SCHUR'S OLD DETERMINANT PROVES A BRAND–NEW THEOREM OF GARRETT–ISMAIL–STANTON

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Abstract. Garrett, Ismail, and Stanton gave a general formula that contains the Rogers–Ramanujan identities as special cases. We show how easy this is when using a determinant that Schur introduced in 1917.

In a recent difficult paper [2] Garrett, Ismail, and Stanton prove, amongst many other things, the following generalization of the celebrated Rogers–Ramanujan identities:

$$\begin{aligned}\sum_{n\ge 0}\frac{q^{n^2+mn}}{(1-q)(1-q^2)\dots(1-q^n)} &= (-1)^m q^{-\binom{m}{2}} E_{m-2}\prod_{n\ge 0}\frac{1}{(1-q^{5n+1})(1-q^{5n+4})}\\ &\quad - (-1)^m q^{-\binom{m}{2}} D_{m-2}\prod_{n\ge 0}\frac{1}{(1-q^{5n+2})(1-q^{5n+3})},\end{aligned} \tag{1}$$

with the *Schur* polynomials, defined by

$$\begin{aligned} D_m &= D_{m-1} + q^m D_{m-2}, \qquad & D_0 = 1,\ D_1 = 1+q,\\ E_m &= E_{m-1} + q^m E_{m-2}, \qquad & E_0 = 1,\ E_1 = 1. \end{aligned}$$

(Another proof, based on generalized Engel expansions, can be found in [1].)

Schur [3] has computed the limits

$$D_\infty = \prod_{n\ge 0}\frac{1}{(1-q^{5n+1})(1-q^{5n+4})}, \qquad E_\infty = \prod_{n\ge 0}\frac{1}{(1-q^{5n+2})(1-q^{5n+3})}.$$

The aim of this note is to give a simple proof of this result, based on the following determinant of Schur:

$$\mathsf{Schur}(x) := \begin{vmatrix} 1 & xq^{1+m} & & & & \dots\\ -1 & 1 & xq^{2+m} & & & \dots\\ & -1 & 1 & xq^{3+m} & & \dots\\ & & -1 & 1 & xq^{4+m} & \dots\\ & & & \ddots & \ddots & \ddots \end{vmatrix}.$$

Expanding the determinant with respect to the first column ("top–recursion") we get

$$\mathsf{Schur}(x) = \mathsf{Schur}(xq) + xq^{1+m}\,\mathsf{Schur}(xq^2).$$

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Setting

$$\mathsf{Schur}(x)=\sum_{n\ge0}a_nx^n,$$

we get, upon comparing coefficients,

$$a_n=q^na_n+q^{1+m}q^{2n-2}a_{n-1},$$

or

$$a_n=\frac{q^{2n-1+m}}{1-q^n}a_{n-1}.$$

Since $a_0=1$, iteration leads to

$$a_n=\frac{q^{n^2+mn}}{(1-q)(1-q^2)\dots(1-q^n)},$$

and thus the left hand side of (1) can be expressed by $\mathsf{Schur}(1)$.

On the other hand, $\mathsf{Schur}(1)$ is the limit of the *finite* determinants

$$\mathsf{Schur}_n:=\begin{vmatrix}1 & q^{1+m} & & & & \dots\\ -1 & 1 & q^{2+m} & & & \dots\\ & -1 & 1 & q^{3+m} & & \dots\\ & & -1 & 1 & q^{4+m} & \dots\\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots\\ & & & -1 & 1 & q^{n+m}\\ & & & & -1 & 1\end{vmatrix}.$$

Expanding this determinant with respect to the last row ("bottom–recursion") we get

$$\mathsf{Schur}_n=\mathsf{Schur}_{n-1}+q^{n+m}\,\mathsf{Schur}_{n-2}\,.$$

We see that the sequences $\langle D_{n+m}\rangle_n$ and $\langle E_{n+m}\rangle_n$ satisfy this recursion, and thus any linear combination; set

$$\mathsf{Schur}_n=\lambda D_{n+m}+\mu E_{n+m}.$$

We can determine the parameters λ and μ using the initial conditions $\mathsf{Schur}_0=1$, $\mathsf{Schur}_1=1+q^{1+m}$, which leads to

$$\lambda=\frac{E_m-E_{m-1}}{D_{m-1}E_m-D_mE_{m-1}}=\frac{q^mE_{m-2}}{D_{m-1}E_m-D_mE_{m-1}}$$

and

$$\mu=\frac{D_m-D_{m-1}}{D_mE_{m-1}-D_{m-1}E_m}=\frac{q^mD_{m-2}}{D_mE_{m-1}-D_{m-1}E_m}.$$

We can find the nicer form

$$\lambda=\frac{q^mE_{m-2}}{D_{m-1}E_m-D_mE_{m-1}}=(-1)^mq^{-\binom{m}{2}}E_{m-2},$$

which amounts to prove that

$$D_{m-1}E_m-D_mE_{m-1}=(-1)^mq^{\binom{m+1}{2}}.$$

Andrews, Knopfmacher, and Paule [1] give a nice formula for $D_iE_j - D_jE_i$. We only need a very particular case of that which can be easily proved by induction on m. The beginning $m = 0$ is trivial; the induction step goes like this:

$$\begin{aligned} D_mE_{m+1} - D_{m+1}E_m &= D_m(E_m + q^{m+1}E_{m-1}) - (D_m + q^{m+1}D_{m-1})E_m \\ &= q^{m+1}(D_mE_{m-1} - D_{m-1}E_m) \\ &= -q^{m+1}(-1)^m q^{\binom{m+1}{2}} = (-1)^{m+1}q^{\binom{m+2}{2}}. \end{aligned}$$

Analogously we obtain

$$\mu = \frac{q^mD_{m-2}}{D_mE_{m-1} - D_{m-1}E_m} = (-1)^{m+1}q^{-\binom{m}{2}}D_{m-2}.$$

Hence

$$\mathsf{Schur}_n = (-1)^m q^{-\binom{m}{2}}E_{m-2}D_{n+m} - (-1)^m q^{-\binom{m}{2}}D_{m-2}E_{n+m}.$$

Performing the limit $n \to \infty$ this turns into

$$\mathsf{Schur}(1) = (-1)^m q^{-\binom{m}{2}}E_{m-2}D_\infty - (-1)^m q^{-\binom{m}{2}}D_{m-2}E_\infty,$$

which is (1).

References

[1] G. Andrews, A. Knopfmacher, and P. Paule. An Infinite Family of Engel Expansions of Rogers-Ramanujan Type. *Adv. in Appl. Math.*, to appear 1999.

[2] K. Garrett, M. Ismail, and D. Stanton, Variants of the Rogers–Ramanujan identities, *Adv. in Appl. Math.* **23** (1999), 274–299.

[3] I. Schur. Ein Beitrag zur additiven Zahlentheorie und zur Theorie der Kettenbrüche. *S.-B. Preuss. Akad. Wiss. Phys.-Math. Kl.*, 1917, 302–321, reprinted in I. Schur, Gesammelte Abhandlungen, vol. 2, pp. 117–136, Springer, 1973.

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