A GENERATING FUNCTIONS PROOF OF A CURIOUS IDENTITY

Alois Panholzer
Department of Algebra and Computer Mathematics, Technical University Vienna, Wiedner Hauptstrasse 8–10, A–1040 Vienna, Austria
Alois.Panholzer@tuwien.ac.at

Helmut Prodinger[1]
School of Mathematics, University of the Witwatersrand, P. O. Wits, 2050 Johannesburg, South Africa
helmut@staff.ms.wits.ac.za
http://www.wits.ac.za/helmut/index.htm

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Abstract

We give an alternative proof of an identity that appeared recently in *Integers*. It is shorter than the original one and uses generating functions.

In the paper [2] that appeared a few days ago the identity

$$S_m := (x+m+1)\sum_{i=0}^{m}(-1)^i\binom{x+y+i}{m-i}\binom{y+2i}{i} - \sum_{i=0}^{m}\binom{x+i}{m-i}(-4)^i = (x-m)\binom{x}{m}$$

(the main result) was proved using double recursions. Here we show this result using generating functions. This proof is perhaps more pleasant and shorter than the original one. The relevant functions can be found in [1, p. 201ff.]. We need

$$\big(\mathcal{B}_t(z)\big)^r = \sum_{k\ge 0}\binom{tk+r}{k}\frac{r}{tk+r}z^k,$$
$$\frac{\big(\mathcal{B}_t(z)\big)^r}{1-t+t\big(\mathcal{B}_t(z)\big)^{-1}} = \sum_{k\ge 0}\binom{tk+r}{k}z^k,$$

and in particular

$$\mathcal{B}_{-1}(z) = \frac{1+\sqrt{1+4z}}{2} \qquad \text{and} \qquad \mathcal{B}_2(z) = \frac{1-\sqrt{1-4z}}{2z}.$$

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Now

$$\sum_{i=0}^{m}(-1)^i\binom{x+y+i}{m-i}\binom{y+2i}{i} = [z^m]\sum_{i\geq 0}\binom{x+y+m-i}{i}z^i\cdot\sum_{i\geq 0}\binom{y+2i}{i}(-z)^i$$
$$= [z^m]\frac{\big(\mathcal{B}_{-1}(z)\big)^{x+y+m}}{2-\big(\mathcal{B}_{-1}(z)\big)^{-1}}\cdot\frac{\big(\mathcal{B}_2(-z)\big)^y}{-1+2\big(\mathcal{B}_2(-z)\big)^{-1}}$$
$$= [z^m]\frac{1}{1+4z}\big(\mathcal{B}_{-1}(z)\big)^{x+m+1}.$$

Further,

$$\sum_{i=0}^{m}\binom{x+i}{m-i}(-4)^i = (-4)^m\sum_{i=0}^{m}\binom{x+m-i}{i}(-\tfrac{1}{4})^i$$
$$= (-4)^m[z^m]\frac{1}{1-z}\sum_{i\geq 0}\binom{x+m-i}{i}(-\tfrac{z}{4})^i$$
$$= [z^m]\frac{1}{1+4z}\sum_{i\geq 0}\binom{x+m-i}{i}z^i$$
$$= [z^m]\frac{1}{(1+4z)^{3/2}}\big(\mathcal{B}_{-1}(z)\big)^{x+m+1}.$$

We use the substitution $z = u/(1-u)^2$ and get:

$$S_m = [z^m]\left(\frac{x+m+1}{1+4z}-\frac{1}{(1+4z)^{3/2}}\right)\big(\mathcal{B}_{-1}(z)\big)^{x+m+1}$$
$$= \frac{1}{2\pi i}\oint\frac{1}{z^{m+1}}\left(\frac{x+m+1}{1+4z}-\frac{1}{(1+4z)^{3/2}}\right)\big(\mathcal{B}_{-1}(z)\big)^{x+m+1}\,dz$$
$$= \frac{1}{2\pi i}\oint\frac{(1-u)^{2m+2}}{u^{m+1}}\frac{(1+u)(2+x+m)-2}{(1+u)^3(1-u)^{x+m-1}}\frac{1+u}{(1-u)^3}\,du$$
$$= [u^m]\frac{(1-u)^{m-x}((1+u)(2+x+m)-2)}{(1+u)^2}$$
$$= (2+x+m)[u^m]\frac{(1-u)^{m-x}}{1+u}-2[u^m]\frac{(1-u)^{m-x}}{(1+u)^2}.$$

Expanding the terms leads now to

$$[u^m]\frac{(1-u)^{m-x}}{1+u} = \sum_{k=0}^{m}\binom{x-k-1}{m-k}(-1)^k$$

and

$$[u^m]\frac{(1-u)^{m-x}}{(1+u)^2} = \sum_{k=0}^{m}(k+1)\binom{x-k-1}{m-k}(-1)^k.$$

This gives

$$\begin{aligned} S_m &= \sum_{k=0}^{m} \left[(x-k)+(m-k)\right]\binom{x-k-1}{m-k}(-1)^k \\ &= (x-m)\sum_{k=0}^{m}\left[\binom{x-k}{m-k}+\binom{x-k-1}{m-k-1}\right](-1)^k \\ &= (x-m)\binom{x}{m}, \end{aligned}$$

as desired (the last sum is telescoping).

References

[1] R. L. Graham, D. E. Knuth, and O. Patashnik. *Concrete Mathematics (Second Edition)*. Addison Wesley, 1994.

[2] Z.-W. Sun. A curious identity involving binomial coefficients. *Integers*, pages A4, 8 pp. (electronic), 2002.