COMBINATORICS OF GEOMETRICALLY DISTRIBUTED RANDOM VARIABLES: NEW q–TANGENT AND q–SECANT NUMBERS

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Abstract. Up–down permutations are counted by tangent resp. secant numbers. Considering words instead, where the letters are produced by independent geometric distributions, there are several ways of introducing this concept; in the limit they all coincide with the classical version. In this way, we get some new q–tangent and q–secant functions. Some of them also have nice continued fraction expansions; in one particular case, we could not find a proof for it. Divisibility results à la Andrews/Foata/Gessel are also discussed.

1. Introduction

Permutations $\pi = \pi_1\pi_2\cdots\pi_n$ are called up–down permutations if $\pi_1 < \pi_2 > \pi_3 < \pi_4 > \pi_5 \cdots$. For odd n, their number is given by $n![z^n]\tan z$, and for even n by $n![z^n]\sec z$. One finds that in many textbooks, e. g. [8, 9].

Instead of speaking about exponential generating functions, we prefer to think of the coefficients of $\tan z$ and $\sec z$ as probabilities.

This paper introduces several q–analogues of the functions $\tan z$ and $\sec z$. There are classical ones, see e. g. [12, 8], but the other ones seem to be new.

If we consider words $a_1a_2\dots a_n$ with letters in $\{1, 2, \dots\}$ with probabilities (weights) $p, pq, pq^2, \dots$, where $p+q = 1$ (independent geometric probilities), then there are several ways to introduce this concept. We can use $<$ or $\le$ for "up," $>$ or $\ge$ for "down," which gives 4 possibilities. Also, it *makes* a difference to consider "up–down" versus "down–up." That gives in principle 8 versions for q–tangent and q–secant numbers. However, reading the word from right to left, the instance "$\le > \le > \dots$" coincides with the "$< \ge < \ge < \dots$," and similarly for "$\ge < \ge < \dots$" and "$> \le > \le \dots$," which gives us 6 q–tangent numbers (probabilities, to be more precise). In the instance of even length (secant numbers), there are more symmetries, and we have only 4 q–secant numbers.

By general principles, the limit $q \to 1$ reduces all the instances to the classical quantities.

Originally, the idea to use geometric probabilities for words came from some combinatorial problems in computer science, see. e. g. [14].

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This papers aims by no means to offer a complete theory of these new numbers. Rather, some results are given, others conjectured, and it is hoped that the papers stimulates other people to do more related research.

A few definitions from q–analysis are needed; consult the books [1] and [2]:

$$\begin{gathered} [n]_q := \frac{1-q^n}{1-q}, \qquad [n]_q! := [1]_q[2]_q \dots [n]_q, \\ (x;q)_n := (1-x)(1-xq)(1-xq^2)\dots(1-xq^{n-1}). \end{gathered} \tag{1.1}$$

2. Recursions

The classical book [8] offers a general framework to deal with words and pattern ("pattern algebra"). However, we have decided to use a different approach that works particularly well in the present context, and is completely elementary. It is called the "adding–a–new–slice" technique. In doing so, recursions are obtained; by iterating them, the generating functions of interest come out almost effortlessly.

We introduce the functions

$$T_n^{\le\ge}(u) \tag{2.1}$$

where the coefficient of u^i in it is the probability that a word of length n satisfies the $\le\ge\le\ge\dots$ condition and ends with the letter i. Also, we define

$$\tau_n^{\le\ge} = T_n^{\le\ge}(1), \tag{2.2}$$

which drops the technical condition about the last letter.

Furthermore, we introduce the generating functions

$$F^{\le\ge}(z,u) = \sum_{n\ge0} T_n^{\le\ge}(u) z^n \qquad \text{and} \qquad f^{\le\ge}(z) = F^{\le\ge}(z,1). \tag{2.3}$$

Quantities like $F^{\le>}(z,u)$ etc. are defined in an obvious way.

For the instance of secant numbers, similar quantities will be defined, but the letters S, σ, G, g are used instead of T, τ, F, f.

Obviously there are only nonzero contributions for odd n in the tangent case and for even n in the secant case.

The reason to operate with a variable u that controls the last letter is the technique of "adding a new slice," that was applied with success in [6] and, more recently, in [13].

Theorem 1. *The functions $T_{2n+1}^{\nabla\triangle}(u)$ satisfy the following recurrences:*

- $$\begin{aligned} T^{\geq\leq}_{2n+1}(u) &= \frac{p^2u}{(1-qu)(1-q^2u)}T^{\geq\leq}_{2n-1}(1) - \frac{p^2u}{(1-qu)(1-q^2u)}T^{\geq\leq}_{2n-1}(q^2u) \\ T^{\geq\leq}_{1}(u) &= \frac{pu}{1-qu}, \end{aligned} \tag{2.4}$$

- $$\begin{aligned} T^{\geq<}_{2n+1}(u) &= \frac{p^2qu^2}{(1-qu)(1-q^2u)}T^{\geq<}_{2n-1}(1) - \frac{p^2qu^2}{(1-qu)(1-q^2u)}T^{\geq<}_{2n-1}(q^2u) \\ T^{\geq<}_{1}(u) &= \frac{pu}{1-qu}. \end{aligned} \tag{2.5}$$

- $$\begin{aligned} T^{><}_{2n+1}(u) &= \frac{p^2qu^2}{(1-qu)(1-q^2u)}T^{><}_{2n-1}(1) - \frac{p^2u}{q(1-qu)(1-q^2u)}T^{><}_{2n-1}(q^2u) \\ T^{><}_{1}(u) &= \frac{pu}{1-qu}, \end{aligned} \tag{2.6}$$

- $$\begin{aligned} T^{\leq\geq}_{2n+1}(u) &= \frac{pu}{q(1-qu)}T^{\leq\geq}_{2n-1}(q) - \frac{p^2u}{q(1-qu)(1-q^2u)}T^{\leq\geq}_{2n-1}(q^2u) \\ T^{\leq\geq}_{1}(u) &= \frac{pu}{1-qu}, \end{aligned} \tag{2.7}$$

- $$\begin{aligned} T^{\leq>}_{2n+1}(u) &= \frac{pu}{q(1-qu)}T^{\leq>}_{2n-1}(q) - \frac{p^2}{q^2(1-qu)(1-q^2u)}T^{\leq>}_{2n-1}(q^2u) \\ T^{\leq>}_{1}(u) &= \frac{pu}{1-qu}, \end{aligned} \tag{2.8}$$

- $$\begin{aligned} T^{<>}_{2n+1}(u) &= \frac{pu}{1-qu}T^{<>}_{2n-1}(q) - \frac{p^2u}{(1-qu)(1-q^2u)}T^{<>}_{2n-1}(q^2u) \\ T^{<>}_{1}(u) &= \frac{pu}{1-qu}. \end{aligned} \tag{2.9}$$

Proof. Since the technique is the same for all the instances, it is enough to discuss e. g. the "$\geq\leq$" case. Adding a new slice means adding a pair (k,j) with $1 \leq k \leq i$, $j \geq k$, replacing u^i by 1 and providing the factor u^j. But

$$\sum_{k=1}^{i} pq^{k-1}\sum_{j\geq k} pq^{j-1}u^j = \frac{p^2u}{(1-qu)(1-q^2u)} - \frac{p^2u}{(1-qu)(1-q^2u)}\big(q^2u\big)^i,$$

which explains the recursion. The starting value is just

$$\sum_{j\geq 1} pq^{j-1}u^j = \frac{pu}{1-qu}.$$

$\square$

Theorem 2. *The numbers $\tau^{\nabla\triangle}_{2n+1}$ have the generating functions $f^{\nabla\triangle}(z) = \tan_q(z) = \sin_q(z)/\cos_q(z)$:*

- $$f^{\geq\leq}(z)=\sum_{n\geq0}\frac{(-1)^n z^{2n+1}}{[2n+1]_q!}q^{n(n+1)}\Bigg/\sum_{n\geq0}\frac{(-1)^n z^{2n}}{[2n]_q!}q^{n(n-1)} \tag{2.10}$$
- $$f^{\geq<}(z)=\sum_{n\geq0}\frac{(-1)^n z^{2n+1}}{[2n+1]_q!}\Bigg/\sum_{n\geq0}\frac{(-1)^n z^{2n}}{[2n]_q!} \tag{2.11}$$
- $$f^{><}(z)=\sum_{n\geq0}\frac{(-1)^n z^{2n+1}}{[2n+1]_q!}q^{n^2}\Bigg/\sum_{n\geq0}\frac{(-1)^n z^{2n}}{[2n]_q!}q^{n^2} \tag{2.12}$$
- $$f^{\leq\geq}(z)=\sum_{n\geq0}\frac{(-1)^n z^{2n+1}}{[2n+1]_q!}q^{n^2}\Bigg/\sum_{n\geq0}\frac{(-1)^n z^{2n}}{[2n]_q!}q^{n(n-1)} \tag{2.13}$$
- $$f^{\leq>}(z)=\sum_{n\geq0}\frac{(-1)^n z^{2n+1}}{[2n+1]_q!}\Bigg/\sum_{n\geq0}\frac{(-1)^n z^{2n}}{[2n]_q!} \tag{2.14}$$
- $$f^{<>}(z)=\sum_{n\geq0}\frac{(-1)^n z^{2n+1}}{[2n+1]_q!}q^{n(n+1)}\Bigg/\sum_{n\geq0}\frac{(-1)^n z^{2n}}{[2n]_q!}q^{n^2} \tag{2.15}$$

Proof. The proofs of the first 3 relations are very similar, and we only sketch the first instance. Summing up we find

$$F^{\geq\leq}(z,u)=\frac{puz}{1-qu}+\frac{p^2uz^2}{(1-qu)(1-q^2u)}F^{\geq\leq}(z,1)-\frac{p^2uz^2}{(1-qu)(1-q^2u)}F^{\geq\leq}(z,q^2u)$$

Iterating that we find for $f(z)=f^{\geq\leq}(z)$:

$$\begin{aligned}f(z)&=\frac{pz}{1-q}+\frac{p^2z^2}{(1-q)(1-q^2)}f(z)-\frac{p^2q^2z^3}{(1-q)(1-q^2)(1-q^3)}\\&\quad-\frac{p^4q^2z^4}{(1-q)(1-q^2)(1-q^3)(1-q^4)}f(z)+\dots\end{aligned}$$

from which the announced formula follows by solving for $f(z)$.

The 3 others are trickier, because of a term $T_{2n-1}^{\nabla\Delta}(q)$. Again, one case is discussed. Observe that

$$T_{2n-1}^{\leq\geq}(q)=q^{-1}S_{2n}^{\leq\geq}(1),$$

because one more "up" step should replace u^i by $\sum_{k\ge i} pq^{k-1} = q^{i-1}$. Now the generating function $g^{\le\ge}(z)$ of the quantities $S_{2n}^{\le\ge}(1)$ (upcoming) is obtained independently, whence we get

$$F^{\le\ge}(z,u) = \frac{puz}{q^2(1-qz)} g^{\le\ge}(z) - \frac{p^2uz^2}{q(1-qu)(1-q^2u)} F^{\le\ge}(z,q^2u).$$

Now iteration as usual derives the desired result. □

Theorem 3. *The functions $S_{2n}^{\nabla\triangle}(u)$ satisfy the following recurrences:*

-
$$\begin{aligned} S_{2n+2}^{\le\ge}(u) &= \frac{p^2u}{(1-qu)(1-q^2u)} S_{2n}^{\le\ge}(1) - \frac{p^2u}{(1-qu)(1-q^2u)} S_{2n}^{\le\ge}(q^2u) \\ S_2^{\le\ge}(u) &= \frac{p^2u}{(1-qu)(1-q^2u)}, \end{aligned} \tag{2.16}$$

-
$$\begin{aligned} S_{2n+2}^{\le>}(u) &= \frac{p^2u}{(1-qu)(1-q^2u)} S_{2n}^{\le>}(1) - \frac{p^2}{q^2(1-qu)(1-q^2u)} S_{2n}^{\le>}(q^2u) \\ S_2^{\le>}(u) &= \frac{p^2u}{(1-qu)(1-q^2u)}, \end{aligned} \tag{2.17}$$

-
$$\begin{aligned} S_{2n+2}^{<\ge}(u) &= \frac{p^2qu^2}{(1-qu)(1-q^2u)} S_{2n}^{<\ge}(1) - \frac{p^2qu^2}{(1-qu)(1-q^2u)} S_{2n}^{<\ge}(q^2u) \\ S_2^{<\ge}(u) &= \frac{p^2qu^2}{(1-qu)(1-q^2u)}, \end{aligned} \tag{2.18}$$

-
$$\begin{aligned} S_{2n+2}^{<>}(u) &= \frac{p^2qu^2}{(1-qu)(1-q^2u)} S_{2n}^{<>}(1) - \frac{p^2u}{q(1-qu)(1-q^2u)} S_{2n}^{<>}(q^2u) \\ S_2^{<>}(u) &= \frac{p^2qu^2}{(1-qu)(1-q^2u)}. \end{aligned} \tag{2.19}$$

Proof. The proof works as in the easy cases of the tangent recursions and is omitted. For the starting value, the first pair of numbers must be considered. □

Theorem 4. *The numbers $\sigma_{2n}^{\nabla\triangle}$ have the generating functions $g^{\nabla\triangle}(z) = 1/\cos_q(z)$:*

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$$g^{\le\ge}(z) = 1 \Big/ \sum_{n\ge0} \frac{(-1)^n z^{2n}}{[2n]_q!} q^{n(n-1)} \tag{2.20}$$

-
$$g^{\le>}(z) = 1 \Big/ \sum_{n\ge0} \frac{(-1)^n z^{2n}}{[2n]_q!} \tag{2.21}$$

- $$g^{<\geq}(z) = 1\Big/\sum_{n\ge0}\frac{(-1)^n z^{2n}}{[2n]_q!}q^{n(2n-1)} \tag{2.22}$$
- $$g^{<>}(z) = 1\Big/\sum_{n\ge0}\frac{(-1)^n z^{2n}}{[2n]_q!}q^{n^2} \tag{2.23}$$

Proof. The proofs are quite similar as before; however, iteration must be done for the function $G^{\nabla\Delta}(z,u)-1$, and 1 must be added at the end. □

3. Jackson's q–sine and q–cosine functions

In this section we are considering a general class of q–sine and q–cosine functions and sort out those that satisfy a natural condition. This condition is even more natural, as all the previously encountered q–sine and q–cosine functions satisfy them.

Jackson in [12] has introduced the functions

$$\begin{aligned}\sin_q(z) &= \sum_{n\ge0}\frac{(-1)^n z^{2n+1}}{[2n+1]_q!},\\ \cos_q(z) &= \sum_{n\ge0}\frac{(-1)^n z^{2n}}{[2n]_q!}.\end{aligned} \tag{3.1}$$

and proved the relation

$$\sin_q(z)\sin_{1/q}(z)+\cos_q(z)\cos_{1/q}(z)=1 \tag{3.2}$$

Since we have here several q–sine and q–cosine functions, we call them a q–sine–cosine pair, if relation (3.2) holds.

Theorem 5. *For the functions*

$$\sin_q(z) := \sum_{n\ge0}\frac{(-1)^n z^{2n+1}}{[2n+1]_q!}q^{An^2+Bn}$$

and

$$\cos_q(z) := \sum_{n\ge0}\frac{(-1)^n z^{2n}}{[2n]_q!}q^{Cn^2+Dn}$$

exactly the 12 pairs in Table 1 are q–sine–cosine pairs:

Proof. The desired relation gives more and more restrictions for the (complex) numbers A,B,C,D when we look at the coefficients of z^{2n} for $n=0,1,2\cdots$. By a tedious search that will not be reported here we find these 12 possibilities, and all others can be excluded. The proof that this indeed works is very similar for all of them, so we give just one, namely the instance $(1,0,1,0)$.

Note the following expansions:

A	B	C	D
0	0	0	0
2	1	0	0
0	0	2	-1
2	1	2	-1
0	1	0	1
2	0	0	1
1	0	1	0
1	1	1	0
0	1	2	-2
2	0	2	-2
1	0	1	-1
1	1	1	-1

TABLE 1.

$$\sin_{1/q} z = \sum_{n\ge 0} \frac{(-1)^n z^{2n+1}}{[2n+1]_q!} q^{\binom{2n+1}{2}-n^2},$$
$$\cos_{1/q} z = \sum_{n\ge 0} \frac{(-1)^n z^{2n}}{[2n]_q!} q^{\binom{2n}{2}-n^2}.$$

So we must prove that for $n \ge 1$

$$\sum_{k=0}^{n} \begin{bmatrix} 2n \\ 2k \end{bmatrix}_q q^{\binom{2k}{2}-k^2+(n-k)^2} = \sum_{k=0}^{n-1} \begin{bmatrix} 2n \\ 2k+1 \end{bmatrix}_q q^{\binom{2k+1}{2}-k^2+(n-k-1)^2}$$

or, reversing the order of summation in the second sum,

$$\sum_{k=0}^{n} \begin{bmatrix} 2n \\ 2k \end{bmatrix}_q q^{2k^2-2k-2nk} = \sum_{k=0}^{n-1} \begin{bmatrix} 2n \\ 2k+1 \end{bmatrix}_q q^{2k^2+k-n-2nk}.$$

We rewrite this again as

$$\sum_{k \text{ even}} \begin{bmatrix} 2n \\ k \end{bmatrix}_q q^{-nk+\binom{k}{2}} = \sum_{k \text{ odd}} \begin{bmatrix} 2n \\ k \end{bmatrix}_q q^{-nk+\binom{k}{2}}.$$

Therefore we have to prove that

$$\sum_{k=0}^{2n} \begin{bmatrix} 2n \\ k \end{bmatrix}_q (-1)^k q^{-nk+\binom{k}{2}} = 0.$$

We use the formula (10.0.9) in [2], see also [1],

$$\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q z^k q^{\binom{k}{2}} = \prod_{j=0}^{n-1} (1+q^j z).$$

The desired result now follows by replacing n by $2n$ and plugging in $z = -q^{-n}$. $\square$

Theorem 6. *The 6 $\tan_q(z)$ functions in Theorem 2 all involve q–sine–cosine pairs.*

Remark. Replacing q by $1/q$ in the q–sine–cosine pairs and rewriting everything again in the q–notation means replacing the vector (A, B, C, D) of exponents by$(2-A, 1-B, 2-C, -1-D)$. This will be called "duality."

A	B	C	D	A'	B'	C'	D'
0	0	0	0	2	1	2	-1
2	1	0	0	0	0	2	-1
0	1	0	1	2	0	2	-2
2	0	0	1	0	1	2	-2
1	0	1	0	1	1	1	-1
1	1	1	0	1	0	1	-1

Table 2.

This reduces the 12 pairs to 6 pairs.

4. continued fractions

Experimenting with Maple, it was found that some of the 12 tangent functions have nice continued fraction expansions. Some could be proved, others not (yet).

A lot of work still needs do be done concerning a *combinatorial theory* of these continued fractions, perhaps in the style of [5].

Theorem 7. *For $(A, B, C, D) = (0, 0, 0, 0)$ and $(A, B, C, D) = (2, 1, 2, -1)$ we have*

$$\tan_q(z) = \cfrac{z}{[1]_q q^{-0} - \cfrac{z^2}{[3]_q q^{-1} - \cfrac{z^2}{[5]_q q^{-2} - \cfrac{z^2}{[7]_q q^{-3} - \cfrac{z^2}{\ddots}}}}} \tag{4.1}$$

The two tangent functions coincide, which is classical, since Jackson [12] *has shown that for his functions*

$$\sin_q z \cos_{1/q} z - \sin_{1/q} z \cos_q z = 0$$

holds.

Proof. For the proof by induction we must do the following: Set $a_n = [2n-1]_q q^{1-n}$ and

$$\begin{aligned} p_n(z) &= a_n p_{n-1}(z) - z^2 p_{n-2}(z), \qquad & p_0(z) = 0,\ p_1(z) = z, \\ q_n(z) &= a_n q_{n-1}(z) - z^2 q_{n-2}(z), \qquad & q_0(z) = 1,\ q_1(z) = a_1. \end{aligned}$$

We must show that

$$[z^k]\Big(p_n(z)\cos_q z - q_n(z)\sin_q z\Big) = 0 \quad \text{for } k \le 2n.$$

Now look at

$$[z^k]\Big((a_n p_{n-1}(z) - z^2 p_{n-2}(z))\cos_q z - (a_n q_{n-1}(z) - z^2 q_{n-2}(z))\sin_q z\Big).$$

By the induction hypothesis we only have to show that

$$[z^{2n-1}]\Big(p_n(z)\cos_q z - q_n(z)\sin_q z\Big) = 0.$$

However, we can easily show by induction that

$$p_n(z) = \sum_k z^{2k+1}(-1)^k \frac{[2n-2k-1]_q!\, q^{k(2k+1)-\binom{n}{2}}}{[n-2k-1]_q![2k+1]_q! \prod_{i=1}^{n-1-2k}(1+q^i)}$$

and

$$q_n(z) = \sum_k z^{2k}(-1)^k \frac{[2n-2k]_q!\, q^{k(2k-1)-\binom{n}{2}}}{[n-2k]_q![2k]_q! \prod_{i=1}^{n-2k}(1+q^i)}$$

holds (the hard part is to find these formulæ). We have to prove that

$$\sum_{k\ge 0}[z^{2k+1}]p_n(z)[z^{2n-2k-2}]\cos_q z = \sum_{k\ge 0}[z^{2k}]q_n(z)[z^{2n-2k-1}]\sin_q z$$

or

$$\sum_{k=0}^{\lfloor\frac{n-1}{2}\rfloor} \frac{[2n-2k-1]_q!\, q^{k(2k+1)}}{[n-2k-1]_q![2k+1]_q! \prod_{i=1}^{n-1-2k}(1+q^i)[2n-2k-2]_q!}$$
$$= \sum_{k=0}^{\lfloor\frac{n}{2}\rfloor} \frac{[2n-2k]_q!\, q^{k(2k-1)}}{[n-2k]_q![2k]_q! \prod_{i=1}^{n-2k}(1+q^i)[2n-2k-1]_q!}.$$

Thus we must prove

$$\sum_{k=0}^{\lfloor\frac{n-1}{2}\rfloor}\frac{(1-q^{2n-2k-1})q^{k(2k+1)}}{[n-2k-1]_q![2k+1]_q!\prod_{i=1}^{n-1-2k}(1+q^i)}=\sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}\frac{(1-q^{2n-2k})q^{k(2k-1)}}{[n-2k]_q![2k]_q!\prod_{i=1}^{n-2k}(1+q^i)}$$

or

$$\sum_{k=0}^{n}\frac{(1-q^{2n-k})q^{\binom{k}{2}}(-1)^k}{[n-k]_q![k]_q!\prod_{i=1}^{n-k}(1+q^i)}=0,$$

or

$$\sum_{k=0}^{n}\frac{1}{(q;q)_k(q^2;q^2)_{n-k}}(1-q^{2n-k})q^{\binom{k}{2}}(-1)^k=0.$$

Now

$$\begin{aligned}\sum_{k=0}^{n}\frac{1}{(q;q)_k(q^2;q^2)_{n-k}}q^{\binom{k}{2}}(-1)^k&=[z^n]\sum_{k\ge0}\frac{q^{\binom{k}{2}}(-1)^kz^k}{(q;q)_k}\sum_{k\ge0}\frac{z^k}{(q^2;q^2)_k}\\&=[z^n]\prod_{k\ge0}(1-zq^k)\Big/\prod_{k\ge0}(1-zq^{2k})\\&=[z^n]\prod_{k\ge0}(1-zq^{2k+1})\\&=q^n[z^n]\prod_{k\ge0}(1-zq^{2k})\\&=\frac{q^nq^{2\binom{n}{2}}(-1)^n}{(q^2;q^2)_n}=\frac{(-1)^nq^{n^2}}{(q^2;q^2)_n}.\end{aligned}$$

Similarly,

$$\begin{aligned}\sum_{k=0}^{n}\frac{q^{2n-k}}{(q;q)_k(q^2;q^2)_{n-k}}q^{\binom{k}{2}}(-1)^k&=q^{2n}[z^n]\sum_{k\ge0}\frac{q^{\binom{k}{2}}(-1)^k(z/q)^k}{(q;q)_k}\sum_{k\ge0}\frac{z^k}{(q^2;q^2)_k}\\&=q^{2n}[z^n]\prod_{k\ge0}(1-zq^{k-1})\Big/\prod_{k\ge0}(1-zq^{2k})\\&=q^{2n}[z^n]\prod_{k\ge0}(1-zq^{2k-1})\\&=q^{2n}q^{-n}[z^n]\prod_{k\ge0}(1-zq^{2k})\\&=\frac{q^nq^{2\binom{n}{2}}(-1)^n}{(q^2;q^2)_n}=\frac{(-1)^nq^{n^2}}{(q^2;q^2)_n}.\end{aligned}$$

This finishes the proof.

The continued fraction for $(2,1,2,-1)$ follows by replacing q by $1/q$. □

Theorem 8. *For* $(A,B,C,D)=(0,1,0,1)$ *we have*

$$\tan_q(z)=\cfrac{z}{[1]_q q^{-0}-\cfrac{z^2}{[3]_q q^{-2}-\cfrac{z^2}{[5]_q q^{-2}-\cfrac{z^2}{[7]_q q^{-4}-\cfrac{z^2}{\ddots}}}}} \tag{4.2}$$

The negative powers of q *go like* $0,2,2,4,4,6,6,8,8,\dots$.

Proof. The proof follows the same lines; this time the polynomials (continuants) are

$$p_n(z)=\sum_k z^{2k+1}(-1)^k\frac{[2n-2k-1]_q!q^{2k(k+1)-\binom{n}{2}-\lfloor\frac{n}{2}\rfloor}}{[n-2k-1]_q![2k+1]_q!\prod_{i=1}^{n-1-2k}(1+q^i)}$$

and

$$q_n(z)=\sum_k z^{2k}(-1)^k\frac{[2n-2k]_q!q^{2k^2-\binom{n}{2}-\lfloor\frac{n}{2}\rfloor}}{[n-2k]_q![2k]_q!\prod_{i=1}^{n-2k}(1+q^i)}.$$

Hence we have to prove that

$$\sum_{k=0}^{\lfloor\frac{n-1}{2}\rfloor}\frac{(1-q^{2n-2k-1})q^{2k(k+1)-k}}{[n-2k-1]_q![2k+1]_q!\prod_{i=1}^{n-1-2k}(1+q^i)}=\sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}\frac{(1-q^{2n-2k})q^{2k^2-k}}{[n-2k]_q![2k]_q!\prod_{i=1}^{n-2k}(1+q^i)};$$

from here on we can use the previous proof.

An alternative proof is by noting that

$$\tan_q^{(0,1,0,1)}(z)=\frac{1}{\sqrt{q}}\tan_q^{(0,0,0,0)}(z\sqrt{q})$$

and using the previous result. ☐

Theorem 9. *For* $(A,B,C,D)=(2,0,2,-2)$ *we have*

$$\tan_q(z)=\cfrac{z}{[1]_q q^{-0}-\cfrac{z^2}{[3]_q q^{-0}-\cfrac{z^2}{[5]_q q^{-2}-\cfrac{z^2}{[7]_q q^{-2}-\cfrac{z^2}{\ddots}}}}} \tag{4.3}$$

The negative powers of q *go like* $0,0,2,2,4,4,6,6,8,8,\dots$.

Proof. This follows from the previous theorem by replacing q by $1/q$. ☐

Conjecture 10. *For* $(A,B,C,D)=(1,0,1,0)$ *we have*

$$\tan_q(z)=\cfrac{z}{[1]_q q^{0}-\cfrac{z^2}{[3]_q q^{-2}-\cfrac{z^2}{[5]_q q^{1}-\cfrac{z^2}{[7]_q q^{-9}-\cfrac{z^2}{\ddots}}}}} \tag{4.4}$$

The positive powers of q *go like* $0,1,6,15,\ldots$ $(k(2k-1))$.

The negative powers of q *go like* $2,9,20,35\ldots$ $((k+1)(2k-1))$.

Comment. It might be useful to rewrite the continued fraction as

$$\cfrac{z}{1-\cfrac{z^2b_1}{1-\cfrac{z^2b_2}{1-\cfrac{z^2b_3}{1-\cfrac{z^2b_4}{\ddots}}}}} \tag{4.5}$$

with

$$\begin{aligned} b_k&=\frac{1}{[k]_q[k+1]_q}q^{-k+(-1)^k(2k-1)}\\ &=\frac{1}{[k]_q[k+1]_q}\Big[\frac12 q^{-3k+1}\big(1+q^{4k-2}\big)-\frac{(-1)^k}{2}q^{-3k+1}\big(1-q^{4k-2}\big)\Big]. \end{aligned}$$

The recursions for the continuants are now

$$\begin{aligned} p_n(z)&=p_{n-1}(z)-b_{n-1}z^2p_{n-2}(z), &\qquad p_0(z)=0,\ p_1(z)=z,\\ q_n(z)&=q_{n-1}(z)-b_{n-1}z^2q_{n-2}(z), &\qquad q_0(z)=1,\ q_1(z)=1. \end{aligned}$$

Unfortunately, even with this form, I am currently unable to guess the coefficients of these polynomials, whence I must leave this expansion as an open problem.

Conjecture 11. *For* $(A,B,C,D)=(1,1,1,-1)$ *we have*

$$\tan_q(z)=\cfrac{z}{[1]_q q^{-0}-\cfrac{z^2}{[3]_q q^{0}-\cfrac{z^2}{[5]_q q^{-5}-\cfrac{z^2}{[7]_q q^{3}-\cfrac{z^2}{\ddots}}}}} \tag{4.6}$$

The positive powers of q go like $0, 3, 10, 21 \dots$ *($(k-1)(2k-1)$).*

The negative powers of q go like $0, 5, 14, 27 \dots$ *($(k-1)(2k+1)$).*

Comment. This would be a corollary of the previous expansion.

Remark. Normally, as e. g. in [5] and [10], the continued fraction expansions of the *ordinary generating function* of the tangent and secant numbers are considered, whereas we stick here to the *exponential (or probability) generating functions.*

5. Divisibility

Theorem 12. *The coefficient*

$$[2n+1]_q [z^{2n+1}] \tan_q(z)$$

is divisible by

$$(1+q)(1+q^2)\dots(1+q^n)$$

for the vectors of exponents $(0,0,0,0)$*,* $(2,1,2,-1)$*,* $(0,1,0,1)$*,* $(2,0,2,-2)$*,* $(1,0,1,0)$*,* $(1,1,1,-1)$*.*

Proof. The proof of [4] covers the first 4 instances, since we note that

$$\tan_q^{(0,1,0,1)}(z) = \frac{1}{\sqrt{q}} \tan_q^{(0,0,0,0)}(z\sqrt{q}).$$

The only open case is thus $(1,0,1,0)$, as the remaining one would follow from duality. Thus, let us now consider

$$\sin_q z = \sum_{n\ge 0} \frac{(-1)^n z^{2n+1}}{[2n+1]_q!} q^{n^2},$$
$$\cos_q z = \sum_{n\ge 0} \frac{(-1)^n z^{2n}}{[2n]_q!} q^{n^2},$$

and $\tan_q z = \sin_q z / \cos_q z$.

We need the following computation that is akin to the one in Theorem 5.

$$
\begin{aligned}
[z^{2n+1}]\sin_{1/q} z \cos_q z &= \sum_{k=0}^{n} \frac{(-1)^k q^{k(k+1)}}{[2k+1]_q!} \frac{(-1)^{n-k} q^{(n-k)^2}}{[2n-2k]_q!}\\
&= \frac{q^{n^2}(-1)^n}{[2n+1]_q!} \sum_{k=0}^{n} \begin{bmatrix} 2n+1 \\ 2k+1 \end{bmatrix}_q q^{2k^2+k-2nk}\\
&= \frac{q^{n(n+1)}(-1)^n}{[2n+1]_q!} \sum_{k \text{ odd}} \begin{bmatrix} 2n+1 \\ k \end{bmatrix}_q q^{\binom{k}{2}-nk}\\
&= \frac{q^{n(n+1)}(-1)^n}{[2n+1]_q!} \frac{1}{2} \sum_{k=0}^{n} \begin{bmatrix} 2n+1 \\ k \end{bmatrix}_q q^{\binom{k}{2}-nk}\\
&= \frac{q^{n(n+1)}(-1)^n}{[2n+1]_q!} \frac{1}{2} \prod_{j=0}^{2n} (1+q^j z)\Big|_{z=q^{-n}}\\
&= \frac{q^{\binom{n+1}{2}}(-1)^n}{[2n+1]_q!} \prod_{i=1}^{n} (1+q^i)^2.
\end{aligned}
$$

Although we do not need it, we also mention the dual formula

$$
[z^{2n+1}]\sin_q z \cos_{1/q} z = \frac{q^{\binom{n}{2}}(-1)^n}{[2n+1]_q!} \prod_{i=1}^{n} (1+q^i)^2.
$$

A similar computation gives the result $(n \ge 1)$

$$
[z^{2n}]\cos_q z \cos_{1/q} z = -[z^{2n}]\sin_q z \sin_{1/q} z = \frac{q^{\binom{n}{2}}(-1)^n}{[2n]_q!} \prod_{i=1}^{n-1} (1+q^i)^2 (1+q^n).
$$

Now we write $\tan_q z = \frac{\sin_q z \cos_{1/q} z}{\cos_q z \cos_{1/q} z}$ and thus

$$
\cos_q z \cos_{1/q} z \sum_{n \ge 0} \frac{T_{2n+1}(q)}{[2n+1]_q!} z^{2n+1} = \sin_q z \cos_{1/q} z.
$$

Comparing coefficients, we find

$$
\begin{aligned}
T_{2n+1}(q) + \sum_{k=1}^{n} \begin{bmatrix} 2n+1 \\ 2k \end{bmatrix}_q q^{\binom{k}{2}} (-1)^k \prod_{i=1}^{k-1} (1+q^i)^2 (1+q^k) T_{2n+1-2k}(q)&\\
= q^{\binom{n}{2}} (-1)^n \prod_{i=1}^{n} (1+q^i)^2.&
\end{aligned}
\tag{5.1}
$$

The induction argument is as in [4]; $T_{2n+1-2k}(q)$ has a factor $\prod_{i=1}^{n-k}(1+q^i)$ and, according again to [4],

$$
\begin{bmatrix} 2n+1 \\ 2k \end{bmatrix}_q \frac{\prod_{i=1}^{k}(1+q^i)}{\prod_{i=n-k+1}^{n}(1+q^i)}
$$

is still a polynomial. The two factors $\prod_{i=n-k+1}^{n}(1+q^i)$ and $\prod_{i=1}^{n-k}(1+q^i)$ mean that everything in (5.1) must be divisible by $\prod_{i=1}^{n}(1+q^i)$, and this finishes the proof.

It is likely that stronger results as in [7] hold, but we have not investigated that. □

The new q–secant numbers do not enjoy any divisibility results that are worthwhile to report; for the classical ones, see [3].

Remark. The paper [11] has a q–exponential function

$$\mathcal{E}_q := \sum_{n\ge 0} \frac{q^{n^2/4} z^n}{(q;q)_n}.$$

Plugging in $iz(1-q)$ for z and taking real parts would result in the q–cosine with factor q^{n^2}. To get the corresponding q–sine, replace z by $izq(1-q)$, take the imaginary part and multiply by $q^{1/4}$. We consider that merely to be a curiosity, not being of much help.

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