} e^{(2k-1)i\theta}\right]$$

Therefore

$$\sum_{k\in\mathbb{Z}} q^{k^2} e^{2ki\theta} \sum_{l\geq k} \frac{q^{l^2}}{(q;q)_{l+k}(q;q)_{l-k}} + \sqrt{q} \sum_{k\in\mathbb{Z}} q^{k(k-1)} e^{(2k-1)i\theta} \sum_{l\geq k-1} \frac{q^{l(l+1)}}{(q;q)_{l+k}(q;q)_{l+1-k}}$$
$$= \frac{1}{(q;q)_\infty} \left[\sum_{k\in\mathbb{Z}} q^{2k^2} e^{2ki\theta} + \sqrt{q} \sum_{k\in\mathbb{Z}} q^{2k(k-1)} e^{(2k-1)i\theta} \right]$$

Now replace $e^{2ki\theta}$ with $(-1)^k q^{k(k-1)/2}$ and $e^{(2k-1)i\theta}$ with 0:

$$\sum_{k\in\mathbb{Z}} q^{k^2} (-1)^k q^{k(k-1)/2} \sum_{l\geq k} \frac{q^{l^2}}{(q;q)_{l+k}(q;q)_{l-k}} = \frac{1}{(q;q)_\infty} \sum_{k\in\mathbb{Z}} q^{2k^2} (-1)^k q^{k(k-1)/2}$$

or

$$\sum_{k\in\mathbb{Z}} (-1)^k q^{k(3k-1)/2} \sum_{l\geq k} \frac{q^{l^2}}{(q;q)_{l+k}(q;q)_{l-k}} = \frac{1}{(q;q)_\infty} \sum_{k\in\mathbb{Z}} (-1)^k q^{k(5k-1)/2}$$

or

$$\sum_{l\in\mathbb{Z}} \sum_{k\leq l} q^{l^2} \frac{(-1)^k q^{k(3k-1)/2}}{(q;q)_{l+k}(q;q)_{l-k}} = \frac{1}{(q;q)_\infty} \sum_{k\in\mathbb{Z}} (-1)^k q^{k(5k-1)/2}.$$

However, we can restrict the range of the summations:

$$\sum_{l\geq 0} q^{l^2} \sum_{k=-l}^{l} \frac{(-1)^k q^{k(3k-1)/2}}{(q;q)_{l+k}(q;q)_{l-k}} = \frac{1}{(q;q)_\infty} \sum_{k\in\mathbb{Z}} (-1)^k q^{k(5k-1)/2}$$

and use the previous lemma to get

$$\sum_{l\geq 0} \frac{q^{l^2}}{(q;q)_l} = \frac{1}{(q;q)_\infty} \sum_{k\in\mathbb{Z}} (-1)^k q^{k(5k-1)/2};$$

this is the first identity of Rogers–Ramanujan.

For the second one replace $e^{2ki\theta}$ with 0 and $e^{(2k-1)i\theta}$ with $(-1)^k q^{k(k+1)/2}$:

$$\sum_{k\in\mathbb{Z}} q^{k(k-1)} (-1)^k q^{k(k+1)/2} \sum_{l\geq k-1} \frac{q^{l(l+1)}}{(q;q)_{l+k}(q;q)_{l+1-k}} = \frac{1}{(q;q)_\infty} \sum_{k\in\mathbb{Z}} q^{2k(k-1)} (-1)^k q^{k(k+1)/2}$$

or

$$\sum_{k\in\mathbb{Z}} (-1)^k q^{k^2} \sum_{l\geq k-1} \frac{q^{l(l+1)}}{(q;q)_{l+k}(q;q)_{l+1-k}} = \frac{1}{(q;q)_\infty} \sum_{k\in\mathbb{Z}} (-1)^k q^{k(5k-3)/2}.$$

Interchange the summations again:

$$\sum_{l\geq 0} q^{l(l+1)} \sum_{k=-l}^{l+1} \frac{(-1)^k q^{k^2}}{(q;q)_{l+k}(q;q)_{l+1-k}} = \frac{1}{(q;q)_\infty} \sum_{k\in\mathbb{Z}} (-1)^k q^{k(5k-3)/2}$$

and apply the lemma to get

$$\sum_{l\geq 0}\frac{q^{l(l+1)}}{(q;q)_l}=\frac{1}{(q;q)_\infty}\sum_{k\in\mathbb{Z}}(-1)^k q^{k(5k-3)/2}.$$

This is the second identity of Rogers–Ramanujan.

Since Fourier coefficients are unique, we get the identity

$$\sum_{l\geq k}\frac{q^{l^2+k^2}}{(q;q)_{l+k}(q;q)_{l-k}}=\frac{q^{2k^2}}{(q;q)_\infty}$$

or

$$\sum_{l\geq k}\frac{q^{(l-k)(l+k)}}{(q;q)_{l+k}(q;q)_{l-k}}=\frac{1}{(q;q)_\infty}$$

Recall an ooold corollary of Cauchy's:

$$\sum_{n\geq 0}\frac{q^{n(n-1)}x^n}{(q;q)_n(x;q)_n}=\frac{1}{(x;q)_\infty},$$

which we rewrite as

$$\sum_{n\geq 0}\frac{q^{n^2}x^n}{(q;q)_n(xq;q)_n}=\frac{1}{(xq;q)_\infty}.$$

Set $x=q^{2k}$. Then

$$\sum_{n\geq 0}\frac{q^{n(n+2k)}}{(q;q)_n(q^{2k+1};q)_n}=\frac{1}{(q^{2k+1};q)_\infty}.$$

Multiplying through with $(q;q)_{2k}$ this turns into

$$\sum_{n\geq 0}\frac{q^{n(n+2k)}}{(q;q)_n(q;q)_{n+2k}}=\frac{1}{(q;q)_\infty}.$$

Replace n with $l-k$:

$$\sum_{l\geq k}\frac{q^{(l-k)(l+k)}}{(q;q)_{l-k}(q;q)_{l+k}}=\frac{1}{(q;q)_\infty},$$

as desired.

23. Bailey's Lemma

Suppose that α_n and β_n are two sequences related by

$$\beta_n = \sum_{r=0}^{n} \frac{\alpha_r}{(q;q)_{n-r}(aq;q)_{n+r}};$$

then

$$\sum_{n\geq 0} a^n q^{n^2} \beta_n = \frac{1}{(aq;q)_\infty} \sum_{n\geq 0} a^n q^{n^2} \alpha_n.$$

Proof.

$$\begin{aligned}
\sum_{n\geq 0} a^n q^{n^2} \beta_n &= \sum_{n\geq 0} a^n q^{n^2} \sum_{r=0}^{n} \frac{\alpha_r}{(q;q)_{n-r}(aq;q)_{n+r}} \\
&= \sum_{r\geq 0} \sum_{n\geq r} \frac{a^n q^{n^2} \alpha_r}{(q;q)_{n-r}(aq;q)_{n+r}} \\
&= \sum_{r\geq 0} \alpha_r \sum_{n\geq 0} \frac{a^{n+r} q^{(n+r)^2}}{(q;q)_n (aq;q)_{n+2r}} \\
&= \sum_{r\geq 0} \alpha_r a^r q^{r^2} \sum_{n\geq 0} \frac{a^n q^{n(n+2r)}}{(q;q)_n (aq;q)_{n+2r}} \\
&= \sum_{r\geq 0} \alpha_r a^r q^{r^2} \frac{1}{(aq;q)_\infty}.
\end{aligned}$$

For that we have used a corollary of Cauchy, viz.

$$\sum_{n\geq 0} \frac{q^{n(n-1)} x^n}{(q;q)_n (x;q)_n} = \frac{1}{(x;q)_\infty};$$

plug in $x = q^{2r+1}a$ to get

$$\sum_{n\geq 0} \frac{q^{n(n-1)+(2r+1)n} a^n}{(q;q)_n (q^{2r+1}a;q)_n} = \frac{1}{(q^{2r+1}a;q)_\infty},$$

or

$$\sum_{n\geq 0} \frac{q^{n(n+2r)} a^n}{(q;q)_n (q^{2r+1}a;q)_n} = \frac{1}{(q^{2r+1}a;q)_\infty}.$$

Divide both sides by $(aq;q)_{2r}$, and we have

$$\sum_{n\geq 0} \frac{q^{n(n+2r)} a^n}{(q;q)_n (qa;q)_{n+2r}} = \frac{1}{(qa;q)_\infty},$$

as desired. —

Take $a = 1$ and

$$\alpha_n = \begin{cases} 1 & n = 0 \\ (-1)^n q^{n(3n-1)/2}(1+q^n) & n > 0. \end{cases}$$

Then

$$\begin{aligned} \beta_n &= \frac{1}{((q;q)_n)^2} + \sum_{r=1}^{n} \frac{(-1)^r q^{r(3r-1)/2}(1+q^r)}{(q;q)_{n-r}(q;q)_{n+r}} \\ &= \sum_{r=-n}^{n} \frac{(-1)^r q^{r(3r-1)/2}}{(q;q)_{n-r}(q;q)_{n+r}} = \frac{1}{(q;q)_n}. \end{aligned}$$

Here we have used Rogers' Lemma.

Now use Bailey's lemma:

$$\begin{aligned} \sum_{n\geq 0} \frac{q^{n^2}}{(q;q)_n} &= \frac{1}{(q;q)_\infty}\left[1 + \sum_{n\geq 1} q^{n^2}(-1)^n q^{n(3n-1)/2}(1+q^n)\right] \\ &= \frac{1}{(q;q)_\infty}\left[1 + \sum_{n\geq 1} (-1)^n q^{n(5n-1)/2}(1+q^n)\right] \\ &= \frac{1}{(q;q)_\infty}\left[1 + \sum_{n\geq 1} (-1)^n q^{n(5n-1)/2} + \sum_{n\geq 1} (-1)^n q^{n(5n+1)/2}\right] \\ &= \frac{1}{(q;q)_\infty}\left[1 + \sum_{n\geq 1} (-1)^n q^{n(5n-1)/2} + \sum_{n\leq -1} (-1)^n q^{n(5n-1)/2}\right] \\ &= \frac{1}{(q;q)_\infty} \sum_{n\in\mathbb{Z}} (-1)^n q^{n(5n-1)/2}. \end{aligned}$$

Wow!

Let us rewrite the Bailey formula for the instance $a = q$:

Suppose that α_n and β_n are two sequences related by

$$\beta_n = (1-q)\sum_{r=0}^{n} \frac{\alpha_r}{(q;q)_{n-r}(q;q)_{n+r+1}};$$

then

$$\sum_{n\geq 0} q^{n(n+1)} \frac{\beta_n}{1-q} = \frac{1}{(q;q)_\infty} \sum_{n\geq 0} q^{n(n+1)} \alpha_n.$$

Now define

$$\alpha_n = (-1)^n q^{n(3n+1)/2}(1-q^{2n+1}).$$

We compute

$$\frac{\beta_n}{1-q} = \sum_{r=0}^{n} \frac{(-1)^r q^{r(3r+1)/2}(1-q^{2r+1})}{(q;q)_{n-r}(q;q)_{n+r+1}}$$

$$= \sum_{r=0}^{n} \frac{(-1)^r q^{r(3r+1)/2}}{(q;q)_{n-r}(q;q)_{n+r+1}} - \sum_{r=0}^{n} \frac{(-1)^r q^{r(3r+1)/2} q^{2r+1}}{(q;q)_{n-r}(q;q)_{n+r+1}}$$
$$= \sum_{r=-n}^{0} \frac{(-1)^r q^{r(3r-1)/2}}{(q;q)_{n+r}(q;q)_{n-r+1}} + \sum_{r=1}^{n+1} \frac{(-1)^r q^{(r-1)(3r-2)/2+2r-1}}{(q;q)_{n-r+1}(q;q)_{n+r}}$$
$$= \sum_{r=-n}^{n+1} \frac{(-1)^r q^{r(3r-1)/2}}{(q;q)_{n+r}(q;q)_{n-r+1}} = \frac{1}{(q;q)_n},$$

by Rogers' Lemma.

Therefore

$$\sum_{n\geq 0} q^{n(n+1)} \beta_n = (1-q) \sum_{n\geq 0} \frac{q^{n(n+1)}}{(q;q)_n}$$
$$= \frac{1}{(q^2;q)_\infty} \sum_{n\geq 0} q^{n(n+1)} (-1)^n q^{n(3n+1)/2} (1-q^{2n+1})$$

or

$$\sum_{n\geq 0} \frac{q^{n(n+1)}}{(q;q)_n} = \frac{1}{(q;q)_\infty} \sum_{n\geq 0} q^{n(n+1)} (-1)^n q^{n(3n+1)/2} (1-q^{2n+1})$$
$$= \frac{1}{(q;q)_\infty} \left[\sum_{n\geq 0} (-1)^n q^{n(5n+3)/2} - \sum_{n\geq 0} (-1)^n q^{(n+1)(5n+2)/2} \right]$$
$$= \frac{1}{(q;q)_\infty} \left[\sum_{n\geq 0} (-1)^n q^{n(5n+3)/2} + \sum_{n\geq 1} (-1)^n q^{n(5n-3)/2} \right]$$
$$= \frac{1}{(q;q)_\infty} \left[\sum_{n\geq 0} (-1)^n q^{n(5n+3)/2} + \sum_{n\leq -1} (-1)^n q^{n(5n+3)/2} \right]$$
$$= \frac{1}{(q;q)_\infty} \sum_{n\in\mathbb{Z}} (-1)^n q^{n(5n+3)/2}$$
$$= \frac{1}{(q;q)_\infty} \sum_{n\in\mathbb{Z}} (-1)^n q^{n(5n-3)/2}.$$

24. Agarwal's formula

$$\sum_{j=0}^{M} \frac{(1-aq^{2j})(q^{-n};q)_j (a;q)_j q^{nj}}{(1-a)(aq^{n+1};q)_j (q;q)_j} = \frac{(aq;q)_M (q^{1-n};q)_M q^{nM}}{(q;q)_M (aq^{n+1};q)_M}.$$

The proof is by induction, the instance $M = 0$ being obvious.

Consider the $M+1$st term

$$\frac{(1-aq^{2M+2})(q^{-n};q)_{M+1} (a;q)_{M+1} q^{nM+n}}{(1-a)(aq^{n+1};q)_{M+1} (q;q)_{M+1}};$$

let us divide it by the right hand side (to cancel as much as possible):

$$\frac{(1-aq^{2M+2})(q^{-n};q)_{M+1}(a;q)_{M+1}q^{nM+n}(q;q)_M(aq^{n+1};q)_M}{(1-a)(aq^{n+1};q)_{M+1}(q;q)_{M+1}(aq;q)_M(q^{1-n};q)_Mq^{nM}}$$
$$=\frac{(1-q^{-n})q^n(1-aq^{2M+2})}{(1-q^{M+1})(1-aq^{n+1+M})}.$$

We are done if we can prove that

$$\frac{(aq;q)_M(q^{1-n};q)_Mq^{nM}}{(q;q)_M(aq^{n+1};q)_M}\left(1+\frac{(1-q^{-n})q^n(1-aq^{2M+2})}{(1-q^{M+1})(1-aq^{n+1+M})}\right)$$
$$=\frac{(aq;q)_{M+1}(q^{1-n};q)_{M+1}q^{n(M+1)}}{(q;q)_{M+1}(aq^{n+1};q)_{M+1}}$$

or, after cancellations,

$$1+\frac{(1-q^{-n})q^n(1-aq^{2M+2})}{(1-q^{M+1})(1-aq^{n+1+M})}=\frac{(1-aq^{M+1})(1-q^{M+1-n})q^n}{(1-q^{M+1})(1-aq^{M+n+1})}$$

or

$$(1-q^{M+1})(1-aq^{n+1+M})+(1-q^{-n})q^n(1-aq^{2M+2})=(1-aq^{M+1})(1-q^{M+1-n})q^n,$$

which can be confirmed by a computer.

25. q–Pfaff–Saalschütz

We need a definition.

$${}_2\phi_1\left(\begin{matrix}a,b\\c\end{matrix}\;\middle|\;q;z\right):=\sum_{n\geq 0}\frac{(a;q)_n(b;q)_n\,z^n}{(c;q)_n(q;q)_n}.$$

Let us rewrite the transformation formula of Heine

$$\sum_{n\geq 0}\frac{(a;q)_n(b;q)_nt^n}{(q;q)_n(c;q)_n}=\frac{(b;q)_\infty(at;q)_\infty}{(c;q)_\infty(t;q)_\infty}\sum_{m\geq 0}\frac{(c/b;q)_m(t;q)_mb^m}{(q;q)_m(at;q)_m}$$

with this handy notation:

$${}_2\phi_1\left(\begin{matrix}a,b\\c\end{matrix}\;\middle|\;q;t\right)=\frac{(b;q)_\infty(at;q)_\infty}{(c;q)_\infty(t;q)_\infty}{}_2\phi_1\left(\begin{matrix}c/b,t\\at\end{matrix}\;\middle|\;q;b\right).$$

Now do 2 applications of this:

$${}_2\phi_1\left(\begin{matrix}a,b\\c\end{matrix}\;\middle|\;q;x\right)=\frac{(b;q)_\infty(ax;q)_\infty}{(c;q)_\infty(x;q)_\infty}{}_2\phi_1\left(\begin{matrix}x,c/b\\ax\end{matrix}\;\middle|\;q;b\right)$$
$$=\frac{(b;q)_\infty(ax;q)_\infty}{(c;q)_\infty(x;q)_\infty}\frac{(c/b;q)_\infty(bx;q)_\infty}{(ax;q)_\infty(b;q)_\infty}{}_2\phi_1\left(\begin{matrix}b,abx/c\\bx\end{matrix}\;\middle|\;q;c/b\right)$$

$$= \frac{(c/b;q)_\infty (bx;q)_\infty}{(c;q)_\infty (x;q)_\infty} {}_2\phi_1\left(\begin{matrix} b, abx/c \\ bx \end{matrix} \,\middle|\, q; c/b\right).$$

Now do another application!

$$\begin{aligned} {}_2\phi_1\left(\begin{matrix} a, b \\ c \end{matrix} \,\middle|\, q; x\right) &= \frac{(c/b;q)_\infty (bx;q)_\infty}{(c;q)_\infty (x;q)_\infty} {}_2\phi_1\left(\begin{matrix} b, abx/c \\ bx \end{matrix} \,\middle|\, q; c/b\right) \\ &= \frac{(c/b;q)_\infty (bx;q)_\infty}{(c;q)_\infty (x;q)_\infty} \frac{(abx/c;q)_\infty (c;q)_\infty}{(bx;q)_\infty (c/b;q)_\infty} {}_2\phi_1\left(\begin{matrix} c/b, c/a \\ c \end{matrix} \,\middle|\, q; abx/c\right) \\ &= \frac{(abx/c;q)_\infty}{(x;q)_\infty} {}_2\phi_1\left(\begin{matrix} c/b, c/a \\ c \end{matrix} \,\middle|\, q; abx/c\right). \end{aligned}$$

From Cauchy we know that

$$\frac{(abx/c;q)_\infty}{(x;q)_\infty} = \sum_{n\geq 0} \frac{(ab/c;q)_n x^n}{(q;q)_n}.$$

Thus

$${}_2\phi_1\left(\begin{matrix} a, b \\ c \end{matrix} \,\middle|\, q; x\right) = \sum_{n\geq 0} \frac{(ab/c;q)_n x^n}{(q;q)_n} \, {}_2\phi_1\left(\begin{matrix} c/b, c/a \\ c \end{matrix} \,\middle|\, q; abx/c\right).$$

Equate coefficients of x^n:

$$\begin{aligned} \frac{(a;q)_n (b;q)_n}{(c;q)_n (q;q)_n} &= \sum_{k=0}^{n} \frac{(c/a;q)_k (c/b;q)_k}{(c;q)_k (q;q)_k} (ab/c)^k \frac{(ab/c;q)_{n-k}}{(q;q)_{n-k}} \\ &= \frac{(ab/c;q)_n}{(q;q)_n} \sum_{k=0}^{n} \frac{(c/a;q)_k (c/b;q)_k (q^{-n};q)_k q^k}{(c;q)_k (q;q)_k (q^{1-n}c/ab;q)_k}, \end{aligned}$$

or

$$\sum_{k=0}^{n} \frac{(c/a;q)_k (c/b;q)_k (q^{-n};q)_k q^k}{(c;q)_k (q;q)_k (q^{1-n}c/ab;q)_k} = \frac{(a;q)_n (b;q)_n}{(c;q)_n (ab/c;q)_n}.$$

The sum can be extended to infinity. We need another notation:

$${}_3\phi_2\left(\begin{matrix} a, b, c \\ d, e \end{matrix} \,\middle|\, q; z\right) := \sum_{n\geq 0} \frac{(a;q)_n (b;q)_n (c;q)_n \, z^n}{(d;q)_n (e;q)_n (q;q)_n}.$$

Then we found that

$${}_3\phi_2\left(\begin{matrix} c/a, c/b, q^{-n} \\ c, q^{1-n}c/ab \end{matrix} \,\middle|\, q; q\right) = \frac{(a;q)_n (b;q)_n}{(c;q)_n (ab/c;q)_n}.$$

Write $a = c/A$ and $b = c/B$. Then

$${}_3\phi_2\left(\begin{matrix} A, B, q^{-n} \\ c, q^{1-n}AB/c \end{matrix} \,\middle|\, q; q\right) = \frac{(c/A;q)_n (c/B;q)_n}{(c;q)_n (c/AB;q)_n}.$$

Now replace A by a and B by b to get

$${}_3\phi_2\left(\begin{matrix} q^{-n}, a, b \\ c, q^{1-n}ab/c \end{matrix} \,\middle|\, q; q\right) = \frac{(c/a;q)_n (c/b;q)_n}{(c;q)_n (c/ab;q)_n}.$$

This is the desired formula.

South African Sears gave a more general result that we don't need.

26. Generalization of Rogers–Ramanujan

Define

$$J_{k,i}(x) = \sum_{n\geq 0} \frac{(-1)^n x^{kn} q^{kn^2+n+kn-in+\binom{n}{2}}(1-x^i q^{2ni+i})}{(q;q)_n (xq^{n+1};q)_\infty}.$$

Claim:

$$J_{k,i}(x) = J_{k,i-1}(x) + (xq)^{i-1} J_{k,k-i+1}(xq).$$

Proof:

$$\begin{aligned}
J_{k,i}(x) - J_{k,i-1}(x) &= \sum_{n\geq 0} \frac{(-1)^n x^{kn} q^{kn^2+n+kn+\binom{n}{2}}\left[q^{-in}(1-q^n) + x^{i-1}q^{ni-n+i-1}(1-xq^{n+1})\right]}{(q;q)_n (xq^{n+1};q)_\infty} \\
&= \sum_{n\geq 1} \frac{(-1)^n x^{kn} q^{kn^2+n+kn+\binom{n}{2}} q^{-in}(1-q^n)}{(q;q)_n (xq^{n+1};q)_\infty} \\
&+ \sum_{n\geq 0} \frac{(-1)^n x^{kn} q^{kn^2+n+kn+\binom{n}{2}} x^{i-1} q^{ni-n+i-1}(1-xq^{n+1})}{(q;q)_n (xq^{n+1};q)_\infty} \\
&= \sum_{n\geq 1} \frac{(-1)^n x^{kn} q^{kn^2+n+kn+\binom{n}{2}} q^{-in}}{(q;q)_{n-1} (xq^{n+1};q)_\infty} \\
&+ \sum_{n\geq 0} \frac{(-1)^n x^{kn} q^{kn^2+n+kn+\binom{n}{2}} x^{i-1} q^{ni-n+i-1}}{(q;q)_n (xq^{n+2};q)_\infty} \\
&= -\sum_{n\geq 0} \frac{(-1)^n x^{kn+k} q^{k(n+1)^2+n+1+kn+k+\binom{n+1}{2}} q^{-in-i}}{(q;q)_n (xq^{n+2};q)_\infty} \\
&+ \sum_{n\geq 0} \frac{(-1)^n x^{kn} q^{kn^2+n+kn+\binom{n}{2}} x^{i-1} q^{ni-n+i-1}}{(q;q)_n (xq^{n+2};q)_\infty} \\
&= \sum_{n\geq 0} \frac{(-1)^n x^{kn} q^{kn^2+n+kn+\binom{n}{2}}\left[x^{i-1}q^{ni-n+i-1} - x^k q^{2kn+k+1+k+n-in-i}\right]}{(q;q)_n (xq^{n+2};q)_\infty} \\
&= (xq)^{i-1} \sum_{n\geq 0} \frac{(-1)^n x^{kn} q^{kn^2+n+kn+\binom{n}{2}}\left[q^{ni-n} - x^{k+1-i} q^{2kn+k+2+k+n-in-2i}\right]}{(q;q)_n (xq^{n+2};q)_\infty} \\
&= (xq)^{i-1} \sum_{n\geq 0} \frac{(-1)^n x^{kn} q^{kn^2+ni+kn+\binom{n}{2}}\left[1 - x^{k+1-i} q^{2kn+2k+2+2n-2in-2i}\right]}{(q;q)_n (xq^{n+2};q)_\infty}
\end{aligned}$$

$$= (xq)^{i-1} \sum_{n\geq 0} \frac{(-1)^n (xq)^{kn} q^{kn^2+ni+\binom{n}{2}} \left[1-(xq)^{k+1-i} q^{2n(k+1-i)+k+1-i}\right]}{(q;q)_n (xq^{n+2};q)_\infty}$$

$$= (xq)^{i-1} \sum_{n\geq 0} \frac{(-1)^n (xq)^{kn} q^{kn^2-n(k+1-i)+nk+n+\binom{n}{2}} \left[1-(xq)^{k+1-i} q^{2n(k+1-i)+k+1-i}\right]}{(q;q)_n (xq^{n+2};q)_\infty}$$

$$= (xq)^{i-1} J_{k,k-i+1}(xq),$$

as desired.

Note that

$$J_{k,i}(1) = \sum_{n\geq 0} \frac{(-1)^n q^{kn^2+n+kn-in+\binom{n}{2}} (1-q^{2ni+i})}{(q;q)_n (q^{n+1};q)_\infty}$$

$$= \frac{1}{(q;q)_\infty} \sum_{n\geq 0} (-1)^n q^{(2k+1)\binom{n+1}{2}-in} (1-q^{(2n+1)i})$$

$$= \frac{1}{(q;q)_\infty} \prod_{n\geq 0} (1-q^{(2k+1)(n+1)})(1-q^{(2k+1)n+i})(1-q^{(2k+1)(n+1)-i})$$

$$= \prod_{\substack{n\geq 1 \\ n\not\equiv 0,\ \pm i \pmod{2k+1}}} \frac{1}{1-q^n};$$

here, we have used a formula that we derived long time ago without any idea what it might be good for!

Claim:

$$J_{k,i}(x) = \sum_{n\geq 0} \frac{x^{(k-1)n} q^{(k-1)n^2+(k-i)n}}{(q;q)_n} J_{k-1,i}(xq^{2n}).$$

Proof: Consider the recursion

$$J_{k,i}(x) = J_{k,i-1}(x) + (xq)^{i-1} J_{k,k-i+1}(xq)$$

and plug in $i = k+1$:

$$J_{k,k+1}(x) = J_{k,k}(x) + (xq)^k J_{k,0}(xq).$$

But look at

$$J_{k,i}(x) = \sum_{n\geq 0} \frac{(-1)^n x^{kn} q^{kn^2+n+kn-in+\binom{n}{2}} (1-x^i q^{2ni+i})}{(q;q)_n (xq^{n+1};q)_\infty}$$

and see that $i = 0$ produces 0. Thus

$$J_{k,k+1}(x) = J_{k,k}(x).$$

The recursion also tells us

$$J_{1,1}(x) = J_{1,1}(xq) = J_{1,1}(xq^2) = \cdots = J_{1,1}(0) = 1.$$

The proof is by induction on k. Define

$$R_{k,i}(x) := \sum_{n\geq 0} \frac{x^{(k-1)n} q^{(k-1)n^2+(k-i)n}}{(q;q)_n} J_{k-1,i}(xq^{2n}).$$

For $1 \leq i \leq k$ we have

$$R_{k,i}(0) = J_{k-1,i}(0) = 1;$$

also

$$R_{k,0}(x) = 0.$$

We will show that the $R_{k,i}$'s satisfy the same recursion as the $J_{k,i}$'s; thus they must coincide.

$$\begin{aligned}
&R_{k,i}(x) - R_{k,i-1}(x) \\
&= \sum_{n\geq 0} \frac{x^{(k-1)n} q^{(k-1)n^2+(k-i)n}}{(q;q)_n} \Big[J_{k-1,i}(xq^{2n}) - q^n J_{k-1,i-1}(xq^{2n}) \Big] \\
&= \sum_{n\geq 0} \frac{x^{(k-1)n} q^{(k-1)n^2+(k-i)n}}{(q;q)_n} \Big[J_{k-1,i-1}(xq^{2n}) + (xq^{2n+1})^{i-1} J_{k-1,k-i}(xq^{2n+1}) - q^n J_{k-1,i-1}(xq^{2n}) \Big] \\
&= \sum_{n\geq 0} \frac{x^{(k-1)n} q^{(k-1)n^2+(k-i)n}}{(q;q)_n} (1-q^n) J_{k-1,i-1}(xq^{2n}) \\
&\qquad + \sum_{n\geq 0} \frac{x^{(k-1)n} q^{(k-1)n^2+(k-i)n}}{(q;q)_n} (xq^{2n+1})^{i-1} J_{k-1,k-i}(xq^{2n+1}) \\
&= \sum_{n\geq 1} \frac{x^{(k-1)n} q^{(k-1)n^2+(k-i)n}}{(q;q)_n} (1-q^n) J_{k-1,i-1}(xq^{2n}) \\
&\qquad + (xq)^{i-1} \sum_{n\geq 0} \frac{x^{(k-1)n} q^{(k-1)n^2+(k+i-2)n}}{(q;q)_n} J_{k-1,k-i}(xq^{2n+1}) \\
&= \sum_{n\geq 0} \frac{x^{(k-1)(n+1)} q^{(k-1)(n+1)^2+(k-i)(n+1)}}{(q;q)_n} J_{k-1,i-1}(xq^{2n+2}) \\
&\qquad + (xq)^{i-1} \sum_{n\geq 0} \frac{x^{(k-1)n} q^{(k-1)n^2+(k+i-2)n}}{(q;q)_n} J_{k-1,k-i}(xq^{2n+1}) \\
&= x^{k-1} q^{2k-i-1} \sum_{n\geq 0} \frac{x^{(k-1)n} q^{(k-1)n^2+(3k-i-2)n}}{(q;q)_n} J_{k-1,i-1}(xq^{2n+2}) \\
&+ (xq)^{i-1} \sum_{n\geq 0} \frac{x^{(k-1)n} q^{(k-1)n^2+(k+i-2)n}}{(q;q)_n} J_{k-1,k-i+1}(xq^{2n+1}) \\
&- x^{k-1} q^{2k-i-1} \sum_{n\geq 0} \frac{x^{(k-1)n} q^{(k-1)n^2+(3k-i-2)n}}{(q;q)_n} J_{k-1,i-1}(xq^{2n+2})
\end{aligned}$$

$$= (xq)^{i-1} \sum_{n\geq 0} \frac{x^{(k-1)n} q^{(k-1)n^2+(k+i-2)n}}{(q;q)_n} J_{k-1,k-i+1}(xq^{2n+1})$$

$$= (xq)^{i-1} R_{k,k-i+1}(xq).$$

The induction is finished.—

The recursion

$$J_{k,i}(x) = \sum_{n\geq 0} \frac{x^{(k-1)n} q^{(k-1)n^2+(k-i)n}}{(q;q)_n} J_{k-1,i}(xq^{2n})$$

can be iterated, with the result

$$J_{k,i}(x) = \sum_{n_1,\ldots,n_{k-1}\geq 0} \frac{x^{N_1+\cdots+N_{k-1}} q^{N_1^2+\cdots+N_{k-1}^2+N_i+\cdots+N_{k-1}}}{(q;q)_{n_1}\ldots(q;q)_{n_{k-1}}},$$

with $N_j = n_j + \cdots + n_{k-1}$.

Plug in $x = 1$ to get

$$\sum_{n_1,\ldots,n_{k-1}\geq 0} \frac{q^{N_1^2+\cdots+N_{k-1}^2+N_i+\cdots+N_{k-1}}}{(q;q)_{n_1}\ldots(q;q)_{n_{k-1}}} = \prod_{\substack{n\geq 1\\ n\not\equiv 0,\ \pm i \pmod{2k+1}}} \frac{1}{1-q^n}.$$

Set $k = 2$, $i = 2$:

$$\sum_{n_1\geq 0} \frac{q^{n_1^2}}{(q;q)_{n_1}} = \prod_{\substack{n\geq 1\\ n\not\equiv 0,\ \pm 2 \pmod 5}} \frac{1}{1-q^n} = \prod_{n\geq 0} \frac{1}{(1-q^{5n+1})(1-q^{5n+4})}.$$

This is the first Rogers–Ramanujan identity.

Set $k = 2$, $i = 1$:

$$\sum_{n_1\geq 0} \frac{q^{n_1^2+n_1}}{(q;q)_{n_1}} = \prod_{\substack{n\geq 1\\ n\not\equiv 0,\ \pm 1 \pmod 5}} \frac{1}{1-q^n} = \prod_{n\geq 0} \frac{1}{(1-q^{5n+2})(1-q^{5n+3})}.$$

This is the second Rogers–Ramanujan identity.

Note

$$J_{3,1}(x) = \sum_{n\geq 0} \frac{(-1)^n x^{3n} q^{3n^2+3n+\binom{n}{2}} (1-xq^{2n+1})}{(q;q)_n (xq^{n+1};q)_\infty},$$

$$J_{3,2}(x) = \sum_{n\geq 0} \frac{(-1)^n x^{3n} q^{3n^2+2n+\binom{n}{2}} (1-x^2q^{4n+2})}{(q;q)_n (xq^{n+1};q)_\infty},$$

$$J_{3,3}(x) = \sum_{n\geq 0} \frac{(-1)^n x^{3n} q^{3n^2+n+\binom{n}{2}} (1-x^3q^{6n+3})}{(q;q)_n (xq^{n+1};q)_\infty}.$$

Define

$$R_1(x) = (-xq;q)_\infty \sum_{n\geq 0} \frac{q^{2n^2+2n}x^{2n}}{(q^2;q^2)_n(-xq;q)_{2n+1}},$$
$$R_2(x) = (-xq;q)_\infty \sum_{n\geq 0} \frac{q^{2n^2+2n}x^{2n}}{(q^2;q^2)_n(-xq;q)_{2n}},$$
$$R_3(x) = (-xq;q)_\infty \sum_{n\geq 0} \frac{q^{2n^2}x^{2n}}{(q^2;q^2)_n(-xq;q)_{2n}}.$$

Then

$$\begin{aligned}
R_3(x) - R_2(x) &= (-xq;q)_\infty \sum_{n\geq 0} \frac{q^{2n^2}x^{2n}}{(q^2;q^2)_n(-xq;q)_{2n}} - (-xq;q)_\infty \sum_{n\geq 0} \frac{q^{2n^2+2n}x^{2n}}{(q^2;q^2)_n(-xq;q)_{2n}} \\
&= (-xq;q)_\infty \sum_{n\geq 0} \frac{q^{2n^2}x^{2n}(1-q^{2n})}{(q^2;q^2)_n(-xq;q)_{2n}} \\
&= (-xq^2;q)_\infty \sum_{n\geq 1} \frac{q^{2n^2}x^{2n}}{(q^2;q^2)_{n-1}(-xq^2;q)_{2n-1}} \\
&= x^2q^2(-xq^2;q)_\infty \sum_{n\geq 0} \frac{q^{2n^2+4n}x^{2n}}{(q^2;q^2)_n(-xq^2;q)_{2n+1}}.
\end{aligned}$$

But

$$x^2q^2R_1(xq) = x^2q^2(-xq^2;q)_\infty \sum_{n\geq 0} \frac{q^{2n^2+2n+2n}x^{2n}}{(q^2;q^2)_n(-xq^2;q)_{2n+1}},$$

so

$$R_3(x) - R_2(x) = x^2q^2R_1(xq).$$

Similarly,

$$\begin{aligned}
R_2(x) - R_1(x) &= (-xq;q)_\infty \sum_{n\geq 0} \frac{q^{2n^2+2n}x^{2n}}{(q^2;q^2)_n(-xq;q)_{2n}} - (-xq;q)_\infty \sum_{n\geq 0} \frac{q^{2n^2+2n}x^{2n}}{(q^2;q^2)_n(-xq;q)_{2n+1}} \\
&= (-xq;q)_\infty \sum_{n\geq 0} \frac{q^{2n^2+2n}x^{2n}xq^{2n+1}}{(q^2;q^2)_n(-xq;q)_{2n+1}} \\
&= xq(-xq;q)_\infty \sum_{n\geq 0} \frac{q^{2n^2+4n}x^{2n}}{(q^2;q^2)_n(-xq;q)_{2n+1}} \\
&= xqR_2(xq).
\end{aligned}$$

Finally,

$$R_3(xq) = (-xq^2;q)_\infty \sum_{n\geq 0} \frac{q^{2n^2+2n}x^{2n}}{(q^2;q^2)_n(-xq^2;q)_{2n}}$$
$$= (-xq;q)_\infty \sum_{n\geq 0} \frac{q^{2n^2+2n}x^{2n}}{(q^2;q^2)_n(-xq;q)_{2n+1}} = R_1(x).$$

These are the celebrated recursions, thus

$$R_1(1) = J_{3,1}(1),$$
$$R_2(1) = J_{3,2}(1),$$
$$R_3(1) = J_{3,3}(1).$$

Thus

$$(-q;q)_\infty \sum_{n\geq 0} \frac{q^{2n^2+2n}}{(q^2;q^2)_n(-q;q)_{2n+1}} = \prod_{\substack{n\geq 1\\ n\not\equiv 0,\,\pm 1 \pmod 7}} \frac{1}{1-q^n},$$
$$(-q;q)_\infty \sum_{n\geq 0} \frac{q^{2n^2+2n}}{(q^2;q^2)_n(-q;q)_{2n}} = \prod_{\substack{n\geq 1\\ n\not\equiv 0,\,\pm 2 \pmod 7}} \frac{1}{1-q^n},$$
$$(-q;q)_\infty \sum_{n\geq 0} \frac{q^{2n^2}}{(q^2;q^2)_n(-q;q)_{2n}} = \prod_{\substack{n\geq 1\\ n\not\equiv 0,\,\pm 3 \pmod 7}} \frac{1}{1-q^n}.$$

27. Some "Constant term" proofs

We will use the Rogers–Ramanujan identities:

$$G(q) = \sum_{n\geq 0} \frac{q^{n^2}}{(q;q)_n} = \prod_{n\geq 1} \frac{1}{(1-q^{5n-4})(1-q^{5n-1})},$$
$$H(q) = \sum_{n\geq 0} \frac{q^{n^2+n}}{(q;q)_n} = \prod_{n\geq 1} \frac{1}{(1-q^{5n-3})(1-q^{5n-2})}.$$

Remember Ramanujan's identity?

$$\sum_{n\in\mathbb{Z}} \frac{(a;q)_n t^n}{(b;q)_n} = \frac{(b/a;q)_\infty (at;q)_\infty (q/at;q)_\infty (q;q)_\infty}{(q/a;q)_\infty (b/at;q)_\infty (b;q)_\infty (t;q)_\infty}$$

We need this:

$$\sum_{n\in\mathbb{Z}} \frac{(a^{-1};q)_n (at)^n}{(b;q)_n} = \frac{(ba;q)_\infty (t;q)_\infty (q/t;q)_\infty (q;q)_\infty}{(qa;q)_\infty (b/t;q)_\infty (b;q)_\infty (at;q)_\infty}$$

Now let $a \to 0$:

$$\sum_{n\in\mathbb{Z}} \frac{(-1)^n q^{\binom{n}{2}} t^n}{(b;q)_n} = \frac{(t;q)_\infty (q/t;q)_\infty (q;q)_\infty}{(b/t;q)_\infty (b;q)_\infty}. \tag{8}$$

Replace q by q^2:

$$\sum_{n\in\mathbb{Z}} \frac{(-1)^n q^{n^2-n} t^n}{(b;q^2)_n} = \frac{(t;q^2)_\infty (q^2/t;q^2)_\infty (q^2;q^2)_\infty}{(b/t;q^2)_\infty (b;q^2)_\infty}.$$

Replace t by tq:

$$\sum_{n\in\mathbb{Z}} \frac{(-1)^n q^{n^2} t^n}{(b;q^2)_n} = \frac{(tq;q^2)_\infty (q/t;q^2)_\infty (q^2;q^2)_\infty}{(b/tq;q^2)_\infty (b;q^2)_\infty}.$$

Replace t by $-z$:

$$\sum_{n\in\mathbb{Z}} \frac{q^{n^2} z^n}{(b;q^2)_n} = \frac{(-zq;q^2)_\infty (-q/z;q^2)_\infty (q^2;q^2)_\infty}{(-b/zq;q^2)_\infty (b;q^2)_\infty}. \tag{9}$$

Replace b by q:

$$\sum_{n\in\mathbb{Z}} \frac{q^{n^2} z^n}{(q;q^2)_n} = \frac{(-zq;q^2)_\infty (-q/z;q^2)_\infty (q^2;q^2)_\infty}{(-1/z;q^2)_\infty (q;q^2)_\infty}.$$

And now:

$$\begin{aligned}
\sum_{n\geq 0} \frac{q^{n(3n-1)/2}}{(q;q)_n (q;q^2)_n} &= [z^0] \sum_{n\in\mathbb{Z}} \frac{q^{n^2} z^n}{(q;q^2)_n} \sum_{m\geq 0} \frac{z^{-m} q^{\binom{m}{2}}}{(q;q)_m} \\
&= [z^0] \frac{(-zq;q^2)_\infty (-z^{-1}q;q^2)_\infty (q^2;q^2)_\infty}{(-z^{-1};q^2)_\infty (q;q^2)_\infty} \cdot (-z^{-1};q)_\infty \\
&= \frac{1}{(q;q^2)_\infty} [z^0] \frac{(-zq;q^2)_\infty (-z^{-1}q;q^2)_\infty (q^2;q^2)_\infty (-z^{-1};q)_\infty}{(-z^{-1};q^2)_\infty} \\
&= \frac{1}{(q;q^2)_\infty} [z^0] (-zq;q^2)_\infty (-z^{-1}q;q^2)_\infty (q^2;q^2)_\infty \cdot (-z^{-1}q;q^2)_\infty \\
&= \frac{1}{(q;q^2)_\infty} [z^0] \sum_{n\in\mathbb{Z}} q^{n^2} z^n \sum_{m\geq 0} \frac{z^{-m} q^{m^2}}{(q^2;q^2)_m} \\
&= \frac{1}{(q;q^2)_\infty} \sum_{n\geq 0} \frac{q^{2n^2}}{(q^2;q^2)_n} \\
&= \frac{1}{(q;q^2)_\infty} G(q^2) \\
&= \prod_{n\geq 1} \frac{1}{1-q^{2n-1}} \prod_{n\geq 1} \frac{1}{(1-q^{10n-8})(1-q^{10n-2})} \\
&= \prod_{n\geq 1,\ n\not\equiv 0,\pm 4 \pmod{10}} \frac{1}{1-q^n}
\end{aligned}$$

Go back to

$$\sum_{n\in\mathbb{Z}}\frac{q^{n^2}z^n}{(b;q^2)_n}=\frac{(-zq;q^2)_\infty(-q/z;q^2)_\infty(q^2;q^2)_\infty}{(-b/zq;q^2)_\infty(b;q^2)_\infty}.$$

Replace b by q^3:

$$\sum_{n\in\mathbb{Z}}\frac{q^{n^2}z^n}{(q^3;q^2)_n}=\frac{(-zq;q^2)_\infty(-q/z;q^2)_\infty(q^2;q^2)_\infty}{(q^2/z;q^2)_\infty(q^3;q^2)_\infty}.$$

Divide by $1-q$:

$$\sum_{n\in\mathbb{Z}}\frac{q^{n^2}z^n}{(q;q^2)_{n+1}}=\frac{(-zq;q^2)_\infty(-q/z;q^2)_\infty(q^2;q^2)_\infty}{(-q^2/z;q^2)_\infty(q;q^2)_\infty}.$$

The next one:

$$\begin{aligned}
\sum_{n\geq 0}\frac{q^{3n(n+1)/2}}{(q;q)_n(q;q^2)_{n+1}}&=[z^0]\sum_{n\in\mathbb{Z}}\frac{q^{n^2}z^n}{(q;q^2)_{n+1}}\sum_{m\geq 0}\frac{z^{-m}q^{\binom{m}{2}+2m}}{(q;q)_m}\\
&=[z^0]\frac{(-zq;q^2)_\infty(-z^{-1}q;q^2)_\infty(q^2;q^2)_\infty}{(-z^{-1}q^2;q^2)_\infty(q;q^2)_\infty}\cdot(-z^{-1}q^2;q)_\infty\\
&=\frac{1}{(q;q^2)_\infty}[z^0](-zq;q^2)_\infty(-z^{-1}q;q^2)_\infty(q^2;q^2)_\infty\cdot(-z^{-1}q^3;q^2)_\infty\\
&=\frac{1}{(q;q^2)_\infty}[z^0]\sum_{n\in\mathbb{Z}}q^{n^2}z^n\sum_{m\geq 0}\frac{z^{-m}q^{m^2+2m}}{(q^2;q^2)_m}\\
&=\frac{1}{(q;q^2)_\infty}\sum_{n\geq 0}\frac{q^{2n^2+2n}}{(q^2;q^2)_n}\\
&=\frac{1}{(q;q^2)_\infty}H(q^2)\\
&=\prod_{n\geq 1}\frac{1}{1-q^{2n-1}}\prod_{n\geq 1}\frac{1}{(1-q^{10n-6})(1-q^{10n-4})}\\
&=\prod_{n\geq 1,\ n\not\equiv 0,\pm 2\pmod{10}}\frac{1}{1-q^n}.
\end{aligned}$$

Go back to

$$\sum_{n\in\mathbb{Z}}\frac{q^{n^2}z^n}{(b;q^2)_n}=\frac{(-zq;q^2)_\infty(-q/z;q^2)_\infty(q^2;q^2)_\infty}{(-b/zq;q^2)_\infty(b;q^2)_\infty}.$$

Replace b by $-q^3$:

$$\sum_{n\in\mathbb{Z}}\frac{q^{n^2}z^n}{(-q^3;q^2)_n}=\frac{(-zq;q^2)_\infty(-q/z;q^2)_\infty(q^2;q^2)_\infty}{(q^2/z;q^2)_\infty(-q^3;q^2)_\infty}.$$

Divide by $1+q$:

$$\sum_{n\in\mathbb{Z}} \frac{q^{n^2}z^n}{(-q;q^2)_{n+1}} = \frac{(-zq;q^2)_\infty(-q/z;q^2)_\infty(q^2;q^2)_\infty}{(q^2/z;q^2)_\infty(-q;q^2)_\infty}.$$

The next one:

$$\begin{aligned}
\sum_{n\geq 0} \frac{q^{n(n+1)}}{(q^2;q^2)_n(-q;q^2)_{n+1}} &= [z^0]\sum_{n\in\mathbb{Z}} \frac{q^{n^2}z^n}{(-q;q^2)_{n+1}} \sum_{m\geq 0} \frac{z^{-m}q^m}{(q^2;q^2)_m} \\
&= [z^0]\frac{(-zq;q^2)_\infty(-q/z;q^2)_\infty(q^2;q^2)_\infty}{(q^2/z;q^2)_\infty(-q;q^2)_\infty} \cdot \frac{1}{(q/z;q^2)_\infty} \\
&= \frac{1}{(-q;q^2)_\infty}[z^0]\frac{(-zq;q^2)_\infty(-q/z;q^2)_\infty(q^2;q^2)_\infty}{(q^2/z;q^2)_\infty(q/z;q^2)_\infty} \\
&= \frac{1}{(-q;q^2)_\infty}[z^0]\frac{(-zq;q^2)_\infty(-q/z;q^2)_\infty(q^2;q^2)_\infty}{(q/z;q)_\infty} \\
&= \frac{1}{(-q;q^2)_\infty}[z^0]\sum_{n\in\mathbb{Z}} q^{n^2}z^n \sum_{m\geq 0} \frac{z^{-m}q^m}{(q;q)_m} \\
&= \frac{1}{(-q;q^2)_\infty}\sum_{n\geq 0} \frac{q^{n^2+n}}{(q;q)_n} \\
&= \frac{1}{(-q;q^2)_\infty}H(q).
\end{aligned}$$

Now change q to $-q$:

$$\begin{aligned}
\sum_{n\geq 0}\frac{q^{n(n+1)}}{(q;q)_{2n+1}} &= \frac{1}{(q;q^2)_\infty}H(-q) \\
&= \prod_{n\geq 1} \frac{1}{1-q^{2n-1}}\frac{1}{1-(-q)^{5n-2}}\frac{1}{1-(-q)^{5n-3}} \\
&= \prod_{n\geq 1} \frac{1}{1-q^{2n-1}}\frac{1}{1-(-q)^{10n-2}}\frac{1}{1-(-q)^{10n-3}}\frac{1}{1-(-q)^{10n-7}}\frac{1}{1-(-q)^{10n-8}} \\
&= \prod_{n\geq 1} \frac{1}{1-q^{2n-1}}\frac{1}{1-q^{10n-2}}\frac{1}{1+q^{10n-3}}\frac{1}{1+q^{10n-7}}\frac{1}{1-q^{10n-8}} \\
&= \prod_{n\geq 1} \frac{1-q^{2n}}{1-q^n}\frac{1}{1-q^{10n-2}}\frac{1-q^{10n-3}}{1-q^{20n-6}}\frac{1-q^{10n-7}}{1-q^{20n-14}}\frac{1}{1-q^{10n-8}} \\
&= \prod_{n\geq 1} \frac{(1-q^{10n})(1-q^{10n-2})(1-q^{10n-4})(1-q^{10n-6})(1-q^{10n-8})}{1-q^n}\frac{1}{1-q^{10n-2}}\frac{1-q^{10n-3}}{1-q^{20n-6}}\frac{1-q^{10n-7}}{1-q^{20n-14}}\frac{1}{1-q^{10n-8}} \\
&= \prod_{n\geq 1} \frac{(1-q^{10n})(1-q^{20n-14})(1-q^{20n-4})(1-q^{20n-16})(1-q^{20n-6})(1-q^{10n-3})(1-q^{10n-7})}{(1-q^n)(1-q^{20n-6})(1-q^{20n-14})} \\
&= \prod_{n\geq 1} \frac{(1-q^{10n})(1-q^{20n-4})(1-q^{20n-16})(1-q^{10n-3})(1-q^{10n-7})}{1-q^n}.
\end{aligned}$$

Go back to

$$\sum_{n\in\mathbb{Z}}\frac{q^{n^2}z^n}{(b;q^2)_n}=\frac{(-zq;q^2)_\infty(-q/z;q^2)_\infty(q^2;q^2)_\infty}{(-b/zq;q^2)_\infty(b;q^2)_\infty}.$$

Replace b by $-q$:

$$\sum_{n\in\mathbb{Z}}\frac{q^{n^2}z^n}{(-q;q^2)_n}=\frac{(-zq;q^2)_\infty(-q/z;q^2)_\infty(q^2;q^2)_\infty}{(1/z;q^2)_\infty(-q;q^2)_\infty}.$$

The next one:

$$\begin{aligned}
\sum_{n\geq 0}\frac{q^{n(n+1)}}{(q^2;q^2)_n(-q;q^2)_n}&=[z^0]\sum_{n\in\mathbb{Z}}\frac{q^{n^2}z^n}{(-q;q^2)_n}\sum_{m\geq 0}\frac{z^{-m}q^m}{(q^2;q^2)_m}\\
&=[z^0]\frac{(-zq;q^2)_\infty(-q/z;q^2)_\infty(q^2;q^2)_\infty}{(1/z;q^2)_\infty(-q;q^2)_\infty}\cdot\frac{1}{(q/z;q^2)_\infty}\\
&=\frac{1}{(-q;q^2)_\infty}[z^0]\frac{(-zq;q^2)_\infty(-q/z;q^2)_\infty(q^2;q^2)_\infty}{(1/z;q)_\infty}\\
&=\frac{1}{(-q;q^2)_\infty}[z^0]\sum_{n\in\mathbb{Z}}q^{n^2}z^n\sum_{m\geq 0}\frac{z^{-m}}{(q;q)_m}\\
&=\frac{1}{(-q;q^2)_\infty}\sum_{n\geq 0}\frac{q^{n^2}}{(q;q)_n}\\
&=\frac{1}{(-q;q^2)_\infty}G(q).
\end{aligned}$$

Now change q to $-q$:

$$\begin{aligned}
\sum_{n\geq 0}\frac{q^{n(n+1)}}{(q;q)_{2n}}&=\frac{1}{(q;q^2)_\infty}G(-q)\\
&=\prod_{n\geq 1}\frac{1}{1-q^{2n-1}}\frac{1}{1-(-q)^{5n-1}}\frac{1}{1-(-q)^{5n-4}}\\
&=\prod_{n\geq 1}\frac{1}{1-q^{2n-1}}\frac{1}{1-(-q)^{10n-1}}\frac{1}{1-(-q)^{10n-6}}\frac{1}{1-(-q)^{10n-4}}\frac{1}{1-(-q)^{10n-9}}\\
&=\prod_{n\geq 1}\frac{(1-q^{2n})(1-q^{10n-1})(1-q^{10n-9})}{(1-q^n)(1-q^{20n-2})(1-q^{10n-6})(1-q^{10n-4})(1-q^{20n-18})}\\
&=\prod_{n\geq 1}\frac{(1-q^{10n-2})(1-q^{10n-8})(1-q^{10n})(1-q^{10n-1})(1-q^{10n-9})}{(1-q^n)(1-q^{20n-2})(1-q^{20n-18})}\\
&=\prod_{n\geq 1}\frac{(1-q^{20n-2})(1-q^{20n-12})(1-q^{20n-8})(1-q^{20n-18})(1-q^{10n})(1-q^{10n-1})(1-q^{10n-9})}{(1-q^n)(1-q^{20n-2})(1-q^{20n-18})}
\end{aligned}$$

$$= \prod_{n\geq 1} \frac{(1-q^{20n-12})(1-q^{20n-8})(1-q^{10n})(1-q^{10n-1})(1-q^{10n-9})}{(1-q^n)}$$

Go back to

$$\sum_{n\in\mathbb{Z}} \frac{q^{n^2}z^n}{(b;q^2)_n} = \frac{(-zq;q^2)_\infty(-q/z;q^2)_\infty(q^2;q^2)_\infty}{(-b/zq;q^2)_\infty(b;q^2)_\infty}.$$

Replace b by q:

$$\sum_{n\in\mathbb{Z}} \frac{q^{n^2}z^n}{(q;q^2)_n} = \frac{(-zq;q^2)_\infty(-q/z;q^2)_\infty(q^2;q^2)_\infty}{(-1/z;q^2)_\infty(q;q^2)_\infty}.$$

The next one:

$$\begin{aligned}
\sum_{n\geq 0} \frac{q^{n^2}}{(q^2;q^2)_n(q;q^2)_n} &= [z^0]\sum_{n\in\mathbb{Z}} \frac{q^{n^2}z^n}{(q;q^2)_n}\sum_{m\geq 0}\frac{z^{-m}}{(q^2;q^2)_m} \\
&= [z^0]\frac{(-zq;q^2)_\infty(-q/z;q^2)_\infty(q^2;q^2)_\infty}{(-1/z;q^2)_\infty(q;q^2)_\infty}\cdot\frac{1}{(1/z;q^2)_\infty} \\
&= \frac{1}{(q;q^2)_\infty}[z^0]\frac{(-zq;q^2)_\infty(-q/z;q^2)_\infty(q^2;q^2)_\infty}{(1/z^2;q^4)_\infty} \\
&= \frac{1}{(q;q^2)_\infty}[z^0]\sum_{n\in\mathbb{Z}} q^{n^2}z^n\sum_{m\geq 0}\frac{z^{-2m}}{(q^4;q^4)_m} \\
&= \frac{1}{(q;q^2)_\infty}\sum_{n\geq 0}\frac{q^{4n^2}}{(q^4;q^4)_n} \\
&= \frac{1}{(q;q^2)_\infty}G(q^4) \\
&= \prod_{n\geq 1}\frac{1}{(1-q^{2n-1})(1-q^{20n-16})(1-q^{20n-4})}.
\end{aligned}$$

The next one:

$$\begin{aligned}
\sum_{n\geq 0} \frac{q^{n^2+2n}}{(q^2;q^2)_n(q;q^2)_{n+1}} &= [z^0]\sum_{n\in\mathbb{Z}} \frac{q^{n^2}z^n}{(q;q^2)_{n+1}}\sum_{m\geq 0}\frac{z^{-m}q^{2m}}{(q^2;q^2)_m} \\
&= [z^0]\frac{(-zq;q^2)_\infty(-q/z;q^2)_\infty(q^2;q^2)_\infty}{(-q^2/z;q^2)_\infty(q;q^2)_\infty}\cdot\frac{1}{(q^2/z;q^2)_\infty} \\
&= \frac{1}{(q;q^2)_\infty}[z^0]\frac{(-zq;q^2)_\infty(-q/z;q^2)_\infty(q^2;q^2)_\infty}{(q^4/z^2;q^4)_\infty} \\
&= \frac{1}{(q;q^2)_\infty}[z^0]\sum_{n\in\mathbb{Z}} q^{n^2}z^n\sum_{m\geq 0}\frac{z^{-2m}q^{4m}}{(q^4;q^4)_m}
\end{aligned}$$

$$= \frac{1}{(q;q^2)_\infty} \sum_{n\geq 0} \frac{q^{4n^2+4n}}{(q^4;q^4)_n}$$
$$= \frac{1}{(q;q^2)_\infty} H(q^4)$$
$$= \prod_{n\geq 1} \frac{1}{(1-q^{2n-1})(1-q^{20n-8})(1-q^{20n-12})}$$

Multiply this by q:

$$\sum_{n\geq 0} \frac{q^{(n+1)^2}}{(q;q)_{2n+1}} = q \prod_{n\geq 1} \frac{1}{(1-q^{2n-1})(1-q^{20n-8})(1-q^{20n-12})}$$

or, nicer,

$$\sum_{n\geq 1} \frac{q^{n^2}}{(q;q)_{2n-1}} = q \prod_{n\geq 1} \frac{1}{(1-q^{2n-1})(1-q^{20n-8})(1-q^{20n-12})}$$

The next one:

$$\sum_{n\geq 0} \frac{q^{n^2}}{(q^4;q^4)_n} = \sum_{n\geq 0} \frac{q^{n^2}}{(q^2;q^2)_n(-q^2;q^2)_n}$$
$$= [z^0] \sum_{n\in\mathbb{Z}} \frac{q^{n^2} z^n}{(-q^2;q^2)_n} \sum_{m\geq 0} \frac{z^{-m}}{(q^2;q^2)_m}$$
$$= [z^0] \frac{(-zq;q^2)_\infty (-q/z;q^2)_\infty (q^2;q^2)_\infty}{(q/z;q^2)_\infty (-q^2;q^2)_\infty} \cdot \frac{1}{(1/z;q^2)_\infty}$$
$$= \frac{1}{(-q^2;q^2)_\infty} [z^0] \frac{(-zq;q^2)_\infty (-q/z;q^2)_\infty (q^2;q^2)_\infty}{(q/z;q^2)_\infty (1/z;q^2)_\infty}$$
$$= \frac{1}{(-q^2;q^2)_\infty} [z^0] \frac{(-zq;q^2)_\infty (-q/z;q^2)_\infty (q^2;q^2)_\infty}{(1/z;q)_\infty}$$
$$= \frac{1}{(-q^2;q^2)_\infty} [z^0] \sum_{n\in\mathbb{Z}} q^{n^2} z^n \sum_{m\geq 0} \frac{z^{-m}}{(q;q)_m}$$
$$= \frac{1}{(-q^2;q^2)_\infty} \sum_{n\geq 0} \frac{q^{n^2}}{(q;q)_n}$$
$$= \frac{1}{(-q^2;q^2)_\infty} G(q)$$
$$= \prod_{n\geq 1} \frac{1}{(1+q^{2n})(1-q^{5n-1})(1-q^{5n-4})}$$
$$= \prod_{n\geq 1} \frac{1-q^{4n}}{(1-q^{2n})(1-q^{5n-1})(1-q^{5n-4})}.$$

The last one:

$$\begin{aligned}
\sum_{n\geq 0}\frac{q^{n^2+2n}}{(q^4;q^4)_n} &= \sum_{n\geq 0}\frac{q^{n^2+2n}}{(q^2;q^2)_n(-q^2;q^2)_n}\\
&= [z^0]\sum_{n\in\mathbb{Z}}\frac{q^{n^2}z^n}{(-q^2;q^2)_n}\sum_{m\geq 0}\frac{z^{-m}q^{2m}}{(q^2;q^2)_m}\\
&= [z^0]\frac{(-zq;q^2)_\infty(-q/z;q^2)_\infty(q^2;q^2)_\infty}{(q/z;q^2)_\infty(-q^2;q^2)_\infty}\cdot\frac{1}{(q^2/z;q^2)_\infty}\\
&= \frac{1}{(-q^2;q^2)_\infty}[z^0]\frac{(-zq;q^2)_\infty(-q/z;q^2)_\infty(q^2;q^2)_\infty}{(q/z;q^2)_\infty(q^2/z;q^2)_\infty}\\
&= \frac{1}{(-q^2;q^2)_\infty}[z^0]\frac{(-zq;q^2)_\infty(-q/z;q^2)_\infty(q^2;q^2)_\infty}{(q/z;q)_\infty}\\
&= \frac{1}{(-q^2;q^2)_\infty}[z^0]\sum_{n\in\mathbb{Z}}q^{n^2}z^n\sum_{m\geq 0}\frac{z^{-m}q^m}{(q;q)_m}\\
&= \frac{1}{(-q^2;q^2)_\infty}\sum_{n\geq 0}\frac{q^{n^2+n}}{(q;q)_n}\\
&= \frac{1}{(-q^2;q^2)_\infty}H(q)\\
&= \prod_{n\geq 1}\frac{1}{(1+q^{2n})(1-q^{5n-2})(1-q^{5n-3})}\\
&= \prod_{n\geq 1}\frac{1-q^{2n}}{(1-q^{4n})(1-q^{5n-2})(1-q^{5n-3})}.
\end{aligned}$$

28. Representation of numbers as sums of squares

We start with Ramanujan's identity:

$$\sum_{n\in\mathbb{Z}}\frac{(a;q)_n x^n}{(b;q)_n} = \frac{(b/a;q)_\infty(ax;q)_\infty(q/ax;q)_\infty(q;q)_\infty}{(q/a;q)_\infty(b/ax;q)_\infty(b;q)_\infty(x;q)_\infty}.$$

Take $b = aq$:

$$\sum_{n\in\mathbb{Z}}\frac{(a;q)_n x^n}{(aq;q)_n} = \frac{(q;q)_\infty(ax;q)_\infty(q/ax;q)_\infty(q;q)_\infty}{(q/a;q)_\infty(q/x;q)_\infty(aq;q)_\infty(x;q)_\infty}.$$

Divide by $1-a$:

$$\sum_{n\in\mathbb{Z}}\frac{x^n}{1-aq^n} = \frac{(ax;q)_\infty(q/ax;q)_\infty(q;q)_\infty^2}{(q/a;q)_\infty(q/x;q)_\infty(a;q)_\infty(x;q)_\infty}.$$

Replace q with q^2:

$$\sum_{n\in\mathbb{Z}} \frac{x^n}{1-aq^{2n}} = \frac{(ax;q^2)_\infty (q^2/ax;q^2)_\infty (q^2;q^2)_\infty^2}{(q^2/a;q^2)_\infty (q^2/x;q^2)_\infty (a;q^2)_\infty (x;q^2)_\infty}.$$

Set $x = q$:

$$\sum_{n\in\mathbb{Z}} \frac{q^n}{1-aq^{2n}} = \frac{(aq;q^2)_\infty (q/a;q^2)_\infty (q^2;q^2)_\infty^2}{(q^2/a;q^2)_\infty (q;q^2)_\infty (a;q^2)_\infty (q;q^2)_\infty}.$$

Set $a = -1$:

$$\sum_{n\in\mathbb{Z}} \frac{q^n}{1+q^{2n}} = \frac{(-q;q^2)_\infty^2 (q^2;q^2)_\infty^2}{(-q^2;q^2)_\infty (q;q^2)_\infty^2 (-1;q^2)_\infty}.$$

or

$$2\sum_{n\in\mathbb{Z}} \frac{q^n}{1+q^{2n}} = \frac{(-q;q^2)_\infty^2 (q^2;q^2)_\infty^2}{(-q^2;q^2)_\infty^2 (q;q^2)_\infty^2}.$$

Observe that

$$\left(\sum_{n\in\mathbb{Z}} q^{n^2}\right)^s = \sum_{n\geq 0} r_s(n) q^n,$$

where $r_s(n)$ is the number of ways n can be written as the sum of s squares. Let us concentrate first on $s = 2$.

Let

$$d_{i,j}(n) = \sum_{k|n,\ k\equiv i \pmod{j}} 1.$$

We need the following version of the triple product

$$\sum_{n\in\mathbb{Z}} q^{n^2} = (q;q^2)_\infty (-q;q^2)_\infty^2.$$

Then

$$\begin{aligned}\left(\sum_{n\in\mathbb{Z}} q^{n^2}\right)^2 &= \sum_{n\geq 0} r_2(n) q^n = (q^2;q^2)_\infty^2 (-q;q^2)_\infty^4 \\ &= \frac{(q;q^2)_\infty^2 (-q;q^2)_\infty^2 (q^2;q^2)_\infty^2}{(q;q^2)_\infty^2 (q^4;q^4)_\infty^2} \\ &= \frac{(q;q^2)_\infty^2 (-q;q^2)_\infty^2}{(q;q^2)_\infty^2 (-q^2;q^2)_\infty^2}\end{aligned}$$

Hence

$$\begin{aligned}\sum_{n\geq 0} r_2(n)q^n &= 2\sum_{n\in\mathbb{Z}} \frac{q^n}{1+q^{2n}} \\ &= 1+4\sum_{n\geq 1} \frac{q^n}{1+q^{2n}} \\ &= 1+4\sum_{n\geq 1,m\geq 0} (-1)^m q^{n+2mn} \\ &= 1+4\sum_{n\geq 1,m\geq 0} (-1)^m q^{n(1+2m)} \\ &= 1+4\sum_{n\geq 1,m\geq 0} \Big(q^{n(1+4m)} - q^{n(3+4m)}\Big) \\ &= 1+4\sum_{n\geq 1} \Big(d_{1,4}(n) - d_{3,4}(n)\Big)q^n.\end{aligned}$$

Comparing coefficients we find

$$\boxed{r_2(n) = 4\big(d_{1,4}(n) - d_{3,4}(n)\big)}$$

29. Basics of Cigler's operator approach

We present the first section of [9], to show how several identities can be obtained using linear operators.

The q-differentiation operator D_q, defined by

$$(D_q f)(x) = \frac{f(qx)-f(x)}{qx-x},$$

is a discrete analogue of the differentiation operator, as is the operator Δ_h, defined by

$$(\Delta_h f)(x) = \frac{f(x+h)-f(x)}{h}.$$

The formal analogy to calculus is even closer and more interesting. For every formula, $q=1$ will give the classical case.

P is the vector space of polynomials over $\mathcal{C}$, and Q the set of formal power series over $\mathcal{C}$. The symbol $f(x)$ always means a formal power series $f(x) = a_0 + a_1 x + a_2 x^2 + \cdots$.

We define the *q-differentiation operator* D for $f(x) = \sum_{k=0}^{\infty} a_k x^k \in Q$ by

$$(Df)(x) = \frac{f(qx)-f(x)}{(q-1)x} = \sum_{k=1}^{\infty} [k]_q a_k x^{k-1}.$$

(We prefer the shorter D instead of D_q.)

The limit $q=1$ leads to the ordinary differentiation operator; we denote it by D_0.

For $(Df)(x)$ we also write $f'(x)$. The operator ε on Q is defined by

$$(\varepsilon f)(x) = f(qx).$$

Then

$$D = \frac{1}{(q-1)\mathbf{x}}(\varepsilon - 1).$$

Here, 1 is the identity mapping and $a(\mathbf{x})$ the multiplication operator, defined by

$$a(\mathbf{x})f(x) = a(x)f(x).$$

Note that

$$f(qx) - f(x) = \sum_{k=1}^{\infty} \left(q^k - 1\right) a_k x^k$$

is a multiple of x, so we can divide by x.

D is a linear operator on Q. Because of

$$\frac{a(qx)b(qx) - a(x)b(x)}{(q-1)x} = \frac{a(qx)\big(b(qx) - b(x)\big)}{(q-1)x} + \frac{\big(a(qx) - a(x)\big)b(x)}{(q-1)x}$$

we have

$$\big(a(x)b(x)\big)' = a(qx)b'(x) + a'(x)b(x),$$

which can be written as

$$D\,a(\mathbf{x}) = a(q\mathbf{x})D + a'(\mathbf{x}),$$

or, interchanging a and b,

$$D\,a(\mathbf{x}) = a(\mathbf{x})D + a'(\mathbf{x})\varepsilon.$$

Note that $Dx^k = [k]_q x^{k-1}$, thus

$$D\mathbf{x}^k - q^k\mathbf{x}^k D = [k]_q \mathbf{x}^{k-1}, \quad k = 1, 2, 3, \ldots$$

and

$$D\mathbf{x}^k - \mathbf{x}^k D = [k]_q \mathbf{x}^{k-1}\varepsilon, \quad k = 1, 2, 3, \ldots.$$

For $k = 1$ that means $D\mathbf{x} - q\mathbf{x}D = \mathbf{1}$ and $D\mathbf{x} - \mathbf{x}D = \varepsilon$.

Apply that to x^n, $n = 0, 1, 2, \ldots$:

$$[n+k]_q - q^k[n]_q = [k]_q$$

and

$$[n+k]_q - [n]_q = [k]_q q^n.$$

We introduce an inner product on P by

$$\left\langle x^k, x^l \right\rangle = [k]_q!\,\delta_{kl}, \quad k, l = 0, 1, 2, \ldots,$$

which can be described also as follows:

Let L be the linear functional on Q, defined by

$$Lf(x) = f(0),$$

Then

$$\left\langle x^k, x^l \right\rangle = LD^k x^l.$$

It is easy to see that the inner product can be extended to the case where one factor is in Q. Then

$$\langle a(x), b(x)\rangle = \langle b(x), a(x)\rangle\,, \quad a \in P, b \in Q.$$

If A is a linear operator, then the transposed operator A^t is defined via

$$\langle A\,a(x), b(x)\rangle = \left\langle a(x), A^t b(x)\right\rangle.$$

Then

$$(\alpha A + \beta B)^t = \alpha A^t + \beta B^t, \quad (AB)^t = B^t A^t$$
$$x^t = D, \quad D^t = x, \quad \varepsilon^t = \varepsilon.$$

In particular for $a(x) = \sum a_k x^k$, the multiplication operator $a(\mathbf{x})$ on Q becomes the operator $a(\mathbf{x})^t = \sum a_k D^k$ on P, defined by

$$\left(\sum a_k D^k\right) p(x) = \sum a_k p^{(k)}(x).$$

Formula:

$$La(D)b(x) = Lb(D)a(x) \text{ for } a \in P, b \in Q$$

If we interpret $a'(D)$ as $(a'(\mathbf{x}))^t$, then

$$Lf(D)x = LDf(x) = Lf'(x) = Lf'(D)1.$$

Furthermore the coefficient a_k in the formal power series $f(x) = \sum a_k x^k$ is given by

$$a_k = L\frac{D^k}{[k]_q!}f(x) = Lf(D)\frac{x^k}{[k]_q!}.$$

Every operator identity gives another one by transposition. In particular

$$a(D)\,\mathbf{x} = \mathbf{x}\,a(qD) + a'(D)$$

and

$$a(D)\,\mathbf{x} = \mathbf{x}\,a(D) + \varepsilon\,a'(D).$$

Analogously

$$D^k\mathbf{x} - q^k\mathbf{x}D^k = [k]_q D^{k-1}$$

and

$$D^k\mathbf{x} - \mathbf{x}D^k = [k]_q\,\varepsilon D^{k-1}.$$

Applying that to x^n, we get the recursions for the Gaussian coefficients:

$$\begin{bmatrix} n+1 \\ k \end{bmatrix}_q - q^k \begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{bmatrix} n \\ k-1 \end{bmatrix}_q$$

and

$$\begin{bmatrix} n+1 \\ k \end{bmatrix}_q - \begin{bmatrix} n \\ k \end{bmatrix}_q = q^{n-k+1}\begin{bmatrix} n \\ k-1 \end{bmatrix}_q.$$

In the sequel we need the q-analogue of the Leibniz' formula.

Since $Df(ax) = af'(ax)$, induction shows immediately that

$$\big(a(x)b(x)\big)^{(n)} = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q a^{(n-k)}\left(q^k x\right) b^{(k)}(x)$$

holds: For $n = 1$ it is true. Therefore

$$\begin{aligned}(a(x)b(x))^{(n+1)} &= \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q a^{(n-k)}\left(q^{k+1}x\right) b^{(k+1)}(x) \\ &\quad + \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q q^k a^{(n+1-k)}(q^k x) b^{(k)}(x) \\ &= \sum_{k=0}^{n+1} \left(\begin{bmatrix} n \\ k \end{bmatrix}_q q^k + \begin{bmatrix} n \\ k-1 \end{bmatrix}_q \right) a^{(n+1-k)}(q^k x) b^{(k)}(x).\end{aligned}$$

We get in operator notation

$$D^n a(\mathbf{x}) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q a^{(n-k)}(q^k \mathbf{x}) D^k$$

and

$$D^n a(\mathbf{x}) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q a^{(k)}(\mathbf{x})\, \varepsilon^k D^{n-k}$$

30. Bailey revisited

We need the following operator version of the q–binomial theorem: Assume that two linear operators A and B satisfy $BA = qAB$, then

$$(A + B)^n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q A^k B^{n-k}.$$

Almost needless to say that

$$\varepsilon^{-1} f(x) = f(\tfrac{x}{q}).$$

Bailey has a general lemma; we only present a special case.

Notice first that

$$\begin{aligned}(\varepsilon^{-1} + \mathbf{x})(xq;q)_\infty &= (x;q)_\infty + x(xq;q)_\infty \\ &= (1-x)(xq;q)_\infty + x(xq;q)_\infty \\ &= (xq;q)_\infty.\end{aligned}$$

Hence also

$$(\varepsilon^{-1} + \mathbf{x})^M (xq;q)_\infty = (xq;q)_\infty.$$

Now

$$\varepsilon^{-1}\mathbf{x} f(x) = \varepsilon^{-1} x f(x) = \tfrac{x}{q} f(\tfrac{x}{q}).$$

On the other hand,

$$\mathbf{x}\varepsilon^{-1}f(x) = \mathbf{x}f(\tfrac{x}{q}) = xf(\tfrac{x}{q}).$$

Thus these operators can be used in the q–binomial theorem; $A = \varepsilon^{-1}$ and $B = \mathbf{x}$. Now

$$\begin{aligned}
\sum_{k=0}^{n} \frac{a_k x^k}{(q;q)_{n-k}(xq;q)_{n+k}} &= \sum_{k=0}^{n} \frac{a_k x^k}{(q;q)_{n-k}} \frac{(xq^{n+k+1};q)_\infty}{(xq;q)_\infty} \\
&= \frac{1}{(xq;q)_\infty} \sum_{k=0}^{n} \frac{a_k x^k}{(q;q)_{n-k}} \varepsilon^{n+k} (xq;q)_\infty \\
&= \frac{1}{(xq;q)_\infty} \sum_{k=0}^{n} \frac{a_k x^k}{(q;q)_{n-k}} \varepsilon^{n+k} (\varepsilon^{-1} + \mathbf{x})^{n-k} (xq;q)_\infty \\
&= \frac{1}{(xq;q)_\infty} \sum_{k=0}^{n} \frac{a_k x^k}{(q;q)_{n-k}} \varepsilon^{n+k} \sum_{j=0}^{n-k} \begin{bmatrix} n-k \\ j \end{bmatrix}_q (\varepsilon^{-1})^{n-k-j} \mathbf{x}^j (xq;q)_\infty \\
&= \frac{1}{(xq;q)_\infty} \sum_{k=0}^{n} \frac{a_k x^k}{(q;q)_{n-k}} \sum_{j=0}^{n-k} \begin{bmatrix} n-k \\ j \end{bmatrix}_q \varepsilon^{2k+j} x^j (xq;q)_\infty \\
&= \frac{1}{(xq;q)_\infty} \sum_{k=0}^{n} \frac{a_k x^k}{(q;q)_{n-k}} \sum_{j=0}^{n-k} \begin{bmatrix} n-k \\ j \end{bmatrix}_q q^{(2k+j)j} x^j (xq^{2k+j+1};q)_\infty \\
&= \frac{1}{(xq;q)_\infty} \sum_{k=0}^{n} \frac{a_k x^k}{(q;q)_{n-k}} \sum_{j=k}^{n} \begin{bmatrix} n-k \\ j-k \end{bmatrix}_q q^{(k+j)(j-k)} x^{j-k} (xq^{k+j+1};q)_\infty \\
&= \sum_{k=0}^{n} \frac{a_k x^k}{(q;q)_{n-k}} \sum_{j=k}^{n} \begin{bmatrix} n-k \\ j-k \end{bmatrix}_q q^{j^2-k^2} x^{j-k} \frac{1}{(xq;q)_{j+k}} \\
&= \sum_{j=0}^{n} \sum_{k=0}^{j} \frac{a_k x^k}{(q;q)_{n-k}} \begin{bmatrix} n-k \\ j-k \end{bmatrix}_q q^{j^2-k^2} x^{j-k} \frac{1}{(xq;q)_{j+k}} \\
&= \sum_{j=0}^{n} \frac{q^{j^2} x^j}{(q;q)_{n-j}} \sum_{k=0}^{j} \frac{a_k q^{-k^2}}{(q;q)_{j-k}(xq;q)_{j+k}}.
\end{aligned}$$

This is Bailey's lemma; let us write it again:

$$\sum_{k=0}^{n} \frac{a_k x^k}{(q;q)_{n-k}(xq;q)_{n+k}} = \sum_{j=0}^{n} \frac{q^{j^2} x^j}{(q;q)_{n-j}} \sum_{k=0}^{j} \frac{a_k q^{-k^2}}{(q;q)_{j-k}(xq;q)_{j+k}}.$$

As always, we are interested in $x = 1$ and $x = q$.

Set $x=1$, $a_0=c_0$ and for $k\geq 1$, $a_k=c_k+c_{-k}$:

$$\sum_{k\in\mathbb{Z}}\frac{c_k}{(q;q)_{n-k}(q;q)_{n+k}}=\sum_{j\geq 0}\frac{q^{j^2}}{(q;q)_{n-j}}\sum_{k\in\mathbb{Z}}\frac{c_kq^{-k^2}}{(q;q)_{j-k}(q;q)_{j+k}}.$$

Let us do the limit $n\to\infty$ of this:

$$\frac{1}{(q;q)_\infty}\sum_{k\in\mathbb{Z}}c_k=\sum_{j\geq 0}q^{j^2}\sum_{k\in\mathbb{Z}}\frac{c_kq^{-k^2}}{(q;q)_{j-k}(q;q)_{j+k}}.$$

Specialize:

$$\begin{aligned}\frac{1}{(q;q)_\infty}\sum_{k\in\mathbb{Z}}(-1)^kq^{\frac{5}{2}k^2-\frac{1}{2}k}&=\sum_{j\geq 0}q^{j^2}\sum_{k\in\mathbb{Z}}\frac{(-1)^kq^{\frac{3}{2}k^2-\frac{1}{2}k}}{(q;q)_{j-k}(q;q)_{j+k}}\\&=\sum_{j\geq 0}q^{j^2}\sum_{l\geq 0}\frac{q^{l^2}}{(q;q)_{j-l}}\sum_{k\in\mathbb{Z}}\frac{(-1)^kq^{\frac{1}{2}k^2-\frac{1}{2}k}}{(q;q)_{l-k}(q;q)_{l+k}}\\&=\sum_{j\geq 0}q^{j^2}\sum_{l\geq 0}\frac{q^{l^2}}{(q;q)_{j-l}}\frac{(q;q)_l(1;q)_l}{(q;q)_{2l}}\\&=\sum_{j\geq 0}\frac{q^{j^2}}{(q;q)_j}.\end{aligned}$$

We used here the finite form for the inner sum, and then Jacobi's triple product (finite version) to evaluate the next inner sum, and then only one term ($l=0$) survived.

So the first Rogers–Ramanujan identity was *computed!*

31. Representation of numbers as sums of squares

We start with Ramanujan's identity:

$$\sum_{n\in\mathbb{Z}}\frac{(a;q)_nx^n}{(b;q)_n}=\frac{(b/a;q)_\infty(ax;q)_\infty(q/ax;q)_\infty(q;q)_\infty}{(q/a;q)_\infty(b/ax;q)_\infty(b;q)_\infty(x;q)_\infty}.$$

Take $b=aq$:

$$\sum_{n\in\mathbb{Z}}\frac{(a;q)_nx^n}{(aq;q)_n}=\frac{(q;q)_\infty(ax;q)_\infty(q/ax;q)_\infty(q;q)_\infty}{(q/a;q)_\infty(q/x;q)_\infty(aq;q)_\infty(x;q)_\infty}.$$

Divide by $1-a$:

$$\sum_{n\in\mathbb{Z}}\frac{x^n}{1-aq^n}=\frac{(ax;q)_\infty(q/ax;q)_\infty(q;q)_\infty^2}{(q/a;q)_\infty(q/x;q)_\infty(a;q)_\infty(x;q)_\infty}.$$

Replace q with q^2:

$$\sum_{n\in\mathbb{Z}} \frac{x^n}{1-aq^{2n}} = \frac{(ax;q^2)_\infty (q^2/ax;q^2)_\infty (q^2;q^2)_\infty^2}{(q^2/a;q^2)_\infty (q^2/x;q^2)_\infty (a;q^2)_\infty (x;q^2)_\infty}.$$

Set $x=q$:

$$\sum_{n\in\mathbb{Z}} \frac{q^n}{1-aq^{2n}} = \frac{(aq;q^2)_\infty (q/a;q^2)_\infty (q^2;q^2)_\infty^2}{(q^2/a;q^2)_\infty (q;q^2)_\infty (a;q^2)_\infty (q;q^2)_\infty}.$$

Set $a=-1$:

$$\sum_{n\in\mathbb{Z}} \frac{q^n}{1+q^{2n}} = \frac{(-q;q^2)_\infty^2 (q^2;q^2)_\infty^2}{(-q^2;q^2)_\infty (q;q^2)_\infty^2 (-1;q^2)_\infty}.$$

or

$$2\sum_{n\in\mathbb{Z}} \frac{q^n}{1+q^{2n}} = \frac{(-q;q^2)_\infty^2 (q^2;q^2)_\infty^2}{(-q^2;q^2)_\infty (q;q^2)_\infty^2}.$$

Observe that

$$\left(\sum_{n\in\mathbb{Z}} q^{n^2}\right)^s = \sum_{n\geq 0} r_s(n) q^n,$$

where $r_s(n)$ is the number of ways n can be written as the sum of s squares. Let us concentrate first on $s=2$.

Let

$$d_{i,j}(n) = \sum_{k|n,\ k\equiv i \mod j} 1.$$

We need the following version of the triple product

$$\sum_{n\in\mathbb{Z}} q^{n^2} = (q;q^2)_\infty (-q;q^2)_\infty^2.$$

Then

$$\begin{aligned}\left(\sum_{n\in\mathbb{Z}} q^{n^2}\right)^2 &= \sum_{n\geq 0} r_2(n) q^n = (q;q^2)_\infty^2 (-q;q^2)_\infty^4 \\ &= \frac{(q;q^2)_\infty^2 (-q;q^2)_\infty^2 (q^2;q^2)_\infty^2}{(q;q^2)_\infty^2 (q^4;q^4)_\infty^2} \\ &= \frac{(q;q^2)_\infty^2 (-q;q^2)_\infty^2}{(q;q^2)_\infty^2 (-q^2;q^2)_\infty^2}\end{aligned}$$

Hence

$$\begin{aligned}\sum_{n\geq 0} r_2(n)q^n &= 2\sum_{n\in\mathbb{Z}} \frac{q^n}{1+q^{2n}}\\ &= 1+4\sum_{n\geq 1} \frac{q^n}{1+q^{2n}}\\ &= 1+4\sum_{n\geq 1,m\geq 0} (-1)^m q^{n+2mn}\\ &= 1+4\sum_{n\geq 1,m\geq 0} (-1)^m q^{n(1+2m)}\\ &= 1+4\sum_{n\geq 1}\Big(q^{n(1+4m)}-q^{n(3+4m)}\Big)\\ &= 1+4\sum_{n\geq 1}\Big(d_{1,4}(n)-d_{3,4}(n)\Big).\end{aligned}$$

Comparing coefficients we find

$$\boxed{r_2(n)=4\big(d_{1,4}(n)-d_{3,4}(n)\big)}$$

32. Slater(38)

This identity from Slater's list can be obtained in a simple way:

$$\begin{aligned}\sum_{n\geq 0}\frac{q^{2n^2}}{(q;q)_{2n}} &= \sum_{n\geq 0}\frac{q^{(2n)^2/2}}{(q;q)_{2n}}\\ &= \sum_{n\geq 0,\ n \text{ even}}\frac{q^{n^2/2}}{(q;q)_n}\\ &= \frac{1}{2}\sum_{n\geq 0}\frac{q^{n^2/2}}{(q;q)_n}\Big(1+(-1)^n\Big)\\ &= \frac{1}{2}\Big((-\sqrt{q};q)_\infty+(\sqrt{q};q)_\infty\Big) \qquad \text{[Euler's first partition identity]}\\ &= \frac{1}{2}\Big((-q^{1/2};q^2)_\infty(-q^{3/2};q^2)_\infty+(q^{1/2};q^2)_\infty(q^{3/2};q^2)_\infty\Big)\\ &= \frac{1}{2(q^2;q^2)_\infty}\Big((-q^{1/2};q^2)_\infty(-q^{3/2};q^2)_\infty(q^2;q^2)_\infty+(q^{1/2};q^2)_\infty(q^{3/2};q^2)_\infty(q^2;q^2)_\infty\Big)\\ &= \frac{1}{2(q^2;q^2)_\infty}\Big(\sum_{n\in\mathbb{Z}} q^{n^2-n/2}+\sum_{n\in\mathbb{Z}}(-1)^n q^{n^2-n/2}\Big)\\ &= \frac{1}{(q^2;q^2)_\infty}\sum_{n\in\mathbb{Z}} q^{4n^2-n}\\ &= \frac{(-q^3;q^8)_\infty(-q^5;q^8)_\infty(q^8;q^8)_\infty}{(q^2;q^2)_\infty}\end{aligned}$$

$$= \frac{(q^6;q^{16})_\infty (q^{10};q^{16})_\infty (q^8;q^8)_\infty}{(q^3;q^8)_\infty (q^5;q^8)_\infty (q^2;q^2)_\infty}$$

$$= \prod_{n \equiv \pm 2, \pm 3, \pm 4, \pm 5 \pmod{16}} \frac{1}{1-q^n}.$$

This bibliography needs some reworking.

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