HOW TO ADVANCE ON A STAIRWAY BY COIN FLIPPINGS

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ABSTRACT. Assume that N players start at stair 1 and flip a fair coin to advance one step. From the surviving people, a demon additionally kills one with probability $1-p$, i.e. with probability p he does not interfer. The average level that such a party will reach is of interest. In this paper we show that this quantity behaves like $\log_2 N$ and give a tutorial survey how to attack such problems. This problem is related to themes in Theoretical Computer Sciences, namely *Approximate Counting* and *Tries*.

Assume that N persons want to advance on a stairway. The rule is as follows: They start at level 1. The $k \geq 1$ persons who advanced to the level j flip a (fair) coin; the persons with the "1" advance; the others (with a "0") die. However, there is a demon supervising the game. Usually, he "consumes" one of the survivers. But, with a probability p, he resigns and does not interfer. (The demon can only interfer at levels 2 of larger.) The question is, how far can the party advance, on the average?

In this example the party gets to level 4; at one stage the demons resigns.

This recreational description has of course a serious background. If $p = 0$, the model is exactly equivalent to an algorithmic procedure called *Approximate counting* [1], as was shown in [7]. If $p = 1$ the resulting recurrences (see the discussion below) resembles recurrences that are customary in the context of *tries*; see for instance the brandnew book of H.Mahmoud [5]. The idea of introducing a probability p of escaping the demon is borrowed from [8]; in this thesis U. Schmid studied the collision of N transmitted data. Usually they are all destroyed, but with a probability p one package survives.

In this paper we want to solve this problem. The main interest is not so much in the actual answer that we will finally obtain; its main aim is to give a rather tutorial introduction to the machinery that is usually used within the context of such problems.

To simplify the notation, let, as usual, $q = 1 - p$. We introduce probability generating functions $F_N(z)$. The coefficient of z^k is the probability that N persons came to level k (and not farther). There is a recursion relating these functions:

$$F_N(z) = z\sum_{k=1}^{N} 2^{-N}\binom{N}{k}\Big[qF_{k-1}(z) + pF_k(z)\Big] + z2^{-N}, \quad N \ge 1,\ F_0(z) = z$$

To understand it, note that $2^{-N}\binom{N}{k}$ is the probability that k of N have flipped a "1". With z we mark a level that is already passed. With probability p, k people go on, with probability q, only $k-1$. (If $k=1$, and the demon takes him, we nevertheless say that the level was reached!) If $k=0$, the recursion stops.

The desired average C_N may be obtained via $F_N'(1)$. By differentiation, we get the recursion

$$C_N = 1 + q\sum_{k=1}^{N} 2^{-N}\binom{N}{k}C_{k-1} + p\sum_{k=1}^{N} 2^{-N}\binom{N}{k}C_k, \quad N \ge 1,\ C_0 = 1.$$

Now it is customary to study the exponential generating function

$$C(z) = \sum_{N\ge 0} C_N \frac{z^N}{N!}.$$

Multiplying the recursion by $(2z)^N/N!$ and summing up we get

$$C(2z) = e^{2z} + pe^zC(z) - pe^z + q\sum_{N\ge 1}\frac{z^N}{N!}\sum_{k=1}^{N}\binom{N}{k}C_{k-1}.$$

To get rid of the unpleasant sum, we differentiate this equation and consider the difference of the new and the old equation. This gives us

$$2C'(2z) = 2e^{2z} + pe^zC(z) + pe^zC'(z) - pe^z + q\sum_{N\ge 1}\frac{z^{N-1}}{(N-1)!}\sum_{k=1}^{N}\binom{N}{k}C_{k-1},$$

or

$$2C'(2z) - C(2z) = e^{2z} + pe^zC'(z) + qe^zC(z).$$

Here we have used that $\binom{N+1}{k} - \binom{N}{k} = \binom{N}{k-1}$. The next step is the introduction of the *Poisson transformed* function $D(z) = e^{-z}C(z)$. Then $D'(z)+D(z) = e^{-z}C'(z)$, and we obtain the easier equation

$$2D'(2z) + D(2z) = 1 + pD'(z) + D(z).$$

Now we consider the coefficients D_N of this (exponential) generating function. Equating the coefficients of $z^N/N!$, we find

$$2^{N+1}D_{N+1} + 2^ND_N = \delta_{N,0} + pD_{N+1} + D_N, \quad N \ge 1, D_0 = 1.$$

It means $D_0 = 1$, $D_1 = \frac{1}{2-p}$ and for $N \geq 1$

$$\left(1 - \frac{p}{2^{N+1}}\right) D_{N+1} = -\frac{1}{2}\left(1 - \frac{p}{2^N}\right) D_N.$$

We rewrite it as

$$D_N = -\frac{1}{2} D_{N-1} \frac{1 - \frac{1}{2^{N-1}}}{1 - \frac{p}{2^N}}$$

and iterate it to get

$$D_N = \frac{1}{2-p}(-1)^{N-1} 2^{1-N} \frac{Q_{N-1}}{Q_{N-1}(\frac{p}{2})}, \qquad N \geq 1.$$

Here,

$$Q_m(a) = \left(1 - \frac{a}{2}\right)\left(1 - \frac{a}{2^2}\right) \dots \left(1 - \frac{a}{2^m}\right)$$

and $Q_m = Q_m(1)$. Since $C(z) = e^z D(z)$, we have

$$\begin{aligned} C_N &= \sum_{k=0}^{N} \binom{N}{k} D_k \\ &= 1 + \frac{1}{2-p} \sum_{k=1}^{N} \binom{N}{k} (-1)^{k-1} 2^{1-k} \frac{Q_{k-1}}{Q_{k-1}(\frac{p}{2})} \\ &= 1 - \frac{1}{2-p} \sum_{k=1}^{N} \binom{N}{k} (-1)^k f(k) \end{aligned}$$

with

$$f(k) = 2^{1-k} \frac{Q_{k-1}}{Q_{k-1}(\frac{p}{2})}.$$

The method to evaluate such an alternating sum is called *Rice's method*, [2], [4], [5], [6] .

Lemma. *Let $\mathcal{C}$ be a curve surrounding the points $1, 2, \dots, N$ in the complex plane and let $f(z)$ be analytic inside $\mathcal{C}$. Then*

$$\sum_{k=1}^{N} \binom{N}{k} (-1)^k f(k) = -\frac{1}{2\pi i} \int_{\mathcal{C}} [N; z] f(z) dz,$$

where

$$[N; z] = \frac{(-1)^{N-1} N!}{z(z-1)\dots(z-N)} = \frac{\Gamma(N+1)\Gamma(-z)}{\Gamma(N+1-z)}. \quad \square$$

Extending the contour of integration it turns out that under suitable growth conditions on $f(z)$ (compare [8]) the asymptotic expansion of the alternating sum is given by

$$\sum \operatorname{Res}\big([N; z] f(z)\big) + \text{smaller order terms}$$

where the sum is taken over all poles z_0 different from $1,\dots,N$.

Thus we must find a complex function $f(z)$, such that $f(k)$ are the numbers $2^{1-k}\frac{Q_{k-1}}{Q_{k-1}(\frac{p}{2})}$. For this task it is convenient to introduce the infinite product

$$Q(x)=\big(1-\frac{x}{2}\big)\big(1-\frac{x}{4}\big)\big(1-\frac{x}{8}\big)\cdots .$$

Then it is easy to see that

$$Q_m=\frac{Q(1)}{Q(2^{-m})},$$

and, more generally,

$$Q_m(a)=\frac{Q(a)}{Q(a2^{-m})}.$$

It makes sense to replace m in the last equality by a complex variable z. Now we have to consider the residues of

$$[N;z]2^{1-z}\frac{Q_{z-1}}{Q_{z-1}(\frac{p}{2})}$$

at $z=0$ and then also at $z=2k\pi i/L$, were we set here and later $L=\log 2$. Also, the notion $\chi_k=2k\pi i/L$ is very useful. The meaning of Q_{z-1} etc. is as it was just described. Let's start as follows:

$$Q_{z-1}=\frac{Q(1)}{Q(2^{1-z})}=\frac{1}{1-2^{-z}}\cdot\frac{Q(1)}{Q(2^{-z})}.$$

Now

$$1-2^{-z}=1-e^{-Lz}\sim 1-\left(1-Lz+\frac{L^2z^2}{2}\mp\dots\right)\sim Lz\left(1-\frac{Lz}{2}\right)$$

and thus

$$\frac{1}{1-2^{-z}}\sim\frac{1}{Lz}\left(1+\frac{Lz}{2}\right).$$

Furthermore,

$$Q(2^{-z})\sim Q(1)+Q'(1)\cdot(-L)\cdot z,$$

and

$$Q'(x)=-Q(x)\cdot\sum_{k\ge1}\frac{1}{2^k-x},$$

so that

$$Q(2^{-z})\sim Q(1)\cdot(1+Lz\alpha),$$

where we use the abbreviation

$$\alpha=\sum_{k\ge1}\frac{1}{2^k-1}.$$

Alltogether we find

$$Q_{z-1} \sim \frac{1}{Lz}\left(1+\frac{Lz}{2}\right)(1-Lz\alpha).$$

The computations for $Q_{z-1}(\frac{p}{2})$ are entirely similar, so that we just mention the result:

$$Q_{z-1}(\frac{p}{2}) \sim \frac{1}{1-\frac{p}{2}} \cdot \big(1-p\alpha(p)Lz\big)$$

with

$$\alpha(p)=\sum_{k\ge 1}\frac{1}{2^k-p}.$$

We need one more local expansion:

$$[N;z] \sim -\frac{1}{z}\big(1+zH_N\big),$$

with the N-th *harmonic number*

$$H_N = 1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{N} \sim \log N+\gamma+\dots\ .$$

Plugging things together we have

$$\begin{aligned} f(z) &\sim 2\big(1-Lz\big)\frac{1}{Lz}\left(1+\frac{Lz}{2}\right)(1-Lz\alpha)\big(1-\frac{p}{2}\big)\big(1+p\alpha(p)Lz\big)\\ &\sim \frac{2-p}{Lz}\Big(1+z\big(-\frac{L}{2}-\alpha L+p\alpha(p)L\big)\Big). \end{aligned}$$

Therefore the residue of $[N;z]f(z)$ at $z=0$ is

$$-\frac{2-p}{L}\Big(H_N-\frac{L}{2}-\alpha L+p\alpha(p)L\Big).$$

We can use the asymptotics for H_N and then go back to the formula for the average C_N and get the main contribution

$$1-\frac{1}{2-p}\cdot\left(-\frac{2-p}{L}\Big(\log N+\gamma-\frac{L}{2}-\alpha L+p\alpha(p)L\Big)\right).$$

This readily simplifies to

$$\log_2 N+\frac{\gamma}{L}+\frac{1}{2}-\alpha+p\alpha(p).$$

From this we can easily see the dependency of p as p varies between 0 and 1. As was to be expected, this quantity increases monotonically and for $p=1$ the α-terms disappear.

Now we have to turn to the residues $z=\chi_k$. This time it is easier because there is only a single pole, originating from Q_{z-1}. We list the local expansions of the factors of

$$f(z)=2^{1-z}\frac{Q_{z-1}}{Q_{z-1}\big(\frac{p}{2}\big)}:$$

$$\begin{aligned}
2^{1-z} &\sim 2 \\
Q_{z-1} &\sim \frac{1}{Lz} \\
\frac{1}{Q_{z-1}\left(\frac{p}{2}\right)} &\sim \frac{Q(p)}{Q\left(\frac{p}{2}\right)} = 1 - \frac{p}{2} \\
[N; z] &\sim N^{\chi_k} \cdot \Gamma(-\chi_k)
\end{aligned}$$

Here, we used the well-known fact that

$$\frac{\Gamma(N+a)}{\Gamma(N+b)} \to N^{a-b}.$$

The residue of $[N; z]f(z)$ at $z = \chi_k$ is thus

$$2 \cdot \frac{1}{L} \cdot \left(1 - \frac{p}{2}\right) \cdot N^{\chi_k}\Gamma(-\chi_k) = (2-p)\frac{1}{L}\Gamma(-\chi_k)N^{\chi_k}.$$

It is now customary to collect all these contributions for $k \in \mathbb{Z}$, $k \neq 0$ in *one* function $\delta(x)$. Set

$$\delta(x) = \frac{1}{L}\sum_{k\neq 0}\Gamma(-\chi_k)e^{2k\pi i\cdot x},$$

then the periodic quantity of our desired quantity C_N is (don't forget the factor $-\frac{1}{2-p}$)

$$-\delta\big(\log_2 N\big).$$

Observe that this quantity is independent of p. However, in the smaller order terms, the parameter p appears.

We summarize our results as follows:

Theorem. *Let C_N be the average level that a random party of N people can reach in presence of a demon (as descibed in the Introduction). Then we have the following asymptotic formula*

$$C_N \sim \log_2 N + \frac{\gamma}{L} + \frac{1}{2} - \alpha + p\alpha(p) - \delta\big(\log_2 N\big)$$

where

$$\alpha(p) = \sum_{k\geq 1}\frac{1}{2^k - p}, \qquad \alpha = \alpha(1)$$

and

$$\delta(x) = \frac{1}{L}\sum_{k\neq 0}\Gamma(-\chi_k)e^{2k\pi i\cdot x}$$

is a periodic function of period 1, mean 0 and small amplitude ($\approx 10^{-6}$). It is given as a Fourier series.

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