MULTIDIMENSIONAL DIGITAL SEARCHING – ALTERNATIVE DATA STRUCTURES

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Abstract. We analyze the average cost of partial match queries in M–dimensional Digital Search Trees and Patricia Tries. Our results allow a precise comparison of the average behaviour of these data structures with the M–dimensional Tries studied by Flajolet and Puech [1]. It turns out that Patricia is superior to Digital Search Trees, the latter ones being superior to Tries.

1. Introduction and Main Results

Random trees under the Bernoulli model occur frequently in connection with data structures in Computer Science. In this paper we concentrate on some aspects of *multidimensional search trees*. The manipulation of *multidimensional data* is of vital importance with respect to the construction of database systems and applications, where we mention e.g. graphical algorithms. See Flajolet/Puech [1] for an exhaustive list of references on the subject. One of the important data structures in this context are the so–called M–dimensional tries, "M–tries" for short:

The principle of the construction of tries (and other related data structures) is that digital properties of keys are used to build up a tree: In the basic case the keys are thought to be 0,1–sequences, where 0 and 1 occur with equal probability, and the keys are stored in the external nodes (leaves) of a binary tree, according to their shortest unique prefix among the other keys. (See Knuth [3] for details.)

The M–tries follow the same principle, but now the keys are M–tuples of 0,1–sequences, e.g.

$$\begin{gathered} K_1 = (K_{11}, \dots, K_{1M}) \\ \vdots \\ K_N = (K_{N1}, \dots, K_{NM}) \end{gathered}$$

with all K_{ij} infinite 0,1–sequences.

To build up the M–trie the M sequences $K_{i1}, \dots, K_{iM}$ corresponding to key K_i are "shuffled" to yield *one* binary sequence K_i', and the sequences $K_1', \dots, K_N'$ are used to build up a regular trie:

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More precisely, if $K_{ij} = \left(K_{ij}^{(0)}, K_{ij}^{(1)}, K_{ij}^{(2)}, \dots\right)$, then

$$K_i' = \left(K_{i1}^{(0)}, \dots, K_{iM}^{(0)}, K_{i1}^{(1)}, \dots, K_{iM}^{(1)}, K_{i1}^{(2)}, \dots\right).$$

Example. *M=2*

$$\begin{aligned}
K_1 &= (01\dots\ ,\ 00\dots) \longrightarrow K_1' = (0010\dots)\\
K_2 &= (10\dots\ ,\ 01\dots) \longrightarrow K_2' = (1001\dots)\\
K_3 &= (11\dots\ ,\ 00\dots) \longrightarrow K_3' = (1010\dots)
\end{aligned}$$

2–trie:

A *partial match query* q asks for all records whose attributes are either specified or not: If e.g. $q = (*, 100\cdots)$, we look for items with unspecified first component, and second component equal to $100\cdots$. The *specification pattern* of a query in an M–trie is a word in $\{S, *\}^M$, where S stands for a specified component and $*$ for an unspecified one; in our example we have the pattern $*S$.

A measure for the *cost of a query* is the number of internal nodes visited during the search. In [1] a precise analysis of the average cost $l_{\omega,N}^{[T]}$ of a query of pattern ω in an M–trie built from N random keys has been presented. Throughout this paper the underlying probability model is the *Bernoulli model.*

Theorem T. ([1])

$$l_{\omega,N}^{[T]} \sim N^{1-\frac{s}{M}} \cdot \left\{ \frac{s}{M^2 L} \cdot \frac{\Gamma\left(\frac{s}{M}\right)}{1-\frac{s}{M}} \cdot \sum_{j=0}^{M-1} (\delta_1\delta_2\dots\delta_j)\, 2^{-j(1-s/M)} + \delta_\omega^{[T]}\left(\log_2 N^{1/M}\right) \right\},$$

where s is the number of specified attributes in ω, $1 \le s \le M-1$, $L = \log 2$, $\delta_i = 1$ resp. 2 according to the i–th attribute in ω being specified resp. unspecified, and $\delta_\omega^{[T]}(x)$ a continuous periodic function of mean zero with small amplitude. $\square$

Similar results have been developed in [1] for binary search trees and grid file algorithms.

The aim of our paper is twofold: On the one hand we want to establish results like Theorem T for the M–dimensional versions of other prominent data structures, namely: Digital Search Trees, Patricia Tries and b–tries. (See [3] for the definitions.) On the other hand we also present an alternative approach for gaining the asymptotic results in the analysis. In the following we give a survey of our main results. More details and sketches of the derivations may be found in the following sections.

In the case of M–dimensional *Digital Search Trees* (M–DST for short), we follow the construction principle of M–tries, but storing the records in the internal nodes of a binary tree according to the order of their appearance.

Example. *With K_1, K_2, K_3 from the last example we get the 2–DST*

The cost of a query is again measured by the number of nodes visited during the search. Adopting an analogous notation to Theorem T, our result reads:

Theorem D.

$$l^{[D]}_{\omega,N} \sim N^{1-\frac{s}{M}} \cdot \left\{ \frac{Q\left(2^{1-\frac{s}{M}}\right)}{Q_\infty} \cdot \frac{1}{ML} \cdot \frac{\Gamma\left(\frac{s}{M}\right)}{1-\frac{s}{M}} \times \right.$$
$$\times \sum_{j=0}^{M-1} \left(\delta_1 \delta_2 \dots \delta_j\right) 2^{-j(1-s/M)}$$
$$\left. + \, \delta^{[D]}_{\omega}\left(\log_2 N^{1/M}\right) \right\},$$

where $Q(t) = \prod_{j\geq 1}\left(1-\frac{t}{2^j}\right)$ *and* $Q_\infty = Q(1)$. □

The basic idea of *Patricia Tries* is to compress one–way branches in Tries:

Example. *With K_1, K_2, K_3 from above we get the 2–Patricia Trie*

Our result for this instance is

Theorem P.

$$l_{\omega,N}^{[P]} \sim N^{1-\frac{s}{M}} \cdot \Bigg\{ \left(\frac{s}{M}+1\right) \cdot \left(1-2^{-s/M}\right) \cdot \frac{1}{ML} \cdot \frac{\Gamma\left(\frac{s}{M}\right)}{1-\frac{s}{M}} \times \\ \times \sum_{j=0}^{M-1} \left(\delta_1 \delta_2 \dots \delta_j\right) 2^{-j(1-s/M)} \\ + \delta_\omega^{[P]}\left(\log_2 N^{1/M}\right) \Bigg\}. \quad \square$$

Theorems T, D and P allow a precise comparison of the average cost of partial match queries in the concerned data structures: Neglecting the small periodic fluctuations the

Comparison Inequality

$$(1+x)\left(1-2^{-x}\right) < \frac{Q\left(2^{1-x}\right)}{Q_\infty} < x \quad \text{for} \quad 0 < x < 1$$

ensures that

$$l_{\omega,N}^{[P]} \preceq l_{\omega,N}^{[D]} \preceq l_{\omega,N}^{[T]}$$

for all query patterns ω. In Section 5 we give a proof of this comparison result.

The reader should be aware of the fact that the average cost of a partial match query is some sort of mixture of the cost of successful and unsuccessful search in the corresponding digital trees and compare the above result with Knuth's results on the latter parameters in [3].

In the following table we present the constants in question for the instance $M = 2$ and the two possible query patterns $\omega_1 = *S$ and $\omega_2 = S*$: (In both cases we have $s = 1$, so that the order of l_N is $\sqrt{N}$.)

	T	D	P
$\omega_1 = *S$	$3.0867\dots$	$2.7768\dots$	$2.7122\dots$
$\omega_2 = S*$	$2.1826\dots$	$1.9635\dots$	$1.9178\dots$

In the final section we deal with M–dimensional tries, where each external node can keep up to b records. If we measure the cost of a query as with M–tries we get

Theorem B.

$$l_{\omega,N}^{[B]} \sim N^{1-\frac{s}{M}} \cdot \Bigg\{ \binom{\frac{s}{M}+b-1}{b} \cdot \frac{1}{ML} \cdot \frac{\Gamma\left(\frac{s}{M}\right)}{1-\frac{s}{M}} \times \\ \times \sum_{j=0}^{M-1} \left(\delta_1 \delta_2 \dots \delta_j\right) 2^{-j(1-s/M)} \\ + \delta_\omega^{[B]}\left(\log_2 N^{1/M}\right) \Bigg\}. \quad \square$$

This result can be compared with the result for grid–file–algorithms in Flajolet/Puech [1], where they use the *same data structure* but a *different cost measure*.

2. Digital Search Trees

Let ω be a fixed query pattern and ω' the (cyclic) shift of ω, i.e. for $\omega = w_1 w_2 \dots w_M$ we have $\omega' = w_2 \dots w_M w_1$. Obviously, $\omega^{(M)} = \omega$. We further denote the probability generating function for the cost with query pattern ω and N records by $H_{\omega,N}(z)$. From the construction of the M–dimensional Digital Search Trees in Section 1, we find immediately the following pair of recursions (the reader should remember that we always consider the Bernoulli model):

$$H_{\omega,N+1}(z) = z \sum_{k=0}^{N} 2^{-N} \binom{N}{k} H_{\omega',k}(z) \cdot \begin{cases} H_{\omega',N-k}(z) & \text{for} \quad \omega = *\nu \\ 1 & \text{for} \quad \omega = S\nu \end{cases}$$

for $N \geq 0$, $H_{\omega,0}(z) = 1$.

The expectations of cost $l_{\omega,N} = l^{[D]}_{\omega,N}$ are given by $H'_{\omega,N}(1)$, so that the recursion translates into

$$l_{\omega,N+1} = 1 + \delta_1 2^{-N} \sum_{k=0}^{N} \binom{N}{k} l_{\omega',k}, \quad \text{for} \quad N \geq 0, \quad l_{\omega,0} = 0,$$

where $\delta_i = 2$ resp. 1 referring to $w_i = *$ resp. S in ω.

Thus the exponential generating functions

$$L_\omega(z) = \sum_{N \geq 0} l_{\omega,N} \frac{z^N}{N!}$$

fulfill the differential–functional equations

$$L'_\omega(z) = e^z + \delta_1 e^{z/2} L_{\omega'}\left(\frac{z}{2}\right).$$

In order to solve it is convenient to introduce

$$\widetilde{L}_\omega(z) = e^{-z} L_\omega(z),$$

so that

$$\widetilde{L}'_\omega(z) + \widetilde{L}_\omega(z) = 1 + \delta_1 \widetilde{L}_{\omega'}\left(\frac{z}{2}\right).$$

Comparing coefficients we find

$$\widetilde{l}_{\omega,N+1} = -\left(\widetilde{l}_{\omega,N} - \delta_1 2^{-N} \widetilde{l}_{\omega',N}\right), \quad N \geq 1$$

with $\widetilde{l}_{\omega,0} = 0$, $\widetilde{l}_{\omega,1} = 1$. This recurrence has the solution

$$\widetilde{l}_{\omega,N+1} = (-1)^N \left(\prod_{j=1}^{N} (1 - 2^{-j} t) \right) \Bigg|_{t^r \to \delta_1 \delta_2 \dots \delta_r} =: (-1)^N Q_{\omega,N}$$

where $\delta_{A\cdot M+r} = \delta_r$.

We finally get the explicit solution

$$l^{[D]}_{\omega,N} = -\sum_{k=1}^{N} \binom{N}{k} (-1)^k Q_{\omega,k-1}.$$

The asymptotic behaviour of an alternating sum of this type may be determined either by using the Mellin transform (this technique is used by Flajolet and Puech in [1]) or, alternatively, by applying the following Lemma from the calculus of finite differences. (Compare [2], [4].)

Lemma. *Let $\mathcal{C}$ be a path surrounding the points $j, j+1, \dots, N$ and $f(z)$ be analytic inside $\mathcal{C}$ and continuous on $\mathcal{C}$. Then*

$$\sum_{k\ge j} \binom{N}{k} (-1)^k f(k) = -\frac{1}{2\pi i}\int_{\mathcal{C}} [N;z] f(z)\,dz$$

with $[N;z] = \frac{(-1)^{N-1}N!}{z(z-1)\dots(z-N)}$. □

Moving the contour of integration, one can obtain the asymptotic expansion of the alternating sum by Cauchy's residue theorem: Under certain growth conditions for $f(z)$, which are fulfilled in our instance,

$$\sum_{k\ge j} \binom{N}{k} (-1)^k f(k)$$

may be asymptotically evaluated by

$$\sum \operatorname{Res}\bigl([N;z]f(z)\bigr)$$

where the sum is taken over the set of poles z different from $j, j+1, \dots, N$.

In our instance we have

$$f(k) = Q_{\omega,k-1},$$

so that we need an analytic continuation of this discrete sequence. Omitting the details the idea is as follows:

Using partial fraction decomposition we have

$$\prod_{j=1}^{N} \bigl(1 - 2^{-j}t\bigr) = \frac{Q(t)}{Q_\infty} \sum_{i\ge 1} a_i \frac{1}{1 - t2^{-i-N}}$$

where

$$Q(t) = \prod_{j\ge 1} \bigl(1 - 2^{-j}t\bigr), \; Q_\infty = Q(1) \quad \text{and} \quad a_i = (-1)^{i-1} 2^{-\binom{i}{2}} / Q_{i-1}.$$

Using Euler's product identity

$$Q(t) = \sum_{n\geq 0} a_{n+1} t^n$$

and expanding the geometric series in t we get

$$\prod_{j=1}^{N} \left(1 - 2^{-j}t\right) = \frac{1}{Q_\infty} \sum_{m\geq 0} t^m \sum_{i\geq 1} a_i \sum_{r=0}^{m} a_{m-r+1} 2^{-r(N+i)}.$$

Since this is an identity for analytic functions in t, the coefficients of t^m must coincide, and we are allowed to substitute t^m by $\delta_1 \dots \delta_m$ in both sides, simultanously. We get

$$Q_{\omega,N} = \frac{1}{Q_\infty} \sum_{m\geq 0} \delta_1 \dots \delta_m \sum_{i\geq 1} a_i \sum_{r=0}^{m} a_{m-r+1} 2^{-r(N+i)}.$$

Now, for $m = A \cdot M + j$,

$$\delta_1 \dots \delta_m = 2^{A(M-s)} \cdot \delta_1 \dots \delta_j$$

so that, after rearranging the sums,

$$Q_{\omega,N} = \frac{1}{Q_\infty} \sum_{j=0}^{M-1} \delta_1 \dots \delta_j \sum_{i\geq 1} a_i \sum_{h\geq 0} a_{h+1} 2^{(h-j)(N+i)} \times \\ \times \sum_{A\geq \max\left(0, \frac{h-j}{M}\right)} 2^{-A(M(N+i-1)+s)}$$

and, by replacing N by z, we have

$$Q_{\omega,z} = \frac{1}{Q_\infty} \sum_{j=0}^{M-1} \delta_1 \dots \delta_j \sum_{i\geq 1} a_i \frac{1}{1 - 2^{-M(z+i-1)-s}} \times \\ \times \left\{ \sum_{h=0}^{j-1} a_{h+1} 2^{(h-j)(z+i)} + \sum_{h\geq j} a_{h+1} 2^{(h-j)(z+i) - \lceil \frac{h-j}{M} \rceil \cdot (M(z+i-1)-s)} \right\}.$$

Using the Lemma from above we have to look for the residues of $[N; z] \cdot Q_{\omega,z-1}$ with real part < 1: The dominating singularity is the first order pole $z = 1 - \frac{s}{M}$, which occurs for $i = 1$ in the above expression for $Q_{\omega,z-1}$, and the residue can be calculated to be

$$\left[N; 1 - \frac{s}{M}\right] \frac{1}{Q_\infty} \sum_{j=0}^{M-1} \delta_1 \dots \delta_j \sum_{h\geq 0} a_{h+1} 2^{(h-j)\left(1-\frac{s}{M}\right)} \cdot \frac{1}{ML}$$

($L = \log 2$). Using again Euler's product identity we get

$$\left[N; 1 - \frac{s}{M}\right] \frac{1}{Q_\infty} \sum_{j=0}^{M-1} \delta_1 \dots \delta_j \, 2^{-j\left(1-\frac{s}{M}\right)} Q\left(2^{1-\frac{s}{M}}\right) \frac{1}{ML}.$$

Since
$$\Big[N; 1-\frac{s}{M}\Big] = \Gamma\Big(-1+\frac{s}{M}\Big)\cdot\Gamma(N+1)\Big/\Gamma\Big(N+\frac{s}{M}\Big)$$
$$\sim N^{1-\frac{s}{M}}\cdot\Gamma\Big(-1+\frac{s}{M}\Big)$$
we get the main term in Theorem D.

The periodic fluctuation $\delta_\omega^{[D]}$ originates from the simple poles at $1-\frac{s}{M}+\frac{2k\pi i}{LM}$; in principle one can get the Fourier expansion of $\delta_\omega^{[D]}(x)$ by collecting the residues at these poles, but we omit the details here.

3. Patricia Tries

With the notations of Section 2 we may proceed with this instance as follows: The probability generating functions of cost fulfill for $N \geq 2$
$$H_{\omega,N}(z) = z\sum_{k=0}^{N} 2^{-N}\binom{N}{k} H_{\omega',k}(z)\cdot\begin{Bmatrix} H_{\omega',N-k}(z) \\ 1 \end{Bmatrix}$$
$$-(z-1)\delta_1\, 2^{-N} H_{\omega',N}(z),$$
$H_{\omega,0}(z) = H_{\omega,1}(z) = 1$.

The subtraction terms correspond to the compression of ordinary tries as indicated in the Introduction.

The average costs are given by
$$l_{\omega,N} = 1-\delta_1 2^{-N}+\delta_1 2^{-N}\sum_{k=0}^{N}\binom{N}{k} l_{\omega',k}$$
for $N \geq 2$.

This time the functions $\widetilde{L}_\omega(z)$ fulfill a system of functional equations
$$\widetilde{L}_\omega(z) = g_1(z)+\delta_1\widetilde{L}_{\omega'}\Big(\frac{z}{2}\Big),$$
where
$$g_i(z) = 1-(1+z)e^{-z}-\delta_i\Big(e^{-z/2}-\Big(1+\frac{z}{2}\Big)e^{-z}\Big).$$
The system may be solved by iteration and gives us
$$\widetilde{l}_{\omega,N} = \frac{1}{1-2^{M(1-N-s/M)}}\sum_{j=0}^{M-1}\delta_1\ldots\delta_j\, 2^{-jN}\Big[\frac{z^N}{N!}\Big] g_{j+1}(z).$$
Now
$$\Big[\frac{z^N}{N!}\Big] g_{j+1}(z) = (-1)^N\Big(N-1-\delta_{j+1}\Big(2^{-N}+\frac{N}{2}-1\Big)\Big),$$
so that
$$l_{\omega,N} = \sum_{k=1}^{N}\binom{N}{k}(-1)^k f(k)$$

with

$$f(z) = \frac{1}{1-2^{M(1-z-s/M)}} \times$$
$$\times \sum_{j=0}^{M-1} \delta_1 \dots \delta_j \, 2^{-jz} \left(z - 1 - \delta_{j+1} \left(2^{-z} + \frac{z}{2} - 1 \right) \right).$$

Collecting the residues of $[N;z]f(z)$ at $z = 1 - s/M$ we get the main contribution

$$N^{1-\frac{s}{M}} \cdot \frac{-1}{ML} \Gamma\left(\frac{s}{M} - 1\right) \sum_{j=0}^{M-1} \delta_1 \dots \delta_j \, 2^{-j(1-s/M)} \times$$
$$\times \left[\frac{s}{M} - \frac{\delta_{j+1}}{2} \left(1 + \frac{s}{M} - 2^{s/M} \right) \right].$$

The sum can be rearranged as follows:

$$\begin{aligned} &\sum_{j=0}^{M-1} \delta_1 \dots \delta_j \, 2^{-j(1-s/M)} \cdot \frac{s}{M} \\ &- \frac{1}{2} \sum_{j=1}^{M} \delta_1 \dots \delta_j \, 2^{-j(1-s/M)} \cdot 2^{1-s/M} \left(1 + \frac{s}{M} - 2^{s/M} \right) \\ &= \sum_{j=1}^{M-1} \delta_1 \dots \delta_j \, 2^{-j(1-s/M)} \cdot \left(\frac{s}{M} - 2^{-s/M} \left(1 + \frac{s}{M} - 2^{s/M} \right) \right) \\ &+ \frac{s}{M} - \frac{1}{2} 2^{M-s-M(1-s/M)+1-s/M} \left(1 + \frac{s}{M} - 2^{s/M} \right), \end{aligned}$$

since $\delta_1 \dots \delta_M = 2^{M-s}$, and furthermore

$$= \left(\frac{s}{M} + 1 \right) \left(1 - 2^{-s/M} \right) \sum_{j=0}^{M-1} \delta_1 \dots \delta_j \, 2^{-j(1-s/M)}.$$

From the last observation Theorem P is immediate.

5. The Comparison Inequality

Let $f(x) = Q\left(2^{-x}\right)/Q_\infty$. It is easily seen that the left–hand side of the inequality is equivalent to

$$\text{(i)} \qquad 1 + x < f(x) \quad \text{for} \quad 0 < x < 1$$

and the right–hand side to

$$\text{(ii)} \qquad f(x) < 1 + x \quad \text{for} \quad -1 < x < 0.$$

Now

$$f(x) = \prod_{j \geq 1} \left(1 + \frac{1 - 2^{-x}}{2^j - 1} \right) = P(y)$$

where $y = 1 - 2^{-x}$ fulfills (i) $0 < y < \frac{1}{2}$ resp. (ii) $-1 < y < 0$.

Let

$$g(y) = \frac{1+x}{f(x)} = \frac{1 - \log_2(1-y)}{P(y)}.$$

Since $g(0) = g\left(\frac{1}{2}\right) = 1$, it is sufficient to prove that $g(y)$ is a logarithmically convex function in y for $-1 < y < \frac{1}{2}$: With $h(y) = \log g(y)$ we find

$$h''(y) = \frac{1}{(1-y)^2} \cdot \left\{ \frac{1}{\log\left(\frac{2}{1-y}\right)} - \frac{1}{\log^2\left(\frac{2}{1-y}\right)} + \sum_{j \geq 1} \left(\frac{2^j}{1-y} - 1 \right)^{-2} \right\}.$$

The expression in brackets should be positive, which is (with $z = 1 + x = 1 - \log_2(1-y)$) equivalent to

$$\sum_{j \geq 0} \left(2^{j+z} - 1\right)^{-2} > \frac{1}{L^2 z^2} - \frac{1}{Lz}, \quad 0 < z < 2.$$

The last inequality is trivial for $z \geq 1/L$, since the right–hand side is < 0, and is *a fortiori* proved for $0 < z < 1/L$ by

$$\left(\frac{2^z - 1}{Lz} \right)^2 < \frac{1}{1 - Lz},$$

which may be seen directly by comparing the Taylor coefficients of both sides.

6. b–Tries

Here we have for the probability generating function

$$H_{\omega,N}(z) = z \sum_{k=0}^{N} 2^{-N} \binom{N}{k} H_{\omega',k}(z) \cdot \begin{Bmatrix} H_{\omega',N-k}(z) \\ 1 \end{Bmatrix}$$

for $N \geq b+1$, $H_{\omega,0}(z) = \cdots = H_{\omega,b}(z) = 1$.

The functional equations for $\widetilde{L}_\omega(z)$ read

$$\widetilde{L}_\omega(z) = 1 - e_b(z) \cdot e^{-z} + \delta_1 \widetilde{L}_{\omega'}\left(\frac{z}{2}\right)$$

where

$$e_b(z) = \sum_{i=0}^{b} \frac{z^i}{i!}.$$

Iteration of the system and comparing coefficients yields

$$l_{\omega,N} = \sum_{k=1}^{N} \binom{N}{k} (-1)^k f(k)$$

with

$$f(z) = \frac{(-1)^{b+1}\binom{z-1}{b}}{1 - 2^{M(1-z-s/M)}} \sum_{j=0}^{M-1} \delta_1 \ldots \delta_j \, 2^{-jz}.$$

Calculating the residues at the dominating poles we find Theorem B by some easy manipulations.

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