ON STIRLING NUMBERS FOR COMPLEX ARGUMENTS AND HANKEL CONTOURS

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Abstract. Cauchy coefficient integrals and Hankel contours provide a natural generalization of Stirling numbers for unrestricted complex values of their arguments. Many classical identities survive such an extension.

1. Introduction

Richmond and Merlini have introduced in [5] an extension of Stirling's subset numbers $\left\{{x \atop y}\right\}$ and cycle numbers $\left[{x \atop y}\right]$ when $x-y$ is an integer. They also propose a further generalization when $x-y$ is not an integer, but most classical properties are no longer preserved. As the authors say about their most general extension in [5, p. 76]: "*It seems to us that the ideas used to derive identities and recurrences lead to complicated formulas in general. There are significant terms resulting from the fact that the integrands are not single valued and also from the fact that the contours change.*"

In this note, we give an alternative and more natural extension of Stirling numbers of complex arguments for which most classical identities are still satisfied. (We restrict ourselves to the most common properties leaving it to the imagination of the reader to go further.) Like in [5], our approach starts with Cauchy coefficient integrals. However, in contrast to [5], we use a Hankel contour that has the merit of *not* being dependent on particular index values. This intrinsic character of the contour precisely ensures the permanence of identities.

Relevant references for the classical theory are [1, 2, 3, 4], the latter paper being an excellent historical account of Stirling numbers.

2. Stirling numbers of complex index

By definition, the Stirling subset numbers ('of the second kind') $\left\{{n \atop k}\right\}$ are for $n,k\in\mathbb{N}$ given by

$$\left\{{n \atop k}\right\} = \frac{n!}{k!}[z^n]\,(e^z-1)^k\,,$$

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or, by Cauchy's coefficient formula,

$$\left\{ {n \atop k} \right\} = \frac{n!}{k!} \frac{1}{2i\pi} \int_{\gamma} (e^z - 1)^k \frac{dz}{z^{n+1}}, \tag{1}$$

where the integration contour γ is a small contour encircling the origin. As n is nonnegative in (1), the contour γ can be deformed into a *Hankel contour* $\mathcal{H}$ (see [6]) that starts from $-\infty$ below the negative axis, surrounds the origin counterclockwise and returns to $-\infty$ in the half plane $\Im z > 0$. Details of $\mathcal{H}$ are of course immaterial and we need only assume that it is at distance ≤ 1 from the real axis.

This suggests the following definition

Definition 1. *The Stirling numbers of complex arguments ("fractional order") are defined for $\Re(x) > 0$ by*

$$\left\{ {x \atop y} \right\} = \frac{x!}{y!} \frac{1}{2i\pi} \int_{\mathcal{H}} (e^z - 1)^y \frac{dz}{z^{x+1}}, \tag{2}$$

where $s! = \Gamma(s+1)$. The determination of $(e^z - 1)^y$ is the principal determination on the part of the contour $\Re z > 0$ extended by continuity to the whole of $\mathcal{H}$.

This definition extends the Stirling subset numbers to *arbitrary* complex arguments (x, y) satisfying $\Re x > 0$. When $\Re(x) \leq 0$, the integral diverges. However, through integration by parts, one finds, when $\Re(x) > 1$,

$$\left\{ {x \atop y} \right\} = \frac{(x-1)!}{(y-1)!} \frac{1}{2i\pi} \int_{\mathcal{H}} e^z \, (e^z - 1)^{y-1} \frac{dz}{z^x}. \tag{3}$$

The integral in (3) now converges for all values of x and y. The variant form (3) shows that $\left\{ {x \atop y} \right\}$ can be continued for $\Re(x) \leq 0$ into a meromorphic function of x (for any fixed y), with poles at the nonpositive integers. As a function of y (for any fixed x not a negative integer), it is entire.

Our definition of generalized Stirling numbers in (2) and (3) coincides with that of [5] *only* when $x - y$ is an integer. It differs significantly in other cases, since Richmond and Merlini propose to define the general form of $\left\{ {x \atop y} \right\}$ by means of a saddle point circle that, contrary to $\mathcal{H}$, is dependent upon the particular values of x, y.

3. Relations

As announced, we show now that the most common properties are preserved for our generalized Stirling numbers as defined by (2) and (3).

Recurrence. In the integral representation for $\left\{ {x \atop y} \right\}$, perform integration by parts. This gives for $\Re x > 1$

$$\frac{1}{2i\pi} \int_{\mathcal{H}} (e^z - 1)^y \frac{dz}{z^{x+1}} = \left[-\frac{1}{x z^x} (e^z - 1)^y \right]_{\mathcal{H}} + \frac{y}{x} \frac{1}{2i\pi} \int_{\mathcal{H}} (e^z - 1)^{y-1} e^y \frac{dz}{z^x}$$

and, upon writing $e^y = (e^y - 1) + 1$,

$$\frac{1}{2i\pi}\int_{\mathcal{H}}(e^z-1)^y\,\frac{dz}{z^{x+1}} = \frac{y}{x}\,\frac{1}{2i\pi}\int_{\mathcal{H}}(e^z-1)^y\,e^y\frac{dz}{z^x} + \frac{y}{x}\,\frac{1}{2i\pi}\int_{\mathcal{H}}(e^z-1)^{y-1}\,e^y\frac{dz}{z^x}$$

or, in standard notation

$$\left\{ {x \atop y} \right\} = \left\{ {x-1 \atop y-1} \right\} + y\left\{ {x-1 \atop y} \right\}.$$

This relation originally established for $\Re x > 1$ persists for all complex x by uniqueness of analytic continuation.

Binomial formula. A binomial expansion of $(e^y-1)^k$ yields the classical formula $(n, k \in \mathbb{N})$,

$$\left\{ {n \atop k} \right\} = \frac{1}{k!}\sum_{j=1}^{k}\binom{k}{j}(-1)^{k-j}j^n.$$

This process naturally extends to complex x giving for all $k \in \mathbb{N}$,

$$\left\{ {x \atop k} \right\} = \frac{1}{k!}\sum_{j=1}^{k}\binom{k}{j}(-1)^{k-j}j^x,$$

upon using the binomial expansion in (2) and appealing to Hankel's original representation of the gamma function [6].

Bell numbers. The Bell numbers of integral order are defined by their exponential generating function

$$\mathcal{B}_n = n![z^n]e^{e^z-1},$$

and they satisfy the relation

$$\mathcal{B}_n = \sum_{k=0}^{n}\left\{ {n \atop k} \right\}.$$

This suggests to define the Bell numbers of any complex order x, $\Re x > 0$, as

$$\mathcal{B}_x = x!\,\frac{1}{2i\pi}\int_{\mathcal{H}} e^{e^z-1}\frac{dz}{z^{x+1}}. \tag{4}$$

When generalized in this way, the Bell numbers satisfy

$$\mathcal{B}_x = \sum_{k=0}^{\infty}\left\{ {x \atop k} \right\},$$

as results from expanding the integrand of (4),

$$e^{e^z-1} = \sum_{k=0}^{\infty}\frac{(e^z-1)^k}{k!}$$

Dobinski's formula. This classical formula [2] also generalizes. If we expand

$$e^{e^z-1} = e^{-1}\sum_{k=0}^{\infty}\frac{e^{kz}}{k!}$$

we get

$$\mathcal{B}_x = \frac{x!}{e}\sum_{k=0}^{\infty}\frac{1}{k!}\,\frac{1}{2i\pi}\int_{\mathcal{H}} e^{kz}\,\frac{dz}{z^{x+1}}.$$

The integral can now be evaluated by use of Hankel's formula for the gamma function (substitute $kz = t$), and

$$\frac{1}{2i\pi}\int_{\mathcal{H}} e^{kz}\,\frac{dz}{z^{x+1}} = \frac{k^x}{x!};$$

hence, the generalized Dobinski formula,

$$\mathcal{B}_x = e^{-1}\sum_{k=0}^{\infty}\frac{k^x}{k!}. \tag{5}$$

Bernoulli numbers. Given that their exponential generating function is $z/(e^z-1)$, it is natural to expect Bernoulli numbers to be related to Stirling numbers of type $\left\{{x \atop -1}\right\}$. Consider first the case of an integer index n. Then

$$\left\{{n \atop y-1}\right\} = \frac{n!}{(y-1)!}\frac{1}{2i\pi}\int_{\mathcal{H}} (e^z-1)^{y-1}\,\frac{dz}{z^{n+1}}.$$

As $y \to 0$, we have by Cauchy's formula and the fact that $(y-1)! \sim \frac{1}{y}$,

$$\left\{{n \atop y-1}\right\} \sim y\cdot n![z^{n+1}]\frac{z}{e^z-1} = y\,\frac{B_{n+1}}{n+1}.$$

This relation gives

$$\frac{d}{dy}\left\{{n \atop y}\right\}\Big|_{y=-1} = \frac{B_{n+1}}{n+1} = \zeta(-n),$$

and more generally, thanks to Hankel's representation of the ζ function (see e.g. [6]),

$$\frac{d}{dy}\left\{{x \atop y}\right\}\Big|_{y=-1} = \zeta(-x).$$

In other words Bernoulli numbers of complex index that are naturally defined by $B_{x+1} := (x+1)\zeta(-x)$ are also obtained by a simple limiting process applied to generalized Stirling numbers of index -1:

$$\frac{B_{x+1}}{x+1} = \lim_{\epsilon\to 0}\frac{1}{\epsilon}\left\{{x \atop -1+\epsilon}\right\} = \frac{d}{dy}\left\{{x \atop y}\right\}\Big|_{y=-1}.$$

Stirling cycle numbers. Formula (3) and the change of variables $z = \log(1+w)$ provide a logarithmic form of Stirling subset numbers,

$$\left\{ {x \atop y} \right\} = \frac{(x-1)!}{(y-1)!} \frac{1}{2i\pi} \int_{\mathcal{H}^\star} \left(\log \frac{1}{1+w} \right)^{-x-1} w^{y-1}\, dw, \tag{6}$$

where $\mathcal{H}^\star$ is a "raindrop contour" that is the image of $\mathcal{H}$ by $z \mapsto w = e^z - 1$. (Thus, $\mathcal{H}^\star$ starts at -1 in the lower half plane, surrounds 0 anticlockwise and returns to -1 in the upper half plane.)

On the other hand, Cauchy's coefficient formula applied to the exponential generating function of Stirling cycle numbers gives

$$\left[{n \atop k} \right] = (-1)^{n-k} \frac{n!}{k!} \frac{1}{2i\pi} \int_{\mathcal{H}^\star} \big(\log(1+w) \big)^k \frac{dw}{w^{n+1}}. \tag{7}$$

A direct consequence of (6) and (7) is that the Stirling cycle numbers of integral arguments arise as limiting cases of generalized Stirling subset numbers as defined in (2), (3),

$$\left[{n \atop k} \right] = \lim_{\epsilon \to 0} \left\{ {-k+\epsilon \atop -n+\epsilon} \right\}.$$

One encounters once more an instance of the duality relation $\left[{x \atop y} \right] = \left\{ {-y \atop -x} \right\}$ that, together with (6), confirms that Stirling numbers eventually reduce to a single family (see [3, 4]).

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