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Note

Two Selection Problems Revisited

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1. INTRODUCTION

Two classical problems in combinatorial analysis are the enumeration of subsets with linear successions and subsets with circular successions, which were first solved by Kaplansky [4]. Hwang [2] and Hwang, Korner and Wei [3] considered the more general cases of these distribution problems where several circles respective lines are considered simultaneously. The original proofs, which are by direct combinatorial or inductive arguments, are rather lengthy. In this note we want to show that by the exclusive use of generating functions short and easy proofs can be achieved. Also this approach leads in a natural way to alternative and new formulae.

In the following we write $[z^n]\,f(z)$ for the coefficient of z^n in the formal Laurent series $f(z)$.

2. SUBSETS WITH CIRCULAR SUCCESSIONS

Let $g_{n,k,l}$ denote the number of k-subsets of a circle of size n with exactly l circular successions. By Kaplansky [4] one has (except for trivial cases)

$$\begin{aligned} g_{n,k,l} &= \frac{n}{k}\binom{k}{l}\binom{n-k-1}{k-l-1} \\ &= \binom{k}{l}\binom{n-k-1}{k-l-1} + \binom{k-1}{l}\binom{n-k}{k-l}. \end{aligned} \tag{2.1}$$

Let

$$g_{n,k}(t) = \sum_{l \geqslant 0} g_{n,k,l}\, t^l \tag{2.2}$$

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denote a corresponding generating function. Following [1; (2.3.22)],

$$kg_{n,k}(t) = n[w^n](wt + w^2 + w^3 + \cdots)^k, \qquad g_{n,0}(t) = 1. \tag{2.3}$$

(This follows easily by decomposing the circle into parts starting with each selected node; each of the k parts contributes a factor $wt + w^2 + w^3 + \cdots$. Considering all n-node patterns we apply all n rotations to get each of the desired configurations k times.) Hence

$$\begin{aligned} g_n(z; t) &:= \sum_{k \geq 0} g_{n,k}(t)\, z^k \\ &= 1 + n[w^n] \sum_{k \geq 1} \left(wt + \frac{w^2}{1-w}\right)^k \frac{z^k}{k} \\ &= 1 - n[w^n] \log \frac{1 - (1+tz)\, w + z(t-1)\, w^2}{1-w} \\ &= \sigma^n(z; t) + \tau^n(z; t) - \delta_{n,0}, \end{aligned} \tag{2.4}$$

with

$$\sigma + \tau = 1 + tz, \qquad \sigma\tau = z(t-1). \tag{2.5}$$

After these preliminaries we turn to the case of m nonempty circles with $n_1, ..., n_m$ nodes respectively. Let

$$g_{\{n_1, ..., n_m\}}(z; t) = \sum_{k \geq 0} g_{\{n_1, ..., n_m\}, k}(t)\, z^k \tag{2.6}$$

denote the polynomial such that the coefficient of $z^k t^l$ is the number of k-subsets with l circular successions in total. Thus

$$g_{\{n_1, ..., n_m\}}(z; t) = \prod_{i=1}^{m} g_{n_i}(z; t) = \prod_{i=1}^{m} (\sigma^{n_i} + \tau^{n_i}). \tag{2.7}$$

In order to evaluate this product we use some shorthand notations: $n = n_1 + \cdots + n_m$ and, for any subset $S \subseteq M = \{1, ..., m\}$, $n(S) = \sum_{i \in S} n_i$:

$$\begin{aligned} g_{\{n_1, ..., n_m\}}(z; t) &= \sum_{S \subseteq M} \sigma^{n(S)} \tau^{n - n(S)} \\ &= \sum_{\substack{S \subseteq M \\ 2n(S) < n}} (\sigma^{n(S)} \tau^{n - n(S)} + \sigma^{n - n(S)} \tau^{n(S)}) + \sum_{\substack{S \subseteq M \\ 2n(S) = n}} \sigma^{n(S)} \tau^{n(S)} \end{aligned}$$

Regarding that $\sigma\tau = z(t-1)$ this can be written as

$$= \sum_{\substack{S \subseteq M \\ 2n(S) \leqslant n}} (t-1)^{n(S)} z^{n(S)} (\sigma^{n-2n(S)} + \tau^{n-2n(S)} - \delta_{n-2n(S),0})$$

such that

$$g_{\{n_1,\ldots,n_m\},k}(t) = \sum_{S \subseteq M} (t-1)^{n(S)} g_{n-2n(S),k-n(S)}(t), \tag{2.8}$$

which is the main result of Hwang [2].

In order to demonstrate the advantages of the generating functions approach, we deduce the following alternative formula (2.11).

For preparation let σ be the solution of (2.5) with $\sigma(0; t) = 1$. We will use "formal residue composition", i.e., an equivalent to Lagrange inversion formula (compare [1; (1.2.2)]), with the substitution

$$z = \frac{u(1+u)}{1+tu}, \tag{2.9}$$

so that

$$\sigma = 1 + u$$

and

$$\frac{dz}{du} = \frac{1+2u+tu^2}{(1+tu)^2}, \qquad \tau = \frac{(t-1)\,u}{1+tu},$$

$$1 - \tau = \frac{1+u}{1+tu}, \qquad \sigma - \tau = \frac{1+2u+tu^2}{1+tu}. \tag{2.10}$$

Starting from (2.7) we have

$$\begin{aligned} g_{\{n_1,\ldots,n_m\}}(z,t) &= \prod_{i=1}^{m} (\sigma^{n_i} + \tau^{n_i}) \\ &= \prod_{i=1}^{m} (\sigma^{n_i} + \sigma^{-n_i}(t-1)^{n_i} z^{n_i}) \\ &= \sum_{S \subseteq M} (t-1)^{n(S)} z^{n(S)} \sigma^{n-2n(S)}. \end{aligned}$$

With substitution (2.9) from above the calculus of formal residues shows that

$$\begin{aligned}
&[z^k]\, g_{\{n_1,\dots,n_m\}}(z;t) \\
&= [u^k] \left(\frac{u}{z}\right)^{k+1} \frac{dz}{du}\, g_{\{n_1,\dots,n_m\}}(z;t) \\
&= [u^k] \frac{(1+tu)^{k+1}}{(1+u)^{k+1}} \frac{1+2u+tu^2}{(1+tu)^2} \sum_{S\subseteq M} (t-1)^{n(S)} \frac{u^{n(S)}(1+u)^{n(S)}}{(1+tu)^{n(S)}} \\
&\quad \times (1+u)^{n-2n(S)} \\
&= \sum_{S\subseteq M} (t-1)^{n(S)} [u^{k-n(S)}](1+2u+tu^2)(1+u)^{n-n(S)-k-1} \\
&\quad \times (1+tu)^{k-1-n(S)}.
\end{aligned}$$

According to $1+2u+tu^2 = u(1+tu)+(1+u)$ we have

$$\begin{aligned}
&g_{\{n_1,\dots,n_m\}}(t) \\
&= \sum_{S\subseteq M} (t-1)^{n(S)} \sum_{j\geq 0} t^j \left\{\binom{k-n(S)}{j}\binom{n-n(S)-k-1}{k-n(S)-j-1}\right. \\
&\quad \left. + \binom{k-1-n(S)}{j}\binom{n-n(S)-k}{k-n(S)-j}\right\}.
\end{aligned} \tag{2.11}$$

(Note that the coefficient of t^j in the j-sum coincides with $g_{n-2n(S),k-n(S),j}$ whenever these indices are nonnegative, which will not be true for all values of $S\subseteq M$. Thus (2.8) and (2.11) are similar but essentially different.)

3. Subsets with Linear Sucessions

Let $f_{n,k,l}$ denote the number of k-subsets of a chain of size n with exactly l linear successions. By Kaplansky [4]:

$$f_{n,k,l} = \binom{k-1}{l}\binom{n-k+1}{k-l}. \tag{3.1}$$

Let

$$f_{n,k}(t) = \sum_{l\geq 0} f_{n,k,l}\, t^l. \tag{3.2}$$

Then, following [1; (2.3.15)], compare (2.3) for the idea of proof,

$$f_{n,k}(t) = [w^n](w+w^2+\cdots)(wt+w^2+w^3+\cdots)^{k-1}(1+w+w^2\cdots), f_{n,0}(t)=1$$

and further

$$\begin{aligned}
f_n(z;t) &:= \sum_{k\geqslant 0} f_{n,k}(t)\, z^k \\
&= 1+[w^n] \sum_{k\geqslant 1} w^k(1-w)^{-2}\left(t+\frac{w}{1-w}\right)^{k-1} z^k \\
&= [w^n]\left(\frac{1}{1-w}+\frac{zw}{1-w}\,(1-(1+tz)\,w+z(t-1)\,w^2)^{-1}\right) \\
&= [w^n]\,\frac{1-(t-1)\,zw}{w(\sigma-\tau)}\left(\frac{1}{1-\sigma w}-\frac{1}{1-\tau w}\right) \\
&= \frac{1}{\sigma-\tau}\,(\sigma^{n+1}-\tau^{n+1}-(t-1)\,z\sigma^n+(t-1)\,z\tau^n) \\
&= \frac{1}{\sigma-\tau}\,(\sigma^{n+1}(1-\tau)-\tau^{n+1}(1-\sigma)). \qquad (3.3)
\end{aligned}$$

As in Section 2 we consider now the case of k-selections of m nonempty chains with cardinalities $n_1,\ldots, n_m$. Using analogous notations as in Section 2 we find

$$\begin{aligned}
f_{\{n_1,\ldots,n_m\}}(z;t) &= \sum_{k\geqslant 0} f_{\{n_1,\ldots,n_m\},k}(t)\, z^k = \prod_{i=1}^m f_{n_i}(z;t) \\
&= (\sigma-\tau)^{-m}\prod_{i=1}^m (\sigma^{n_i+1}(1-\tau)-\tau^{n_i+1}(1-\sigma)).
\end{aligned}$$

With the substitution (2.9) this yields

$$\begin{aligned}
&[z^k]\, f_{\{n_1,\ldots,n_m\}}(z;t) \\
&= [u^k]\,\frac{(1+tu)^{k+1}}{(1+u)^{k+1}}\,\frac{1+2u+tu^2}{(1+tu)^2}\,\frac{(1+tu)^m}{(1+2u+tu^2)^m} \\
&\quad\times\prod_{i=1}^m\left((1+u)^{n_i+1}\,\frac{1+u}{1+tu}+\frac{u(t-1)^{n_i+1}\,u^{n_i+1}}{(1+tu)^{n_i+1}}\right) \\
&= [u^k]\,\frac{(1+tu)^{k-1-n}}{(1+2u+tu^2)^{m-1}\,(1+u)^{k+1}} \\
&\quad\times\prod_{i=1}^m ((1+u)^{n_i+2}\,(1+tu)^{n_i}+(t-1)^{n_i+1}\,u^{n_i+2}) \\
&= \sum_{S\subseteq M}\,[u^{k-n(S)-2|S|}](1+2u+tu^2)^{-(m-1)}(t-1)^{n(S)+|S|} \\
&\quad\times(1+tu)^{k-1-n(S)}\,(1+u)^{n-n(S)+2m-2|S|-k-1}. \qquad (3.4)
\end{aligned}$$

Writing $1+2u+tu^2$ as $(1+u)^2+(t-1)\,u^2$ we get:

$$f_{\{n_1,\dots,n_m\},k}(t)=\sum_{S\subseteq M}\sum_{i\geqslant 0}\binom{-(m-1)}{i}(t-1)^{n(S)+|S|+i}$$
$$\times[u^{k-n(S)-2|S|-2i}](1+u)^{n-n(S)-2|S|-k-2i+1}(1+tu)^{k-1-n(S)}$$
$$=\sum_{S\subseteq M}\sum_{i,j\geqslant 0}\binom{-(m-1)}{i}(t-1)^{n(S)+|S|+i}t^j\binom{k-1-n(S)}{j}$$
$$\times\binom{n-n(S)-2\,|S|-k-2i+1}{k-n(S)-2\,|S|-2i-j}. \tag{3.5}$$

For $t=0$ this is Theorem 1 of [3]. We note also that

$$[t^l]\,f_{\{n_1,\dots,n_m\},k}\{t\}$$
$$=\sum_{S\subseteq M}\sum_{i,j\geqslant 0}\binom{i+m-2}{i}\binom{n(S)+|S|+i}{l-j}$$
$$\times(-1)^{n(S)+|S|-l+j}\binom{k-1-n(S)}{j}\binom{n-n(S)-2\,|S|-2i-k+1}{k-n(S)-2\,|S|-2i-j}. \tag{3.6}$$

Using the identity $1+tu=1+u+(t-1)\,u$ we get the alternate formula

$$f_{\{n_1,\dots,n_m\},k}(t)=\sum_{S\subseteq M}\sum_{i,j\geqslant 0}\binom{-(m-1)}{i}(t-1)^{n(S)+|S|+i+j}$$
$$\times\binom{k-1-n(S)}{j}\binom{n-2n(S)-2\,|S|-2i-j}{k-n(S)-2\,|S|-2i-j}. \tag{3.7}$$

A third formula of similar type is obtained by writing $1+2u+tu^2$ in (3.4) as $(1+u)+u(1+tu)$:

$$f_{\{n_1,\dots,n_m\},k}(t)=\sum_{S\subseteq M}\sum_{i,j\geqslant 0}\binom{-(m-1)}{i}(t-1)^{n(S)+|S|}\,t^j$$
$$\times\binom{i+k-1-n(S)}{j}\binom{n-n(S)+m-2\,|S|-k-i}{k-n(S)-2\,|S|-i-j}. \tag{3.8}$$

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