HORIZONTAL RUNS IN DOMINO TILINGS

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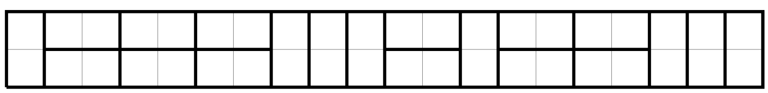
ABSTRACT. We discuss tilings of a grid (of size $n \times 2$) with dominoes of size 2×1. Parameters that might be called "longest run" are investigated, in terms of generating functions and also asymptotically. Extensions are also mentioned.

1. INTRODUCTION

Donald Knuth [3] included *tilings* of an $n \times 2$-rectangle using 2×1-sized tiles ("dominoes") as an introductory example of the use of *generating functions*. If T_n is the number of these tilings, then $T_n = T_{n-1} + T_{n-2}$, since we have two choices to start: one vertical domino, leaving an $(n-1) \times 2$-rectangle, or two horizontal dominos, leaving an $(n-2) \times 2$-rectangle. Since $T_0 = 1$ and $T_1 = 1$, this leads to $T_n = F_{n+1}$ (a *Fibonacci number*).

Here is one particular tiling of a 20×2-rectangle:



Denoting $\mathscr{T}$ the *family* (=set) of *all* tiled $n \times 2$-rectangles, for $n \geq 0$, then we can write a *symbolic equation*:

$$\mathscr{T} = \mid + \;\square\,\mathscr{T} + \;\boxminus\,\mathscr{T}$$

With the generating function

$$T(z) := \sum_{n \geq 0} T_n z^n,$$

the symbolic equation translates directly into

$$T(z) = 1 + zT(z) + z^2 T(z) = \frac{1}{1 - z - z^2}.$$

This equation is simple enough that, with partial fraction decomposition, one finds an explicit form

$$T_n = \frac{1}{\sqrt{5}} \left[\left(\frac{1 + \sqrt{5}}{2} \right)^{n+1} - \left(\frac{1 - \sqrt{5}}{2} \right)^{n+1} \right].$$

The number

$$\alpha := \frac{1 + \sqrt{5}}{2}$$

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is called the *golden ratio*; it is one of the most important constants in *mathematics*. In terms of *asympotics* it is dominating:

$$T_n \sim \frac{1}{\sqrt{5}}\Big(\frac{1+\sqrt{5}}{2}\Big)^{n+1}.$$

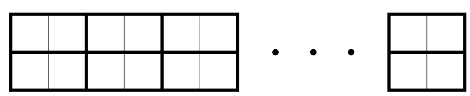
If one cannot resort to an explicit expression as above, one looks at the *dominating singularity* and writes

$$T(z) \sim \frac{C}{1-\alpha z} \qquad \text{as} \quad z \to \frac{1}{\alpha}.$$

The constant C may be computed as

$$C = \lim_{z\to\frac{1}{\alpha}} \frac{1-\alpha z}{1-z-z^2} = \frac{\alpha}{\sqrt{5}}.$$

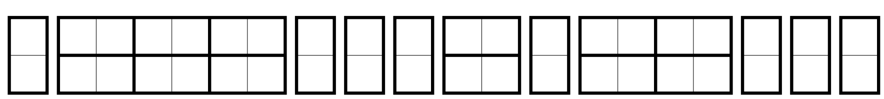
In this paper, we are interested in the (consecutive) sequence of horizontal dominoes in a tiling. We are looking for the longest substructure of the type in the figure.



We will indicate in Section 2 how the expected value of this parameter may be computed. Then, in Section 3, we change our setting to tilings of $n \times 3$ rectangles. Things become more involved, but it is highly instructive to see how one has to deal with the difficulties. For more material of a similar type we refer to [8].

2. The longest horizontal run in tilings of $n \times 2$-rectangles

As the first step of our analysis, we decompose a tiled rectangle according to (maximal) runs of horizontal dominoes. We indicate this for the example from the introduction: We see here runs of 3, 1, and 2 (stacked) horizontal dominoes. So our parameter is



3 for this example. Various runs of length 0 are not indicated. Based on this (unique!) decomposition, we introduce $\mathscr{T}^{<h}$, the family of tiled dominoes where the (maximal) run parameter is $< h$, we find

$$\mathscr{T}^{<h} = \boxminus^{<h}\Big(\Box\boxminus^{<h}\Big)^*.$$

For completeness, we mention that the '*' operator (Kleene's star in language theory) produces sequences. If A is a set, then $A^* = A^0 \cup A^1 \cup A^2 \cup \cdots$, i.e., all sequences that can be formed from elements of A. This operation is also useful when one deals with generating functions. If $f(z)$ is the generating function associated to A, so that the coefficient of z^n (written as $[z^n]f(z)$) counts the number of elements of size n in A, then $\frac{1}{1-f(z)}$ is the generating function associated to A^*. For general background we would like to mention the book [2].

Here,

$$\substack{\square\\\square}^{<h} = \bigcup_{i=0}^{h-1} \substack{\square\\\square}^{i}.$$

The lefthand side describes repetitions of $\substack{\square\\\square}$ with at most $h-1$ elements; the righthand side splits the repetitions into the various possibilities from $i = 0, 1, \dots, h-1$. In terms of generating functions, the last expression translates into

$$1 + z^2 + z^4 + \cdots + z^{2(h-1)} = \frac{1 - z^{2h}}{1 - z^2}.$$

Consequently, we get

$$T^{<h}(z) = \frac{1 - z^{2h}}{1 - z^2} \frac{1}{1 - \frac{z(1-z^{2h})}{1-z^2}} = \frac{1 - z^{2h}}{1 - z - z^2 + z^{2h+1}}.$$

One can see from this that, as $h \to \infty$, which means no more restrictions on the run lengths, we find again the generating function

$$T(z) = \frac{1}{1 - z - z^2}.$$

Further,

$$\begin{aligned} T^{\geq h}(z) := T(z) - T^{<h}(z) &= \frac{1}{1 - z - z^2} - \frac{1 - z^{2h}}{1 - z - z^2 + z^{2h+1}} \\ &= \frac{1 - z^2}{1 - z - z^2} \frac{z^{2h}}{1 - z - z^2 + z^{2h+1}}. \end{aligned}$$

There is a dominant root ρ_h of $1 - z - z^2 + z^{2h+1}$, which is close to $1/\alpha$. So we set $\rho_h := 1/\alpha + \varepsilon_h$ and continue

$$1 - \frac{1}{\alpha} - \varepsilon_h - \frac{1}{\alpha^2} - \frac{2\varepsilon_h}{\alpha} + \alpha^{-2h-1} \sim 0,$$

or

$$\varepsilon_h \sim \sqrt{5}\alpha^{-2h-1}.$$

We write

$$1 - z - z^2 + z^{2h+1} \sim \Big(1 - \frac{z}{\rho_h}\Big) C,$$

and find

$$C = \lim_{z \to \rho_h} \frac{1 - z - z^2 + z^{2h+1}}{1 - \frac{z}{\rho_h}} \sim \frac{\sqrt{5}}{\alpha}.$$

Now we can read off coefficients:

$$\begin{aligned} [z^n]T^{\geq h}(z) &\sim \frac{\alpha^{n+1}}{\sqrt{5}} - [z^n] \frac{\alpha}{\sqrt{5}} \frac{1}{1 - z/\rho_h} \\ &= \frac{\alpha^{n+1}}{\sqrt{5}} - \frac{\alpha}{\sqrt{5}} \rho_h^{-n} \\ &\sim \frac{\alpha^{n+1}}{\sqrt{5}} - \frac{\alpha}{\sqrt{5}} \Big(\frac{1}{\alpha} + \sqrt{5}\alpha^{-2h-1}\Big)^{-n} \end{aligned}$$

$$
\begin{aligned}
&= \frac{\alpha^{n+1}}{\sqrt{5}} - \frac{\alpha^{n+1}}{\sqrt{5}}\Big(1+\sqrt{5}\alpha^{-2h}\Big)^{-n}\\
&\sim \frac{\alpha^{n+1}}{\sqrt{5}} - \frac{\alpha^{n+1}}{\sqrt{5}}\Big(1-\sqrt{5}\alpha^{-2h}\Big)^{n}\\
&\sim \frac{\alpha^{n+1}}{\sqrt{5}} - \frac{\alpha^{n+1}}{\sqrt{5}}e^{-\sqrt{5}n/\alpha^{2h}}.
\end{aligned}
$$

Normalization leads to

$$\frac{[z^n]T^{\ge h}(z)}{T_n} \sim 1 - e^{-\sqrt{5}n/\alpha^{2h}}.$$

In order to compute the *average value* of our parameter, one has to sum this:

$$\sum_{h\ge 1}\Big(1 - e^{-\sqrt{5}n/\alpha^{2h}}\Big).$$

It is very well understood how to get asymptotics for

$$f(x) := \sum_{h\ge 1}\Big(1 - e^{-x/\alpha^{2h}}\Big)$$

as $x\to\infty$, see [1]. One computes the *Mellin transform*

$$f^*(s) = \sum_{h\ge 1}\alpha^{2hs}\cdot\Gamma(s) = -\frac{\alpha^{2s}}{1-\alpha^{2s}}\Gamma(s),$$

valid in $\langle -1, 0\rangle$ (the fundamental strip). Then one employs the *inversion formula*

$$f(x) = -\frac{1}{2\pi i}\int_{-\frac12 - i\infty}^{-\frac12 + i\infty}\frac{\alpha^{2s}}{1-\alpha^{2s}}\Gamma(s)x^{-s}ds,$$

shifts the line of integration to the right and takes the residues (with a negative sign) into account. The contribution from the pole at $s=0$ is

$$\frac12\log_\alpha x + \frac{\gamma}{2\log\alpha} - \frac12.$$

There is also a contribution coming from the simple poles at $s = \frac{k\pi i}{\log\alpha}$:

$$-\frac{1}{2\log\alpha}\sum_{k\ne 0}\Gamma\Big(\frac{k\pi i}{\log\alpha}\Big)e^{-k\pi i\cdot\log_\alpha x}.$$

This is a periodic function of small amplitude. Such functions occur frequently when one analyses *run statistics*, see [8]. Altogether we found for the average of our parameter the asymptotic formula

$$\frac12\log_\alpha n + \frac14\log_\alpha 5 + \frac{\gamma}{2\log\alpha} - \frac12 - \delta(\sqrt{5}n),$$

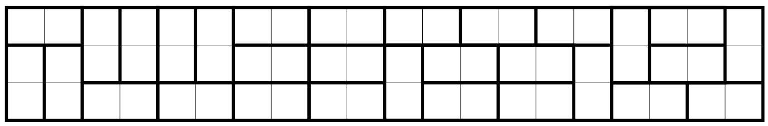
with

$$\delta(x) := -\frac{1}{2\log\alpha}\sum_{k\ne 0}\Gamma\Big(\frac{k\pi i}{\log\alpha}\Big)e^{-k\pi i x}.$$

Let us emphasize that the paper [8] has many similar results, and concrete error terms and estimates are worked out. Here, for the benefit of the reader, we suppress technical details and just stick to the main ideas.

3. Tilings of $n \times 3$-rectangles

Let us start with an example of a 20×3-rectangle:



It decomposes in a natural way as follows:

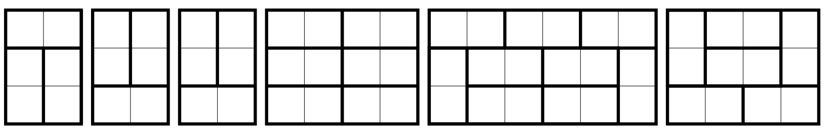


Figure 1. Decomposition into blocks

The individual (maximal) blocks are of 3 types, as indicated in the figure:

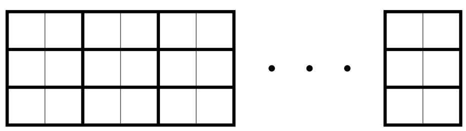


Figure 2. First type

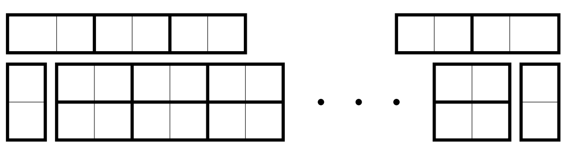


Figure 3. Second type

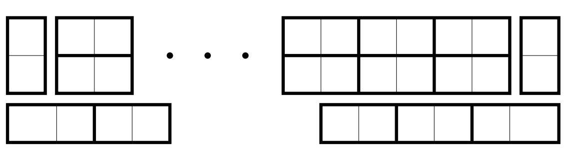


Figure 4. Third type

Again, we will look at

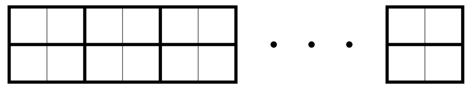


FIGURE 5. Maximal sequence of horizontal dominoes present (2 layers)

Let the three types with a sequence of exactly i horizontal (stacked) dominoes be denoted by $\mathscr{F}^i$, $\mathscr{G}^i$, $\mathscr{H}^i$, then the family of dominoes with our run parameter $< h$ can be described by

$$\mathscr{T}^{<h} = \Big(\bigcup_{i=0}^{h-1}\mathscr{F}^i\Big)\Big[\Big(\bigcup_{i=0}^{h-1}\mathscr{G}^i \cup \bigcup_{i=0}^{h-1}\mathscr{H}^i\Big)\Big(\bigcup_{i=0}^{h-1}\mathscr{F}^i\Big)\Big]^*.$$

In terms of generating functions (the variable z marks the length of the domino), we have the following:

$$\bigcup_{i=0}^{h-1}\mathscr{F}^i \longrightarrow 1 + z^2 + \cdots + z^{2(h-1)} = \frac{1-z^{2h}}{1-z^2},$$

$$\bigcup_{i=0}^{h-1}\mathscr{G}^i \longrightarrow z^2 + \cdots + z^{2h} = \frac{z^2(1-z^{2h})}{1-z^2}$$

and

$$\bigcup_{i=0}^{h-1}\mathscr{H}^i \longrightarrow z^2 + \cdots + z^{2h} = \frac{z^2(1-z^{2h})}{1-z^2}.$$

Consequently

$$\begin{aligned} T^{<h}(z) &= \frac{1-z^{2h}}{1-z^2}\frac{1}{1-2\dfrac{z^2(1-z^{2h})}{1-z^2}\dfrac{1-z^{2h}}{1-z^2}} \\ &= \frac{(1-z^2)(1-z^{2h})}{1-4z^2+z^4+4z^{2h+2}-2z^{4h+2}}. \end{aligned}$$

In the limit $h \to \infty$ (no restrictions), we find

$$T(z) = \frac{1-z^2}{1-4z^2+z^4},$$

which was already derived in [3] using a different method. These functions only depend on z^2, which is clear, since a tiled $n \times 3$-rectangle is only possible for even n. Thus we set $w := z^2$ and work with

$$R^{<h}(w) = \frac{(1-w)(1-w^h)}{1-4w+w^2+4w^{h+1}-2w^{2h+1}}$$

and

$$R(w) = \frac{1-w}{1-4w+w^2}.$$

There is a dominant root at $w = 1/\rho$ with $\rho = 2 + \sqrt{3}$, and

$$R(w) \sim \frac{3-\sqrt{3}}{6} \frac{1}{1-\rho z},$$

and so

$$[w^n]R(w) \sim \frac{3-\sqrt{3}}{6}\rho^n.$$

There is a dominant root $\omega_h = \frac{1}{\rho} + \varepsilon_h$ of

$$1 - 4w + w^2 + 4w^{h+1} - 2w^{2h+1}.$$

One application of *bootstrapping* results in

$$\varepsilon_h = \frac{2}{\sqrt{3}\rho^{h+1}}.$$

From a historical point of view, it is interesting to point out that this procedure appeared for the first time in [6].

A computation that is analogous to the one in the previous section yields

$$\begin{aligned}
[w^n]T^{\ge h}(z) &\sim \frac{3-\sqrt{3}}{6}\rho^n - \frac{3-\sqrt{3}}{6}\omega^{-n} \\
&\sim \frac{3-\sqrt{3}}{6}\rho^n - \frac{3-\sqrt{3}}{6}\Big(\frac{1}{\rho} + \varepsilon_h\Big)^{-n} \\
&= \frac{3-\sqrt{3}}{6}\rho^n - \frac{3-\sqrt{3}}{6}\rho^n\Big(1 + \frac{2}{\sqrt{3}\rho^h}\Big)^{-n} \\
&\sim \frac{3-\sqrt{3}}{6}\rho^n - \frac{3-\sqrt{3}}{6}\rho^n\Big(1 - \frac{2}{\sqrt{3}\rho^h}\Big)^{n}.
\end{aligned}$$

Normalization leads to

$$\frac{[w^n]T^{\ge h}(z)}{[w^n]T(z)} \sim 1 - e^{-\frac{2n}{\sqrt{3}\rho^h}}.$$

To compute the average, we need to evaluate

$$\sum_{h\ge 1}\Big(1 - e^{-x/\rho^h}\Big),$$

with $x = 2n/\sqrt{3}$. The asymptotic evaluation is as before:

$$\log_\rho n + \log_\rho 2 - \frac{1}{2}\log_\rho 3 + \frac{\gamma}{\log\rho} - \frac{1}{2} - \frac{1}{\log\rho}\sum_{k\ne 0}\Gamma\Big(\frac{2k\pi i}{\log\rho}\Big)e^{-2\pi i k\cdot\log_\rho(2n/\sqrt{3})}.$$

4. The longest vertical run in tilings of $n \times 2$-rectangles

For completeness, we briefly discuss how one can attack the parameter "longest vertical run."

We introduce $\mathscr{T}^{<h}$, the family of tiled dominoes where the (maximal) run parameter is $< h$, we find

$$\mathscr{T}^{<h} = \square^{<h}\Big(\boxminus\,\square^{<h}\Big)^*.$$

In terms of generating functions, this means

$$T^{<h}(z)=\frac{1-z^h}{1-z-z^2+z^{h+2}}.$$

We set again $\rho_h=1/\alpha+\varepsilon_h$, and find

$$\varepsilon_h\sim\frac{1}{\sqrt5}\frac{1}{\alpha^{h+2}}.$$

Further,

$$\frac{[z^n]T^{\ge h}(z)}{T_n}\sim 1-e^{\sqrt5 n/\alpha^{h+1}}.$$

The expected value is to be evaluated as

$$\sum_{h\ge1}\Bigl(1-e^{-x/\alpha^k}\Bigr),$$

with $x=\frac{\sqrt5}{\alpha}n$. The asymptotic evaluation is

$$\log_\alpha n+\frac12\log_\alpha 5+\frac{\gamma}{\log\alpha}-\frac32-\frac1\alpha\sum_{k\ne0}\Gamma\Bigl(\frac{2k\pi i}{\log\alpha}\Bigr)e^{-2\pi ik\cdot\log_\alpha(n\sqrt5/\alpha)};$$

n refers to the length of the tiled rectangle.

The case of an $n\times3$-rectangle and runs of vertical tiles can also be done, but is a bit more elaborate. It leads to a similar type of result.

5. Conclusion

We have demonstrated how to deal with runlength parameters: First, symbolic equations lead to explicit forms of the associated generating functions. Then, one identifies the dominant singularity and finds an approximate expression for the coefficients. The average value that one wants to compute is then asymptotically described by a series. To work out asymptotics for this sum, the Mellin transform [1, 2] is used. This series of operations works also in other contexts. A few references for further reading are [4, 5, 7].

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