Analysis of an Optimized Search Algorithm for Skip Lists*

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Abstract

It was suggested in [8] to avoid redundant queries in the skip list search algorithm by marking those elements whose key has already been checked by the search algorithm. We present here a precise analysis of the total search cost (expectation and variance), where the cost of the search is measured in terms of the number of key-to-key comparison.These results are then compared with the corresponding values of the standard search algorithm.

1 Introduction

Skip lists have recently been introduced as a type of list-based data structure that may substitute search trees [9]. A set of n elements is stored in a collection of sorted linear linked lists in the following manner: all elements are stored in increasing order in a linked list called *level* 1 and, recursively, each element which appears in the linked list *level* i is included with independent probability q $(0 < q < 1)$ in the linked list *level* $i+1$.

The *level* of an element x is the number of linked lists it belongs to. For each element in the skip list, we need a node to store its key and as many pointers as its level indicates. The successor of x at the list level i is given by the i-th pointer of x, also called i-th *forward* pointer of x. A *header* refers to the first element in each of the linked lists and it also holds the *height* of the skip list, which is the maximum level among the levels of the elements or total number of non-empty linked lists.

A detailed description of several skip list algorithms, as well as variants of the data structure, can be found in [8]. Interesting analytic aspects of the average-case performance

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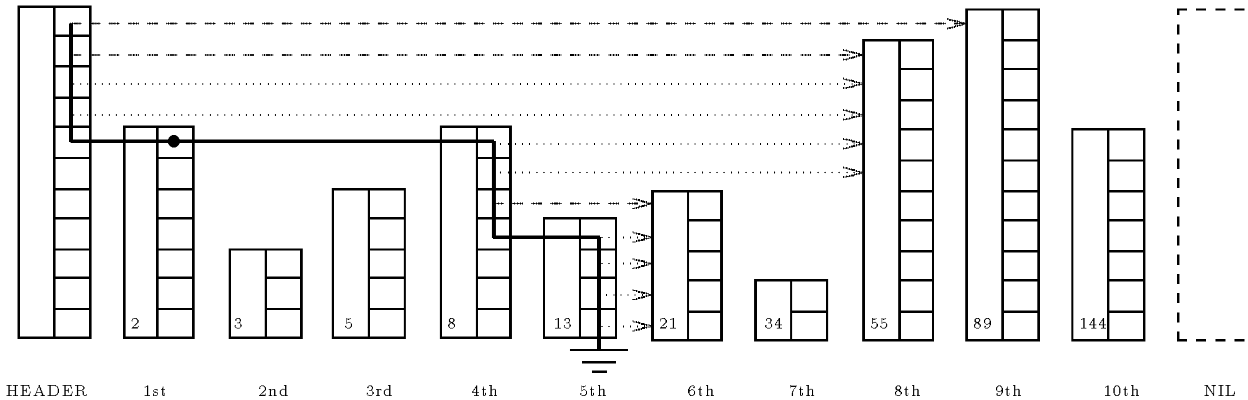

Figure 1: An example of a search path in a skip list

of skip lists algorithms may be found in [1, 5, 6]. In [4] the probabilistic analysis of the search cost was considered in a slightly different way, namely, performing the asymptotic analysis of the *total search cost* or *path length*, i.e. the sum of the successful search costs to find all the elements in the data structure. In particular, the variance of this parameter was analyzed precisely.

The present paper is devoted to the analysis of the total search cost of an *optimized version* of the search algorithm that greatly reduces the number of key comparisons. The optimized version of the search guarantees that the search key will be compared at most once with the key of any element in the skip list; additional pointer comparisons may be needed for those elements whose key has already been compared with the search key. If a key comparison is more expensive than a pointer comparison, this optimized version will be useful. The optimized search algorithm was proposed in [8], together with an estimate of the savings in the number of key comparisons. As we will see later on, only the main term of this estimate is correct.

According to the standard algorithm the search for an element is performed by traversing forward pointers as long as the key of the successor of the current node is smaller than the search key. When this traversal stops at the current level, the search is continued one level below. Clearly the algorithm terminates at level 1 when we are just one node in front of the node that contains the element we are looking for (we assume that the desired element is already in the skip list). In Figure 1 we depict the search path for the 6th element in a skip list of size 10 and height 11.

In the optimized version of the search algorithm the number of comparisons is reduced by assuring that the search key is never compared against the key of an element more than once. To this aim, the variable `alreadyChecked` is introduced. At the beginning this variable is set to “NIL”. In the following loop we follow forward pointers as long as the elements pointed to are different from `alreadyChecked` and the keys of those elements are smaller than the search key. As soon as this horizontal traversal ends, `alreadyChecked` is

```
x := header(S); l := height(S);
alreadyChecked := NIL;
while l > 0 do
    while (x ↑ .forward[l] ≠ alreadyChecked) and
          (x ↑ .forward[l] ↑ .key < search_key) do
        x := x ↑ .forward[l]
    end;
    alreadyChecked := x;
    l := l − 1
end
```

Figure 2: Optimized search algorithm (see [8])

set to the element pointed to at this moment and the search continues one level below (see Figure 2).

Each dashed arrow and thick horizontal line in Figure 1 corresponds to an "expensive" key comparison of the search key against the key which is actually pointed to. The dotted lines correspond to successful "cheap" pointer comparisons. For each of these comparisons, the search path must drop one level. In our example we have 14 comparisons altogether. These split into 6 expensive key-to-key comparisons and 8 cheap pointer-to-pointer comparisons.

In order to analyze the total search cost, i.e. the total number of key comparisons made with this optimized algorithm to search each of the elements in a skip list, it is helpful to describe a skip list of size n as an n-tuple $(a_1, \ldots, a_n)$, where a_i denotes the level of the i-th element. For instance, the skip list in Figure 1 is described by the 10-tuple $(7, 3, 5, 7, 4, 5, 2, 9, 11, 8)$.

The probabilistic model for *random skip lists* describes a random skip list as the outcome of n independent identically distributed random variables. In particular, each $a_i \in \mathbb{N}$ is the outcome of a *geometric* random variable G_i of parameter p, i.e. $\text{Prob}\{G_i = k\} = pq^{k-1}$, where $q = 1 - p$. (Note that in some earlier papers the roles of p and q are interchanged.)

It is of interest to translate this search cost parameter into terms of *order statistics*: The number of key comparisons when searching for element i with the optimized algorithm can be split up into two contributions:

(1) key comparisons where the actual key is larger than or equal to the search key (there is a dashed horizontal arrow in Figure 1 for each comparison of this kind).

(2) key comparisons where the actual key is smaller than the search key (there is a thick horizontal line in Figure 1 for each comparison of this kind).

It is an immediate observation that the comparisons of type (1) correspond bijectively to the strict left-to-right maxima of the sequence $(a_i, \ldots, a_n)$ whereas the comparisons of type (2) correspond bijectively to the weak right–to-left maxima of $(a_1, \ldots, a_{i-1})$.

To be precise, let us say that a_j $(i \leq j \leq n)$ is a strict left-to-right maximum of $(a_i, \ldots, a_n)$ if it is larger than $a_i, \ldots, a_{j-1}$, and that a_k $(1 \leq k \leq i-1)$ is a weak right-to-left maximum of $(a_i, \ldots, a_n)$ if it is larger than or equal to $a_i, \ldots, a_{j-1}$.

Observe that for a fixed element i the two parameters are independent random variables; but this is no longer true for the total search cost, which is given by the sum of the numbers of strict left-to-right maxima of all suffixes of $(a_1, \ldots, a_n)$ and of weak right-to-left maxima of all prefixes of $(a_1, \ldots, a_{n-1})$. This dependency is the reason that we cannot simply add the variances of these two parameters, which where already computed in [4].

A second observation is the fact that the number of weak right-to-left maxima of $(a_1, \ldots, a_n)$ is never counted above. This unpleasant asymmetry between the cumulation of the strict left-to-right maxima and the weak right-to-left maxima would lead to cumbersome recurrences in the probabilistic analysis. Therefore, we shift our attention to the *total unsuccessful search cost*. Let $C_{n,i}$ denote the cost of an unsuccessful search of a key belonging to the interval (x_{i-1}, x_i) in a random skip list of n elements, where x_i denotes the key of the i-th element. By convention, $x_0 = -\infty$ and $x_{n+1} = +\infty$. Then, C_n, the total unsuccesful search cost is

$$C_n = \sum_{1 \leq i \leq n+1} C_{n,i}. \tag{1}$$

Obviously, $C_{n,i}$ is also the successful search cost for the i-th element in a random skip list of n elements, for $i = 1, \ldots, n$. It turns out that C_n fulfills a nice recurrence relation that greatly simplifies the analysis. Furthermore, in Section 4, we will show that the two first moments of C_n are asymptotically equivalent to those of $\widetilde{C}_n$, the total successful search cost in a random skip list of n elements.

Finally, there is a nice and suggestive interpretation for the unsuccessful search cost $C_{n,n+1}$ in terms of the skip list algorithm which will be very helpful in the sequel: The number of weak right-to-left maxima in $(a_1, \ldots, a_n)$ equals the number of key comparisons if we search for a key larger than any other already in the skip list (or equivalently, if we search for "NIL", which is marked as `alreadyChecked` from the very beginning).

Now we are ready to start our analysis. The main part of the paper will be organized as follows:

In Section 2 we start from a combinatorial decomposition of random skip lists in order to get a functional equation for the probability generating function of the total unsuccessful search cost. This allows to compute the asymptotics of the expectation in a straightforward manner. In Section 3 we concentrate on our main result and evaluate the variance of C_n. In Section 4 we prove that our asymptotic results remain true for the total successful search cost $\widetilde{C}_n$. Finally, in Section 5 we discuss some generalizations of the algebraic techniques used in the probabilistic analysis of skip list algorithms.

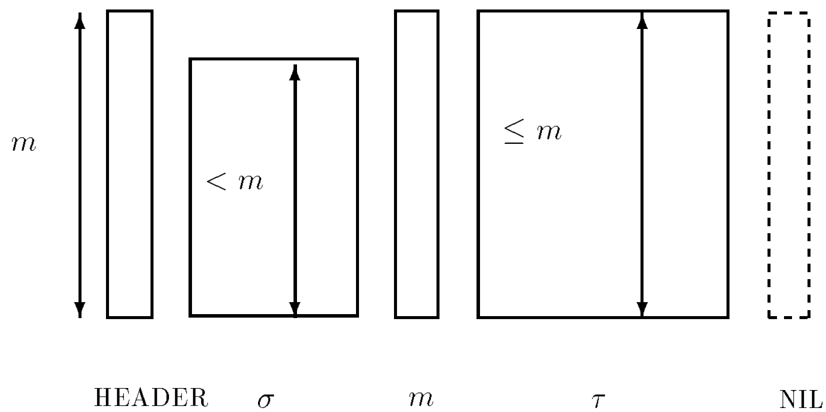

Figure 3: Recursive decomposition of a skip list.

2 Probability Generating Functions and Expectations

In order to derive a recurrence relation for the probability generating function of the total unsuccessful search cost, it is convenient to consider the following combinatorial decomposition of a skip list of height m (see Figure 3): We split up the whole skip list $S = (a_1, \ldots, a_n)$ according to the first appearance of an element $a_i = m$ into the partitioning element m and two skip lists $\sigma = (a_1, \ldots, a_{i-1})$ and $\tau = (a_{i+1}, \ldots, a_n)$. Observe that σ has height less than m and τ has height less than or equal to m; both of them have less elements than the skip list S.

Following the previous discussion, the total unsuccessful search cost $C(S)$ of a skip list S is the sum of two contributions $L(S)$ and $R(S)$, where $L(S)$ is the cumulated number of strict left-to-right maxima of all suffixes of S and $R(S)$ is the cumulated number of weak right-to-left maxima of all prefixes of S.

Now, it is plain to see that

$$L(\sigma m \tau) = L(\sigma) + L(\tau) + |\sigma| + 1, \tag{2}$$

since each element at the left of $a_i = m$, i.e. in the σ-part, contributes as many left-to-right maxima to $L(S)$ as it contributes to $L(\sigma)$ plus 1, the additional left-to-right maximum corresponding to the partitioning element (the latter contribution is $|\sigma|$). The partitioning element itself contributes 1 to $L(S)$, and the elements in the τ-part contribute as many left-to-right maxima to $L(S)$ as they do to $L(\tau)$.

For the second contribution, we can argue as above. It is however easier to imagine —since we are summing up all these numbers— that, when considering some particular element, we are interested in the right-to-left maxima to the left of it, the element itself being contributing as a right-to-left maximum. Note that this approach takes into account the contribution of right-to-left maxima of the whole sequence, corresponding to the unsuccessful search of a key larger than any other in the skip list, without the need of dealing with the "NIL" element.

Then, if an element is in the σ-part, it contributes as many right-to-left maxima to $L(S)$ as it does to $L(\sigma)$. If it is the partitioning element $a_i = m$, the contribution is 1, and if it is in the τ-part, we must add 1 to the number of right-to-left maxima it already contributed to the τ-part. This gives us a similar recursion for $R(S)$

$$R(\sigma m\tau) = R(\sigma) + R(\tau) + |\tau| + 1. \tag{3}$$

Altogether we have

$$C(\sigma m\tau) = C(\sigma) + C(\tau) + |S| + 1. \tag{4}$$

If we forget about the L (resp. R) contributions, the recurrence above may also be obtained by observing that the key of the partitioning element will always be compared with the search key no matter which element we are searching for.

Let us denote by $P^{*}(z,y)$ the bivariate generating function where the coefficient of $z^n y^k$ denotes the probability that a random skip list of size n has height fulfilling condition $*$ and the the total search cost is equal to k. Then Eq. (4) immediately translates to the functional equation

$$\begin{aligned} P^{=m}(z,y) &= pq^{m-1}zy^2P^{<m}(zy,y)P^{\le m}(zy,y), \quad m \ge 1, \\ P^{=0}(z,y) &= 1, \end{aligned} \tag{5}$$

since the probability for a fixed element a_i to have value m is pq^{m-1}, and $|S|+1 = |\sigma| + |\tau| + 2$, so that the contribution of the additional term in Eq. (4) splits up as

$$y^{|S|+1} = y^{|\sigma|}y^{|\tau|}y^2.$$

It is somehow easier to work with $R^{=m}(z,y) := zP^{=m}(z,y)$, because the recursion reads now

$$\begin{aligned} R^{=m}(z,y) &= pq^{m-1}R^{<m}(zy,y)R^{\le m}(zy,y), \quad m \ge 1, \\ R^{=0}(z,y) &= z. \end{aligned} \tag{6}$$

Using the decomposition of the cost $C(S) = L(S) + R(S)$, the asymptotic behavior of the expectation would be a simple corollary of the results in [4], since we may add the expectations of L and R. However, to make the paper more self-contained, we give some details about the techniques and intermediate steps in the derivation of the asymptotic behavior of the expectation of C_n. It is also useful to present these computations here, since we shall apply similar techniques to compute the variance.

Let us introduce some handy abbreviations: $Q := q^{-1}$, $L := \log Q$, and

$$[\![m]\!] := 1 - z(1-q^m).$$

Note that $[\![m]\!] = [\![m-1]\!] - pq^{m-1}z$.

If we deal with the recursion given by Eq. (6), we are in fact interested in $R(z,y) := \lim_{m\to\infty} R^{\le m}(z,y)$. Then, we obtain the generating function $S(z)$ of the expectations in the usual way by deriving $R(z,y)$ w.r.t. y and setting $y=1$. Doing that, we find the recursion

$$S^{=m}(z) = pq^{m-1}\left[\left(\frac{z}{[\![m-1]\!]^2} + S^{<m}(z)\right)\frac{z}{[\![m]\!]} + \left(\frac{z}{[\![m]\!]^2} + S^{\le m}(z)\right)\frac{z}{[\![m-1]\!]}\right].$$

Since $S^{=m}(z) = S^{\le m}(z) - S^{<m}(z)$, we can rewrite the equation above as

$$S^{\le m}(z)[\![m]\!]^2 = S^{<m}(z)[\![m-1]\!]^2 + pq^{m-1}z^2\left(\frac{1}{[\![m]\!]} + \frac{1}{[\![m-1]\!]}\right),$$

which can be solved by *iteration*,

$$S^{\le m}(z) = \frac{p}{q}\frac{z^2}{[\![m]\!]^2}\sum_{i=1}^{m}\frac{q^i}{[\![i]\!]} + p\frac{z^2}{[\![m]\!]^2}\sum_{i=0}^{m-1}\frac{q^i}{[\![i]\!]}. \tag{7}$$

Performing the limit for $m \to \infty$, we find

$$S(z) = \frac{p}{q}\frac{z^2}{(1-z)^2}\sum_{i\ge1}\frac{q^i}{[\![i]\!]} + p\frac{z^2}{(1-z)^2}\sum_{i\ge0}\frac{q^i}{[\![i]\!]}. \tag{8}$$

The expected value of interest is the coefficient of z^{n+1} in this expression; $[z^{n+1}]S(z) = \mathrm{E}(C_n)$. It could be obtained for instance by *partial fraction decomposition*. However, for more complicated expressions, as the ones we shall encounter in the next section, such an approach is not feasible, and we will therefore use a more sophisticated procedure. It was already used in the previous paper [4], but we would like to present it again in a slightly rephrased form. The standard substitution

$$z = \frac{w}{w-1}$$

proves here to be very useful. In general,

$$[z^n]f(z) = (-1)^n[w^n](1-w)^{n-1}f(w/(w-1)).$$

This formula can be easily seen by (formal) residue calculus, as explained for instance in [3]. Our expressions will usually look nicer, when expressed with the variable w, since

$$[\![i]\!] = \frac{1-wq^i}{1-w},$$

leading to expressions that belong to the class of the so-called *harmonic sums*

$$F(w) = \sum_i a_i f(b_i w),$$

where the coefficient of w^n in F satisfies

$$[w^n]F(w) = \sum_i a_i b_i^n \cdot [w^n]f(w).$$

Quite often, the series $\sum_i a_i b_i^n$ has a closed form representation, and $[w^n]f(w)$ can be computed explicitly.

We would like to show this paradigm by considering

$$[z^n]\frac{z^2}{(1-z)^2}\sum_{i\geq 1}\frac{q^i}{[\![i]\!]}.$$

Following the method that we sketched above

$$\begin{aligned}
&[z^n]\frac{z^2}{(1-z)^2}\sum_{i\geq 1}\frac{q^i}{[\![i]\!]} \\
&= (-1)^n[w^n]w(1-w)^n\sum_{i\geq 1}\frac{wq^i}{1-wq^i} \\
&= (-1)^n\sum_{k=1}^{n}\binom{n}{k}(-1)^{n-k}[w^{k-1}]\sum_{i\geq 1}\frac{wq^i}{1-wq^i} \\
&= \sum_{k=1}^{n}\binom{n}{k}(-1)^k\sum_{i\geq 1}q^{(k-1)i}[w^{k-1}]\frac{w}{1-w} \\
&= \sum_{k=2}^{n}\binom{n}{k}(-1)^k\frac{1}{Q^{k-1}-1}.
\end{aligned}$$

This example is typical: the answer comes out as an alternating sum, involving both binomial coefficients and some "known" quantities. While such a form is not very convenient for numerical purposes because of the cancellations that occur, it is very handy for the asymptotic evaluation. Such a sum can be written as a Rice integral, and asymptotics are obtained simply by considering appropriate residues.

The survey paper [2] (in this issue) explains this methodology in detail.

Here, we confine ourselves to the basic formula

$$\sum_{k=a}^{n}\binom{n}{k}(-1)^k f(k) = -\frac{1}{2\pi i}\int_{\mathcal{C}} B(n+1,-z)f(z)\,dz,$$

where $B(x,y)=\dfrac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$ is the classical Beta function, $\mathcal{C}$ is a positively oriented curve encircling the points $a, a+1, \ldots, n$ and $f(z)$ is analytic inside $\mathcal{C}$ and is a continuation to the complex plane of the sequence $f(k)$.

If $f(z)$ decreases sufficiently fast towards $\pm i\infty$, one may perform an asymptotic evaluation by extending the contour of the integral to the left. The newly encountered negative residues will give the terms in the asymptotic expansion. See [10] for detailed analytical information.

In our instance, we find

$$\mathrm{E}(C_n) = [z^{n+1}]S(z) = p(Q+1)\sum_{k=2}^{n}\binom{n}{k}(-1)^k\frac{1}{Q^{k-1}-1} + pn. \tag{9}$$

Rice's method can be applied in this particular case using the obvious continuation $1/(Q^{z-1}-1)$ for the sequence $1/(Q^{k-1}-1)$.

The residue computations are standard and can be done by some computer algebra system, e.g. MAPLE, giving us the asymptotic behavior of $\mathrm{E}(C_n)$.

Theorem 2.1. [Expected total search cost (optimized)]
The expected value of total unsuccesful search cost, C_n, in a random skip list of n elements is, as $n \to \infty$,

$$\mathrm{E}(C_n) = (Q-q)n\log_Q n + n(Q-q)\left(\frac{(\gamma-1)}{L} - \frac{1}{2} + \frac{1}{Q+1} + \frac{1}{L}\delta_1(\log_Q n)\right) + O(\log n),$$

where γ is Euler's constant and $\delta_1(x) = \sum_{k\neq 0}\Gamma\left(-1-\frac{2k\pi i}{L}\right)e^{2k\pi i x}$ is a continuous function of period 1 and mean zero.

This should be compared with the previous result for the standard search algorithm.

Theorem 2.2. [Expected total search cost (standard)]
The expected total search cost, $C_n^{[\text{standard}]}$, in a random skip of n elements is, as $n \to \infty$,

$$\mathrm{E}(C_n^{[\text{standard}]}) = Qn\log_Q n + n\left(\frac{(\gamma-1)Q+1}{L} - \frac{Q}{2} + 1 + \frac{1}{L}\delta_3(\log_Q n)\right) + O(\log n),$$

where δ_3 is a continuous periodic function of period 1 and mean zero [4].

Comparing the leading terms in both instances, we see that, asymptotically, we save about $qn\log_Q n$ comparisons by using the optimized version. It is interesting to study the factor $q/\log Q$ as a function of q. The savings increase as $q \to 1$, but it should be clear that the total number of steps in both versions of the search are the same and that a larger value of q leads to a larger expected number of pointers per element. Therefore, it is not wise to choose a large q, but to look for a value of q that trades off key comparison savings, total number of comparisons and storage requirements. A plot of the coefficients K and K' of the $n\log n$ term in $\mathrm{E}(C_n)$ and $\mathrm{E}(C^{[\text{standard}]})$, respectively, is given in Figure 4.

In [8], Pugh claims that the optimized algorithm saves an average of $q(\log_Q n + q/p^2)$ comparisons per element. Comparing this formula with the average savings that we have computed, it is plain to see that the second term is not correct.

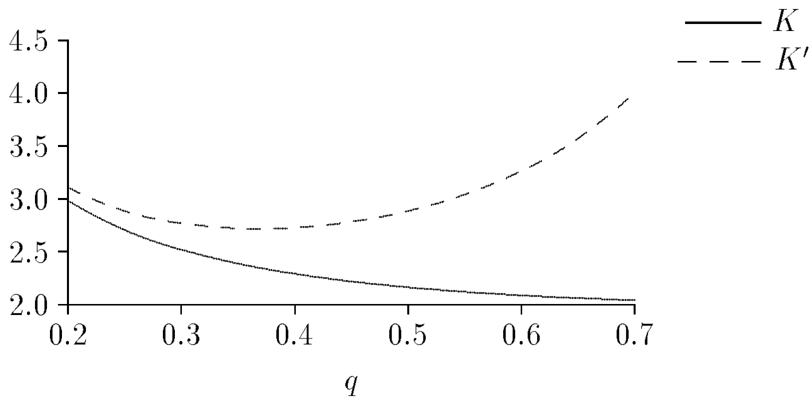

Figure 4: Behavior of K and K' as a function of q

The expectation of $C_{n,i}$ directly follows from the expectation of both the number of left-to-right maxima $\ell_{n,i}$ in $(a_i, \ldots, a_n)$ and the number of right-to-left maxima r_i in $(a_1, \ldots, a_{i-1})$. These expectations are [7]:

$$\begin{aligned} \mathrm{E}(\ell_{n,i}) &= p\left[\log_Q(n-i+1) + \frac{\gamma}{L} + \frac{1}{2} - \frac{1}{L}\delta(\log_Q(n-i+1))\right] + O\left(\frac{1}{n-i+1}\right), \quad n-i \to \infty \\ \mathrm{E}(r_i) &= \frac{p}{q}\left[\log_Q(i-1) + \frac{\gamma}{L} + \frac{1}{2} - \frac{1}{L}\delta(\log_Q(i-1))\right] + O\left(\frac{1}{i-1}\right), \quad i \to \infty \end{aligned}$$

where $\delta(x) = \sum_{k\neq 0} \Gamma(-2k\pi i/L)e^{2k\pi i x}$ is a periodic function of period 1 and mean 0.

3 Variance of the total search cost

In this section, we will compute the asymptotic behavior of the second factorial moment of C_n and of its variance. The generating function for the second factorial moment can be obtained deriving $P^*(z,y)$ twice w.r.t. y and setting $y=1$. We shall proceed starting from $R^*(z,y)$ instead, because the recurrences for R^* are easier.

Let us recall Eq. (6)

$$\begin{aligned} R^{=m}(z,y) &= pq^{m-1}R^{<m}(zy,y)R^{\leq m}(zy,y), \quad m \geq 1, \\ R^{=0}(z,y) &= z. \end{aligned}$$

Let $T^*(z) := R^*_{yy}(z,1)$, where a subscript x means partial derivative with respect to x. It is not difficult to see that

$$R^{\leq m}(z,1) = \frac{z}{[\![m]\!]}, \qquad R^{\leq m}_{z}(z,1) = \frac{1}{[\![m]\!]^2}, \qquad R^{\leq m}_{zz}(z,1) = \frac{2(1-q^m)}{[\![m^3]\!]}$$

$$R^{\leq m}_{y}(z,1) = S^{\leq m}(z), \quad R^{\leq m}_{zy}(z,1) = S^{\leq m}_{z}(z),$$

where $S^{\leq m}$ was defined in Section 2, Eq. (7).

Using these equalities above and collecting terms yields

$$\begin{aligned} T^{\leq m}(z)[\![m]\!]^2 &= T^{<m}(z)[\![m-1]\!]^2 + 2pq^{m-1}\Bigg[\frac{z^2}{[\![m]\!][\![m-1]\!]} + \frac{z^3(1-q^m)}{[\![m]\!]^2} + \frac{z^3(1-q^{m-1})}{[\![m-1]\!]^2} \\ &\quad + \frac{z}{[\![m-1]\!]}[\![m]\!]S^{\leq m}(z) + \frac{z}{[\![m]\!]}[\![m-1]\!]S^{<m}(z) \\ &\quad + [\![m]\!][\![m-1]\!]S^{\leq m}(z)S^{<m}(z) \\ &\quad + z^2[\![m]\!]S_z^{\leq m}(z) + z^2[\![m-1]\!]S_z^{<m}(z)\Bigg]. \end{aligned}$$

We can solve this recurrence by iteration, as we did for $S^{\leq m}(z)$ when dealing with the expectation in Section 2, and then compute the limiting generating function for $m \to \infty$. Finally, dividing by z that limiting generating function gives us $H(z)$, the generating function for second factorial moments:

$$H(z) = 2\frac{p}{q}\frac{z}{(1-z)^2}\left((1+q)\sum_{i\geq 1} a_i(z)q^i + \sum_{i\geq 1} d_i(z)d_{i-1}(z)q^i\right), \tag{10}$$

where

$$\begin{aligned} a_i(z) = \frac{z(1-q^i)}{[\![i]\!]^2} \quad &+ \quad \frac{p}{q}z\left(\frac{2}{[\![i]\!]^2}\sum_{1\leq j\leq i}\frac{q^j}{[\![j]\!]} + \frac{1}{[\![i]\!]}\sum_{1\leq j\leq i}\frac{q^j}{[\![j]\!]^2} - \frac{1}{[\![i]\!]}\sum_{1\leq j\leq i}\frac{q^j}{[\![j]\!]}\right) \\ &+ \quad pz\left(\frac{2}{[\![i]\!]^2}\sum_{0\leq j<i}\frac{q^j}{[\![j]\!]} + \frac{1}{[\![i]\!]}\sum_{0\leq j<i}\frac{q^j}{[\![j]\!]^2} - \frac{1}{[\![i]\!]}\sum_{0\leq j<i}\frac{q^j}{[\![j]\!]}\right), \end{aligned}$$

and

$$d_i(z) = \frac{1+pz}{[\![i]\!]} - \frac{pq^iz}{[\![i]\!]^2} + (Q-q)\frac{z}{[\![i]\!]}\sum_{1\leq j\leq i}\frac{q^j}{[\![j]\!]}.$$

The next step is to plug the values of a_i and d_i in Eq. (10) and express $H(z)$ as a linear combination of "standard" sums $\mathcal{S}_i(z)$, $i = 1, \ldots, 15$, as listed in the Appendix. The argument z in each $\mathcal{S}_i(z)$ is omitted for brevity.

$$\begin{aligned} H(z) &= 2(Q-q)\left[\mathcal{S}_6 + 2pz\mathcal{S}_6 - \mathcal{S}_7 - 3p\mathcal{S}_8 + p\mathcal{S}_9\right] + 2(Q-q)^2\left[\mathcal{S}_3 + 2\mathcal{S}_4 - \mathcal{S}_5\right] \\ &\quad + 2\frac{p}{q}(1+pz)^2\mathcal{S}_{10} + 4(Q-q)\frac{p}{q}\mathcal{S}_2 - 2\frac{p^2}{q^2}(1+pz)\mathcal{S}_{11} + 2\frac{p^2}{q^2}(1+pz)\mathcal{S}_{12} \\ &\quad + 2\frac{p^3}{q^3}(1+q)^2\mathcal{S}_1 + 4\frac{p^3}{q^2}(1+q)z\mathcal{S}_2 - 2\frac{p^3}{q^3}(1+q)\mathcal{S}_{13} - 2\frac{p^3}{q^2}(1+q)\mathcal{S}_{14} + 2\frac{p^3}{q^2}\mathcal{S}_{15}. \end{aligned}$$

Now we give an example how to obtain the list of coefficients for the sums $\mathcal{S}_i$ from the Appendix.

$$[z^n]\mathcal{S}_{12}(z) = (-1)^n[w^n](1-w)^{n+2}\sum_{i\geq 1}\frac{w^2q^{2i}}{(1-wq^i)^2(1-wq^iQ)}$$

$$
\begin{aligned}
&= \sum_{k=3}^{n+2} \binom{n+2}{k} (-1)^k [w^{k-2}] \sum_{i\geq 1} \frac{w^2 q^{2i}}{(1-wq^i)^2(1-wq^iQ)} \\
&= \sum_{k=3}^{n+2} \binom{n+2}{k} (-1)^k \frac{1}{Q^{k-2}-1} [w^{k-2}] \frac{w^2}{(1-w)^2(1-wQ)} \\
&= \sum_{k=3}^{n+2} \binom{n+2}{k} (-1)^k \frac{1}{Q^{k-2}-1} \left(\frac{Q^{k-2}-1}{(Q-1)^2} - \frac{k-2}{Q-1} \right) \\
&= \frac{1}{(Q-1)^2} \left(-1 + (n+2) - \binom{n+2}{2} \right) - \frac{1}{Q-1} \sum_{k=3}^{n+2} \binom{n+2}{k} (-1)^k \frac{k-2}{Q^{k-2}-1}
\end{aligned}
$$

From this, the formula in Appendix A is immediate.

Once we have expressed the n-th coefficient of each $\mathcal{S}_i$ as an alternating sum, we can compute its asymptotic behavior using Rice's method. The continuation to the complex plane of the discrete sequences that appear in the alternating sums is almost straightforward, except for the sequences of the type

$$\sum_{m=1}^{k} \frac{1}{Q^m - 1}. \tag{11}$$

To get an analytic continuation of this kind of sequence, we write it as a difference of infinite series and shift the index in the second summation:

$$\alpha - \sum_{m\geq 1} \frac{1}{Q^{m+k} - 1}, \tag{12}$$

where $\alpha = \sum_{m\geq 1} 1/(Q^m - 1)$ is a constant. For instance, for $Q = 2$, the value of α is $1.606695\ldots$.

Now, it makes sense to replace k by z in Eq. (12) so the continuation of the sequence to the complex plane is

$$\alpha - \sum_{m\geq 1} \frac{1}{Q^{m+z} - 1}.$$

There is a similar sequence of the type $\sum m/(Q^m - 1)$ appearing in the analysis, which could be dealt with in an analogous way, but it turns out that the terms including that kind of sequence cancel out.

The residue computations involved in Rice's method were performed using Maple; for the reader's convenience, we will compute the asymptotic behavior of one of the alternating sums containing a sequence of the type given in Eq. (11). There are eight types of alternating sums occurring in $[z^n]\mathcal{S}_1$ up to $[z^n]\mathcal{S}_{15}$. One of these is

$$\sum_{k=3}^{n+2} \binom{n+2}{k} (-1)^k \frac{1}{Q^{k-2}-1} \sum_{m=1}^{k-3} \frac{1}{Q^m - 1}.$$

We can then write the summation above as a contour integral:

$$\frac{1}{2\pi i}\int_{\mathcal{C}} \frac{\Gamma(n+3)\Gamma(-z)}{\Gamma(n+3-z)} \frac{1}{Q^{z-2}-1} \left(\alpha - \sum_{m\geq 1} \frac{1}{Q^{m+z-3}-1}\right) dz,$$

where $\mathcal{C}$ encloses $\{3, 4, \ldots, n+2\}$. There is a pole of the integrand at $z = 2$ which gives us a main contribution to the asymptotic behavior and there are also poles at $z = 2 + \frac{2\pi i}{L}k$, for k an integer different from 0. These last poles contribute a small periodic fluctuation to the coefficient of the leading term in the asymptotic behavior.

We will only consider the pole at $z = 2$ in the discussion that follows. Hence, we have to compute the residue of

$$-\frac{\Gamma(n+3)\Gamma(-z)}{\Gamma(n+3-z)} \frac{1}{Q^{z-2}-1} \left(\alpha - \sum_{m\geq 1} \frac{1}{Q^{m+z-3}-1}\right).$$

This can be readily done by considering each term separately,

i)

$$-\frac{\Gamma(n+3)\Gamma(-z)}{\Gamma(n+3-z)} \frac{\alpha}{Q^{z-2}-1},$$

ii)

$$\frac{\Gamma(n+3)\Gamma(-z)}{\Gamma(n+3-z)} \frac{1}{(Q^{z-2}-1)^2},$$

iii)

$$\frac{\Gamma(n+3)\Gamma(-z)}{\Gamma(n+3-z)} \frac{1}{Q^{z-2}-1} \frac{1}{Q^{m+z-3}-1}, \quad m \geq 2,$$

and then sum up the corresponding residues,

i)

$$\frac{\alpha}{2L} n^2 \log n + \frac{(2\gamma - L - 3)\alpha}{L} n^2 + O(n \log n),$$

ii)

$$-\frac{1}{4L^2} n^2 \log^2 n + \frac{(2L + 3 - 2\gamma)}{4L^2} n^2 \log n$$
$$+ \left(\frac{\gamma}{2L} + \frac{3\gamma}{4L^2} - \frac{7}{8L^2} - \frac{3}{4L} - \frac{\pi^2}{24L^2} - \frac{\gamma^2}{4L^2} - \frac{5}{24}\right) n^2 + O(n \log^2 n),$$

iii)

$$-\frac{1}{2L} \frac{1}{Q^{m-1}-1} n^2 \log n$$
$$+ n^2 \left(\frac{1}{Q^{m-1}-1} \left(-\frac{\gamma}{2L} + \frac{3}{4L} + \frac{3}{4}\right) + \frac{1}{2(Q^{m-1}-1)^2}\right) + O(n \log n), \quad m \geq 2,$$

yielding

$$
\begin{aligned}
& \operatorname*{Res}_{z=2}\left[-\frac{\Gamma(n+3)\Gamma(-z)}{\Gamma(n+3-z)}\frac{1}{Q^{z-2}-1}\left(\alpha-\sum_{m\geq 1}\frac{1}{Q^{m+z-3}-1}\right)\right] \\
= \; & -\frac{1}{4L^2}n^2\log^2 n+\frac{(2L+3-2\gamma)}{4L^2}n^2\log n \\
& +n^2\left(\frac{\alpha+\beta}{2}+\frac{\gamma}{2L}+\frac{3\gamma}{4L^2}-\frac{7}{8L^2}-\frac{3}{4L}-\frac{\pi^2}{24L^2}-\frac{\gamma^2}{4L^2}-\frac{5}{24}\right)+O(n\log^2 n).
\end{aligned}
$$

It will turn out that the $n^2\log^2 n$ and $n^2\log n$ terms cancel out when we substract the square of the expectation to get the variance and only n^2 and lower order terms remain. However, for the transparency of the result, we will only give the main ($=n^2$)–term of the result, but in principle one could produce as many lower order terms as desired.

Summing up the asymptotic behavior of each of the alternating sums yields the asymptotic behavior of $\mathrm{E}(C_n(C_n-1))$ as n tends to ∞. Finally, we add $\mathrm{E}(C_n)$ and substract $\mathrm{E}^2(C_n)$ in order to get the variance $\mathrm{Var}(C_n)$.

Theorem 3.1. [Variance of the total search cost (optimized)]
The variance of the total unsuccessful search cost, C_n, in a random skip list of n elements, as $n\to\infty$, is

$$
\begin{aligned}
\mathrm{Var}(C_n) \;=\; & n^2(Q-q)^2\left(\frac{\pi^2}{6L^2}+\frac{1}{12}+\frac{1}{L^2}-\frac{1}{2L}-2(\alpha+\beta)\right) \\
+ \; & n^2\left(\frac{Q-q}{QL}+1-[\delta_1^2]_0+\delta_2(\log_Q n)\right)+O(n\log^2 n),
\end{aligned}
$$

where α and β are the constants $\sum_{m\geq1}1/(Q^m-1)$ and $\sum_{m\geq1}1/(Q^m-1)^2$, $[\delta_1^2]_0$ is the mean of the square of the periodic function $\delta_1(x)$ (see Theorem 2.1), and $\delta_2(x)$ is a periodic function of period 1 and mean 0. Moreover,

$$
\alpha+\beta\approx\frac{\pi^2}{6L^2}-\frac{1}{2L}+\frac{1}{24},
$$

for "reasonable" values of q [4].

We now recall the variance of the total search cost for the standard search algorithm, for comparison purposes.

Theorem 3.2. [Variance of the total search cost (standard)]
The variance of the total search cost, $C_n^{[\mathrm{standard}]}$, in a random skip list of n elements, as $n\to\infty$, is

$$
\begin{aligned}
\mathrm{Var}(C_n^{[\mathrm{standard}]}) \;=\; & (Q^2-1)n^2\left(\frac{1}{2L}-\frac{\pi^2}{6L^2}+\frac{1}{L^2}\right)+2(Q-1)n^2\left(\frac{\alpha}{L}-\sum_{m\geq1}\frac{m}{(Q^m-1)^2}\right) \\
+ \; & n^2\left(\frac{\pi^2}{6L^2}+\frac{1}{12}+\varepsilon+\delta_7(\log_Q n)\right)+O(n\log^2 n),
\end{aligned}
$$

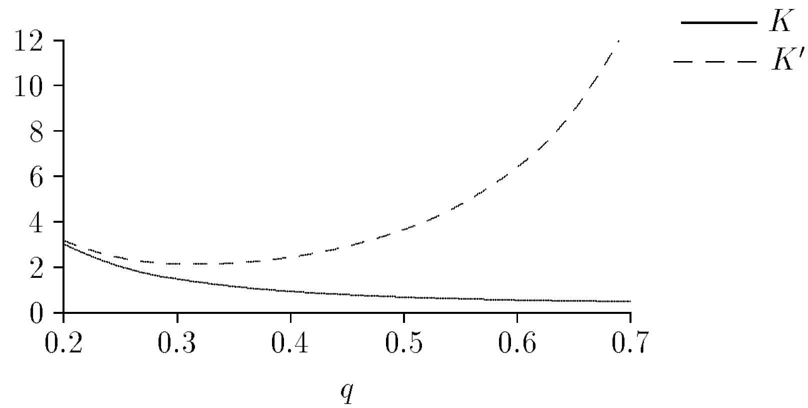

Figure 5: Behavior of K and K' as a function of q

q	K	K'
0.1	10.79...	10.57 ...
0.2	3.01 ...	3.16 ...
0.3	1.48 ...	2.15 ...
0.31...	1.37 ...	2.13 ...
0.4	0.93 ...	2.44 ...
0.5	0.68 ...	3.66 ...
0.6	0.55 ...	6.41 ...
0.7	0.48 ...	12.96 ...
0.8	0.44 ...	33.02 ...
0.9	0.42 ...	148.13...

Table 1: Some numerical values of K and K'

where α is as in the previous theorem, ε is a very small quantity for which a series representation is available, and $\delta_7(x)$ is a periodic function of period 1 and mean 0 [4].

Figure 5 depicts the coefficients K and K' of n^2 in $\mathrm{Var}(C_n)$ and $\mathrm{Var}(C_n^{[\mathrm{standard}]})$, respectively, as a function of q. Table 3 contains several numerical values of both K and K'. The coefficient K' achieves its minimum at $q = 0.31\ldots$.

As in the case of the expectation, the coefficient K of the leading term in the asymptotic behavior of the variance does not reach a local minimum, but takes its minimum value for the degenerate case where $q = 1$.

The variance of $C_{n,i}$ is also available once we compute the variance of the number of strict left-to-right maxima in $(a_i, \ldots, a_n)$, say $\mathrm{Var}(\ell_{n,i})$ and the variance of the weak right-to-left maxima in $(a_1, \ldots, a_{i-1})$, say $\mathrm{Var}(r_i)$, since both random variables are independent.

Both quantities are known [7]:

$$\begin{aligned}
\mathrm{Var}(\ell_{n,i}) &= pq\log_Q(n-i+1)+p^2\left(-\frac{5}{12}+\frac{\pi^2}{6L^2}-\frac{\gamma}{L}-\frac{1}{L^2}[\delta^2]_0\right)+p\left(\frac{\gamma}{L}+\frac{1}{2}\right)\\
&\quad +\delta_3(\log_Q(n-i+1))+O(\frac{1}{n-i+1}),\quad n-i\to\infty\\
\mathrm{Var}(r_i) &= \frac{p}{q^2}\log_Q(i-1)+\frac{p^2}{q^2}\left(-\frac{5}{12}+\frac{\pi^2}{6L^2}-\frac{\gamma}{L}-\frac{2}{L}-\frac{1}{L^2}[\delta^2]_0\right)+\frac{p}{q}\left(\frac{\gamma}{L}-\frac{1}{2}\right)\\
&\quad +\delta_5(\log_Q(i-1))+O(\frac{1}{i-1}),\quad i\to\infty
\end{aligned}$$

where $[\delta^2]_0$ is the mean of the square of δ (see Section 2), and δ_3 and δ_5 are periodic functions of period 1 and mean 0.

4 Transferring the results to the successful search

As we have already described, C_n and $\widetilde{C}_n$ differ only by the number of (weak) right-to-left maxima of the whole sequence $(a_1,\ldots,a_n)$. Our goal is to show that – w.r.t. to the leading terms – the asymptotic behavior of both random variables is the same.

We will conclude this from simple properties of probability theory. Assume that X is a random variable with a mean of order $n\log n$, a standard deviation of order n, and maximum of order n^2. And Y is a random variable with a mean of order $\log n$, a standard deviation of order $\log^{1/2} n$, and maximum of order n. Clearly, X plays the role of $\widetilde{C}_n$, Y that of the number of right-to-left maxima of $(a_1,\ldots,a_n)$ and $X+Y$ that of C_n.

For the expectations there is nothing to be shown, but for the variances we argue as follows: Since

$$\mathrm{Var}(X+Y)=\mathrm{Var}(X)+\mathrm{Var}(Y)+2\mathrm{E}(XY)-2\mathrm{E}(X)\cdot\mathrm{E}(Y),$$

we are done when we can show that $\mathrm{E}(XY)$ is of a neglectible order of growth.

We use Chebyshev's inequality in the form

$$\mathrm{Prob}\Big(|Z-\mathrm{E}(Z)|>r\sigma\Big)<\frac{1}{r^2},\quad r>1.$$

We distinguish cases: First we consider the case where the first parameter is large, $|X-\mathrm{E}(X)|>n^{\alpha+1}$. This has a small probability, and we bound both parameters by their maxima. In the other case, we distinguish the two subcases, where the second parameter is large, $|Y-\mathrm{E}(Y)|>n^{\beta}\log^{1/2} n$, where we bound the second parameter by its maximum, and the remaining subcases, where we cannot say anything about the probabilities, but both parameters are small.

Doing as indicated, we find

$$\begin{aligned}
\mathrm{E}(XY) &= O\Big(\mathrm{Prob}|X-\mathrm{E}(X)|>n^{\alpha+1}\Big)\cdot O(n^2)\cdot O(n)\\
&+ O(n^{1+\alpha})\cdot O\Big(|\mathrm{Prob}|Y-\mathrm{E}(Y)|>n^{\beta}\log^{1/2} n\Big)\cdot O(n)\\
&+ O(n^{1+\alpha})\cdot O(n^{\beta}\log^{1/2} n)\cdot O(n).
\end{aligned}$$

By Chebyshev's inequalities we find

$$\mathrm{E}(XY) = O(n^{3-2\alpha}) + O(n^{2+\alpha-2\beta}) + O(n^{1+\alpha+\beta+\varepsilon}),$$

where we replaced the logarithmic factor by n^{ε}, for simplicity.

Now we can obtain a relatively small error term by balancing the three exponents. It turns out that $\alpha = \frac{5}{9}$ and $\beta = \frac{1}{3}$ is the optimal choice, and then our bound is $O(n^{17/9+\varepsilon})$.

This is probably far from the truth, since the next term in the asymptotic expansion of $\mathrm{Var}(\widetilde{C}_n)$ should also be of order $n \log^2 n$. However, we don't see any other way of seeing this than to do most of the computations also in this instance, and since the recursion is not so nice, they would be even messier, whence we decided to confine ourselves with this elementary bounding technique.

5 Extensions

The total unsuccessful search cost that we have analyzed is just a particular case of what we can call *additive costs.*

Definition 5.1. *A cost C over skip lists is said to be an* additive cost *if, given a skip list S of height m, C satisfies*

$$C(S) = C(\sigma) + C(\tau) + U(|\sigma|, |\tau|, m),$$

where σ is the part of S before the partitioning element, and τ is the part of S after the partitioning element. The function U is called a valuation *function over skip lists.*

The algebraic part of the probabilistic analysis of additive costs can be carried without much difficulty, especially in the case of the analysis of the expectation of a cost C over skip lists of size n. Moreover, it can be carried out even if the levels of the elements of skip lists where generated by independent random variables other than geometric ones. Let us denote $\pi_k = \mathrm{Prob}(a_j = k)$ and $\phi_k = \mathrm{Prob}(a_j \le k)$. Furthermore, let

$$[\![m]\!] := 1 - z\phi_m.$$

Note that $[\![m]\!] = [\![m-1]\!] - \pi_m z$ and that this definition is consistent with the one that we give in Section 2 for the case of geometrically distributed levels.

We first introduce a family of generating functions $C^*(z)$:

$$C^*(z) = \sum_{s \in \mathcal{S}^*} \mathrm{Prob}(s)\, C(s)\, z^{|s|},$$

where $\mathcal{S}^*$ denotes the set of skip lists whose height satisfies condition $(*)$.

If we use the recursive decomposition of skip lists and since C is additive, we have,

$$\begin{aligned}
C^{=m}(z) &= C^{\leq m}(z) - C^{<m}(z) \\
&= \sum_{\substack{\sigma\in\mathcal{S}^{<m}\\ \tau\in\mathcal{S}^{\leq m}}} \mathrm{Prob}(\sigma)\mathrm{Prob}(\tau)\pi_m\left(C(\sigma)+C(\tau)+U(|\sigma|,|\tau|,m)\right) z^{|\sigma|+|\tau|+1} \\
&= \pi_m z\Bigg[\left(\sum_{\tau\in\mathcal{S}^{\leq m}} \mathrm{Prob}(\tau)z^{|\tau|}\right)\left(\sum_{\sigma\in\mathcal{S}^{<m}} \mathrm{Prob}(\sigma)C(\sigma)z^{|\sigma|}\right) \\
&+ \left(\sum_{\sigma\in\mathcal{S}^{<m}} \mathrm{Prob}(\sigma)z^{|\sigma|}\right)\left(\sum_{\tau\in\mathcal{S}^{\leq m}} \mathrm{Prob}(\tau)C(\tau)z^{|\tau|}\right) + U_m(z)\Bigg] \\
&= \pi_m z\left(\frac{C^{<m}(z)}{[\![m]\!]} + \frac{C^{\leq m}(z)}{[\![m-1]\!]} + U_m(z)\right),
\end{aligned}$$

where $U_m(z)$ is defined as follows:

$$U_m(z) = \sum_{\sigma,\tau} \mathrm{Prob}(\sigma)\mathrm{Prob}(\tau)U(|\sigma|,|\tau|,m)z^{|\sigma|+|\tau|}.$$

If we assume w.l.o.g. that $C^{=0}(z) = 0$ then we can solve the linear recurrence using iteration and get

$$C^{\leq m}(z) = \frac{z}{[\![m]\!]^2}\sum_{1\leq i\leq m} [\![i]\!][\![i-1]\!]\pi_i U_i(z).$$

The quantity we are interested in, namely the expectation of C over skip lists of size n, is the n-th coefficient of the limit generating function $C(z)$:

$$C(z) = \lim_{m\to\infty} C^{\leq m}(z) = \frac{z}{(1-z)^2}\sum_{i\geq 1}[\![i]\!][\![i-1]\!]\pi_i U_i(z). \tag{13}$$

If the valuation function U is of the type $U(|\sigma|,|\tau|,m) = f(|\sigma|)+g(|\tau|)$, then,

$$U_m(z) = \frac{F_{m-1}(z)}{[\![m]\!]} + \frac{G_m(z)}{[\![m-1]\!]},$$

where

$$F_m(z) = \sum_{s\in\mathcal{S}^{\leq m}} f(|s|)\mathrm{Prob}(s)z^{|s|}, \qquad G_m(z) = \sum_{s\in\mathcal{S}^{\leq m}} g(|s|)\mathrm{Prob}(s)z^{|s|}.$$

Plugging these expressions in Eq. (13) yields

$$C(z) = \frac{z}{(1-z)^2}\left(\sum_{i\geq 0}[\![i]\!]\pi_{i+1}F_i(z) + \sum_{i\geq 1}[\![i]\!]\pi_i G_i(z)\right). \tag{14}$$

Similar techniques work for additive costs that are defined in terms of a reverse recursive decomposition of the skip lists: skip lists of height m are decomposed into two skip lists of heights $\leq m$ and $< m$, respectively, using the last element of height m of the skip list as the partitioning element.

Furthermore, we can also introduce and apply the former algebraic techniques for costs satisfying linear recurrences, i.e.

$$C(s) = \alpha C(\sigma) + \beta C(\tau) + U(|\sigma|, |\tau|).$$

A particular case of these linear additive costs are the costs where either α or β is 0. Important examples of such type of costs are the number of left-to-right and right-to-left maxima in a skip list S.

References

1. L. Devroye. A limit theory for random skip lists. *The Annals of Applied Probability*, 2(3):597–609, 1992.

2. Ph. Flajolet and R. Sedgewick. Mellin transforms and asymptotics: Finite differences and Rice's integrals. Preprint. Submitted for publication.

3. I. Goulden and D. Jackson. *Combinatorial Enumeration*. J. Wiley, 1983.

4. P. Kirschenhofer and H. Prodinger. The path length of random skip lists. *Acta Informatica*, 1994. To appear.

5. T. Papadakis, J. I. Munro, and P. V. Poblete. Average search and update costs in skip lists. *BIT*, 32:316–332, 1992.

6. Th. Papadakis. *Skip Lists and Probabilistic Analysis of Algorithms*. PhD thesis, University of Waterloo, 1993. Available as Technical Report CS-93-28.

7. H. Prodinger. Combinatorics of geometrically distributed random variables: Left-to-right maxima. In A. Barlotti, M. Delest, and R. Pinzani, editors, *Proc. of the 5th Formal Power Series Conference*, pages 367–380, 1993. Submitted to *Discrete Applied Mathematics*.

8. W. Pugh. A skip list cookbook. Technical Report CS-TR-2286.1, Institute for Advanced Computer Studies, Department of Computer Science, University of Maryland, College Park, MD, June 1990. Also published as UMIACS-TR-89-72.1.

9. W. Pugh. Skip lists: a probabilistic alternative to balanced trees. *Comm. ACM*, 33(6):668–676, 1990.

10. U. Schmid. On a tree collision resolution algorithm in presence of capture. *RAIRO Informatique Théorique et Applications/Theoretical Informatics and Applications*, 2(26):163–197, 1992.

A Appendix

Here is the list of sums that we used in our analysis. They are not sorted in any particular order.

$$\mathcal{S}_1(z) = \frac{z^3}{(1-z)^2} \sum_{\substack{1 \le j \le i \\ 1 \le h < i}} \frac{q^{i+j+h}}{[\![i]\!][\![i-1]\!][\![j]\!][\![h]\!]} \qquad \mathcal{S}_9(z) = \frac{z^2}{(1-z)^2} \sum_{i \ge 1} \frac{q^{2i}}{[\![i]\!]^2}$$

$$\mathcal{S}_2(z) = \frac{z^2}{(1-z)^2} \sum_{1 \le j < i} \frac{q^{i+j}}{[\![i]\!][\![i-1]\!][\![j]\!]} \qquad \mathcal{S}_{10}(z) = \frac{z}{(1-z)^2} \sum_{i \ge 1} \frac{q^{i}}{[\![i]\!][\![i-1]\!]}$$

$$\mathcal{S}_3(z) = \frac{z^2}{(1-z)^2} \sum_{1 \le j \le i} \frac{q^{i+j}}{[\![i]\!][\![j]\!]^2} \qquad \mathcal{S}_{11}(z) = \frac{z^2}{(1-z)^2} \sum_{i \ge 1} \frac{q^{2i}}{[\![i]\!][\![i-1]\!]^2}$$

$$\mathcal{S}_4(z) = \frac{z^2}{(1-z)^2} \sum_{1 \le j \le i} \frac{q^{i+j}}{[\![i]\!]^2[\![j]\!]} \qquad \mathcal{S}_{12}(z) = \frac{z^2}{(1-z)^2} \sum_{i \ge 1} \frac{q^{2i}}{[\![i]\!]^2[\![i-1]\!]}$$

$$\mathcal{S}_5(z) = \frac{z^2}{(1-z)^2} \sum_{1 \le j \le i} \frac{q^{i+j}}{[\![i]\!][\![j]\!]} \qquad \mathcal{S}_{13}(z) = \frac{z^3}{(1-z)^2} \sum_{1 \le j \le i} \frac{q^{2i+j}}{[\![i]\!][\![i-1]\!]^2[\![j]\!]}$$

$$\mathcal{S}_6(z) = \frac{z}{(1-z)^2} \sum_{i \ge 1} \frac{q^{i}}{[\![i]\!]^2} \qquad \mathcal{S}_{14}(z) = \frac{z^3}{(1-z)^2} \sum_{1 \le j < i} \frac{q^{2i+j}}{[\![i]\!]^2[\![i-1]\!][\![j]\!]}$$

$$\mathcal{S}_7(z) = \frac{z}{(1-z)^2} \sum_{i \ge 1} \frac{q^{i}}{[\![i]\!]} \qquad \mathcal{S}_{15}(z) = \frac{z^3}{(1-z)^2} \sum_{i \ge 1} \frac{q^{3i}}{[\![i]\!]^2[\![i-1]\!]^2}$$

$$\mathcal{S}_8(z) = \frac{z^2}{(1-z)^2} \sum_{i \ge 1} \frac{q^{2i}}{[\![i]\!]^3}$$

Next, we will give the list of the coefficients $[z^n]\mathcal{S}_1, \ldots, [z^n]\mathcal{S}_{15}$.

$$[z^n]\mathcal{S}_1(z) = -\frac{1}{Q-1} \sum_{k=3}^{n+2} \binom{n+2}{k} (-1)^k \frac{1}{Q^{k-2}-1} \left[\binom{k-3}{2} + \frac{k-3}{Q-1} + \sum_{m=1}^{k-3} \frac{m-2}{Q^m-1} \right]$$

$$[z^n]\mathcal{S}_2(z) = \frac{1}{Q-1} \sum_{k=3}^{n+2} \binom{n+2}{k} (-1)^k \frac{k-3}{Q^{k-2}-1}$$

$$[z^n]\mathcal{S}_3(z) = \sum_{k=3}^{n+2} \binom{n+2}{k} (-1)^k \frac{1}{Q^{k-2}-1} \left[\binom{k-2}{2} - \sum_{m=1}^{k-3} \frac{m}{Q^m-1} + (k-2) \sum_{m=1}^{k-3} \frac{1}{Q^m-1} \right]$$

$$
\begin{aligned}
[z^n]\mathcal{S}_4(z) &= \sum_{k=3}^{n+2}\binom{n+2}{k}(-1)^k\frac{1}{Q^{k-2}-1}\left[\binom{k-2}{2}+\sum_{m=1}^{k-3}\frac{m}{Q^m-1}\right]\\
[z^n]\mathcal{S}_5(z) &= -\sum_{k=2}^{n+1}\binom{n+1}{k}(-1)^k\frac{1}{Q^{k-1}-1}\left[k-2+\sum_{m=1}^{k-2}\frac{1}{Q^m-1}\right]\\
[z^n]\mathcal{S}_6(z) &= -\sum_{k=3}^{n+2}\binom{n+2}{k}(-1)^k\frac{k-2}{Q^{k-2}-1}\\
[z^n]\mathcal{S}_7(z) &= \sum_{k=2}^{n+1}\binom{n+1}{k}(-1)^k\frac{1}{Q^{k-1}-1}\\
[z^n]\mathcal{S}_8(z) &= \sum_{k=3}^{n+2}\binom{n+2}{k}(-1)^k\frac{1}{Q^{k-2}-1}\binom{k-2}{2}\\
[z^n]\mathcal{S}_9(z) &= -\sum_{k=2}^{n+1}\binom{n+1}{k}(-1)^k\frac{k-2}{Q^{k-1}-1}\\
[z^n]\mathcal{S}_{10}(z) &= \frac{1}{Q-1}\binom{n+1}{2}\\
[z^n]\mathcal{S}_{11}(z) &= \frac{1}{(Q-1)^2}\binom{n+1}{2}-\frac{n}{Q(Q-1)}+\frac{1}{Q(Q-1)}\sum_{k=3}^{n+2}\binom{n+2}{k}(-1)^k\frac{k-2}{Q^{k-2}-1}\\
[z^n]\mathcal{S}_{12}(z) &= -\frac{1}{(Q-1)^2}\binom{n+1}{2}-\frac{1}{Q-1}\sum_{k=3}^{n+2}\binom{n+2}{k}(-1)^k\frac{k-2}{Q^{k-2}-1}\\
[z^n]\mathcal{S}_{13}(z) &= \frac{n^2}{(Q-1)^3}-\frac{n}{Q(Q-1)^3}+\frac{1}{(Q-1)^2}\sum_{k=3}^{n+2}\binom{n+2}{k}(-1)^k\frac{k-3}{Q^{k-2}-1}\\
&- \frac{Q-q}{(Q-1)^3}\sum_{k=3}^{n+2}\binom{n+2}{k}(-1)^k\frac{k-2}{Q^{k-2}-1}\\
&- \frac{1}{Q(Q-1)}\sum_{k=3}^{n+2}\binom{n+2}{k}(-1)^k\frac{1}{Q^{k-2}-1}\left[\binom{k-2}{2}+\sum_{m=1}^{k-3}\frac{m}{Q^m-1}\right]\\
[z^n]\mathcal{S}_{14}(z) &= \frac{1}{Q-1}\sum_{k=3}^{n+2}\binom{n+2}{k}(-1)^k\frac{1}{Q^{k-2}-1}\sum_{m=1}^{k-3}\frac{m}{Q^m-1}\\
&- \frac{1}{(Q-1)^2}\sum_{k=3}^{n+2}\binom{n+2}{k}(-1)^k\frac{k-3}{Q^{k-2}-1}\\
[z^n]\mathcal{S}_{15}(z) &= -\frac{n^2}{(Q-1)^3}-\frac{n}{Q(Q-1)^3}-\frac{Q-q}{(Q-1)^3}\sum_{k=3}^{n+2}\binom{n+2}{k}(-1)^k\frac{k-2}{Q^{k-2}-1}
\end{aligned}
$$