TERNARY SMIRNOV WORDS AND GENERATING FUNCTIONS

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ABSTRACT. We demonstrate that enumeration problems related to words with neighbouring letters being always different (Smirnov words) are most efficiently done using generating functions.

Dedicted to Robert Tichy on the occasion of his 60th birthday.

1. INTRODUCTION

Smirnov words [1] are characterized by the property that the words $x_1 \dots x_n$ must satisfy the condition that $x_i \neq x_{i+1}$ for all $i = 1, \dots, n-1$. To obtain generating functions for them is a well known process [1]; for a recent application of the principle, the reader is referred to [2].

Koshy and Grimaldi [3] treated several enumerations around *ternary* (Smirnov) words (an alphabet with 3 letters is underlying) in an elementary fashion, which involves long computations. The aim of the present note is to show how Smirnov words and generating functions get such results in a painless way. Generalizations to more than 3 letters and additional parameters are immanent.

Many results may be expressed by Jacobsthal and Jacobsthal-Lucas numbers

$$J_n = \frac{2^n - (-1)^n}{3} \quad \text{and} \quad j_n = 2^n + (-1)^n.$$

This is not essential, and is always possible, since one can replace 2^n by $J_{n+1} + J_n$ and $(-1)^n$ by $J_{n+1} - 2J_n$.

2. TERNARY SMIRNOV WORDS STARTING AND ENDING WITH THE LETTER 0

We assume that the 3 letters are $0, 1, 2$ and set up generating functions $f_i(z)$ via the system

$$\begin{aligned} f_0(z) &= z + zf_1(z) + zf_2(z), \\ f_1(z) &= zf_0(z) + zf_2(z), \\ f_2(z) &= zf_0(z) + zf_1(z). \end{aligned}$$

This system is inspired by random walks, and describes how a new letter is appended, which must be different from the previous one. The letter z marks the letters. The functions $f_i(z)$

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count Smirnov words starting with 0 and ending with i. In [3] the main interest was in $f_0(z)$, but the other two are easy as well. The solutions are:

$$\begin{aligned} f_0(z) &= \frac{1}{2} + \frac{1}{6}\frac{1}{1-2z} - \frac{2}{3}\frac{1}{1+z}, \\ f_1(z) = f_2(z) &= -\frac{1}{2} + \frac{1}{6}\frac{1}{1-2z} + \frac{1}{3}\frac{1}{1+z}, \\ f_0(z) + f_1(z) + f_2(z) &= -\frac{1}{2} + \frac{1}{2}\frac{1}{1-2z}. \end{aligned}$$

Reading off the coefficient of z^n, we find for $n \ge 1$:

$$[z^n]f_0(z) = \frac{1}{6}2^n - \frac{2}{3}(-1)^n = 2J_{n-2}, \quad [z^n]f_1(z) = \frac{1}{6}2^n + \frac{1}{3}(-1)^n = J_{n-1}.$$

In total, we get of course 2^{n-1} Smirnov words of length n, starting with 0.

Corollary 1 of [3] gives J_{n-2} for the number of Smirnov words, starting with 01 and ending with 0. This follows without computation from the symmetry of such words having the second letter equal to 1 resp. 2.

3. Ternary Smirnov words starting and ending with the letter 0 — counting the letters 0

We can use essentially the same equations, but now using a second variable u, counting the numbers 0.

$$\begin{aligned} f_0(z,u) &= zu + zf_1(z,u) + zf_2(z,u), \\ f_1(z,u) &= zuf_0(z,u) + zf_2(z,u), \\ f_2(z,u) &= zuf_0(z) + zf_1(z,u). \end{aligned}$$

We find

$$f_0(z,u) = \frac{zu(1-z)}{1-z-2z^2u} = \frac{zu}{1-\frac{2z^2}{1-z}u}.$$

Therefore (for $n \ge 2$ and $k \ge 2$)

$$[u^k]f_0(z,u) = 2^{k-1}\frac{z^{2k-1}}{(1-z)^{k-1}},$$

and

$$[z^nu^k]f_0(z,u) = [z^n]2^{k-1}\frac{z^{2k-1}}{(1-z)^{k-1}} = [z^{n+1-2k}]\frac{2^{k-1}}{(1-z)^{k-1}} = 2^{k-1}\binom{n-1-k}{k-2}.$$

The total number of letters 0 in Smirnov words of length n is given via $\frac{\partial}{\partial u}f_0(z,1)$:

$$\begin{aligned}
&[z^n]\bigg(\frac{1}{18}\frac{1}{(1-2z)^2}+\frac{5}{54}\frac{1}{1-2z}-\frac{4}{9}\frac{1}{(1+z)^2}+\frac{8}{27}\frac{1}{1+z}\bigg)\\
&=\frac{1}{18}(n+1)2^n+\frac{5}{54}2^n-\frac{4}{9}(n+1)(-1)^n+\frac{8}{27}(-1)^n\\
&=\frac{1}{18}n2^n+\frac{4}{27}2^n-\frac{4}{9}n(-1)^n-\frac{4}{27}(-1)^n.
\end{aligned}$$

Analogous computations are as follows:

$$f_1(z,u)=\frac{z^2u^2}{1-z-2z^2u}=\frac{z^2u^2}{1-z}\frac{1}{1-\frac{2z^2}{1-z}u},$$

and

$$[u^k]f_1(z,u)=\frac{z^2}{1-z}[u^{k-2}]\frac{1}{1-\frac{2z^2}{1-z}u}=2^{k-2}\frac{z^{2k-2}}{(1-z)^{k-1}},$$

and

$$[z^nu^k]f_1(z,u)=2^{k-2}[z^{n+2-2k}]\frac{1}{(1-z)^{k-1}}=2^{k-2}\binom{n-k}{k-2}.$$

Furthermore,

$$\frac{\partial}{\partial u}f_1(z,1)=-\frac{1}{2}+\frac{1}{18}\frac{1}{(1-2z)^2}+\frac{4}{27}\frac{1}{1-2z}+\frac{2}{9}\frac{1}{(1+z)^2}+\frac{2}{27}\frac{1}{1+z},$$

and the coefficient of z^n in it is

$$\frac{1}{18}n2^n+\frac{11}{54}2^n+\frac{2}{9}n(-1)^n+\frac{8}{27}(-1)^n.$$

4. Ternary Smirnov words starting and ending with the letter 0 – counting the letters 1

In a very similar way we can count the total number of letters 1. A second variable u is counting the numbers 1.

$$\begin{aligned}
f_0(z,u)&=z+zuf_1(z,u)+zf_2(z,u),\\
f_1(z,u)&=zf_0(z,u)+zf_2(z,u),\\
f_2(z,u)&=zf_0(z)+zuf_1(z,u).
\end{aligned}$$

We find for instance

$$\frac{\partial}{\partial u}f_0(z,1)=\frac{1}{18}\frac{1}{(1-2z)^2}-\frac{7}{54}\frac{1}{1-2z}-\frac{1}{9}\frac{1}{(1+z)^2}+\frac{5}{27}\frac{1}{1+z}$$

and the coefficient of z^n in it:

$$\frac{1}{18}n2^n-\frac{2}{27}2^n-\frac{1}{9}n(-1)^n+\frac{2}{27}(-1)^n.$$

Other results are similar.

5. Words interpreted as decimal (and other) numbers

Let $q \geq 3$ be a base, and define $\mathsf{value}(x_1 \dots x_n) = x_1 q^{n-1} + x_2 q^{n-2} + \cdots + x_n$. As in [3], we consider S_n, the sum over the values of all Smirnov words of length n, rendered by zeros. The recursion $\mathsf{value}(x_1 \dots x_n) = q\mathsf{value}(x_1 \dots x_{n-1}) + x_n$ is obvious.

We can set up the following equations:

$$\begin{aligned} f_0(z,u) &= z + z f_1(z,u^q) + z f_2(z,u^q), \\ f_1(z,u) &= zu f_0(z,u^q) + zu f_2(z,u^q), \\ f_2(z,u) &= zu^2 f_0(z,u^q) + zu^2 f_1(z,u^q). \end{aligned}$$

Then the coefficient of $z^n u^k$ in $f_i(z,u)$ is the number of Smirnov words ending in i, having length n and $\mathsf{value} = k$. We are only aiming at the total value, so in other words we should differentiate $f_i(z,u)$ and evaluate it at $u = 1$. The evaluations of $f_i(z) := f_i(z,1)$ have been given before, so, with $g_i(z) = \frac{\partial}{\partial u} f_i(z,1)$, we get

$$\begin{aligned} g_0(z) &= z + z g_1(z) + z g_2(z), \\ g_1(z) &= qz g_0(z) + qz g_2(z,u) + z f_0(z) + z f_2(z), \\ g_2(z) &= qz g_0(z) + qz g_1(z) + 2z f_0(z) + 2 f_1(z). \end{aligned}$$

This system is readily solved, with result

$$\begin{aligned} g_0(z) = \frac{1}{2(q-1)(1+2q)(1-2qz)} &- \frac{1}{(2+q)(q-1)(1+qz)} \\ &- \frac{q}{2(2+q)(q-1)(1-2z)} + \frac{q}{(q-1)(1+2q)(1+z)}. \end{aligned}$$

Reading off the coefficient of z^n leads to the numbers S_n:

$$\begin{aligned} S_n = \frac{1}{2(q-1)(1+2q)} 2^n q^n &- \frac{1}{(2+q)(q-1)} (-1)^n q^n \\ &- \frac{q}{2(2+q)(q-1)} 2^n + \frac{q}{(q-1)(1+2q)} (-1)^n. \end{aligned}$$

Of course, the coefficients of $g_1(z)$ and $g_2(z)$ could also be determined, as well as higher moments, by more differentiations. We are not doing this here; we just want to indicate how things could be done efficiently.

The values for $q = 3$ are derived in [3] using long computations.

6. Inversions

An inversion (in a Smirnov word as always in this paper) is a pair $x_i > x_j$ and $1 \leq i < j \leq n$. To count the total number of inversions, we proceed as follows. We consider a *marked* letter i, so that the word can be written as $x = wiy$, and we count the inversions with first letter i. They are given by the number of symbols $< i$ in y. So the generating functions decomposes:

$$\frac{1}{z} f_0(z) f_0(z) + \frac{1}{z} f_1(z) g_1(z,u) + \frac{1}{z} f_2(z) h_2(z,u).$$

The first factor refers to the words wi (ending at i), the second factor to the reversed word $y^R i$, where in $g_1(z,u)$ the second variable is used to count the letters 0, and in $h_2(z,u)$ the second variable is used to count the letters 0 or 1. Since the designated letter i is part of both factors, we must divide by z.

Now we have again

$$\begin{aligned}
f_0(z) &= z + zf_1(z) + zf_2(z),\\
f_1(z) &= zf_0(z) + zf_2(z),\\
f_2(z) &= zf_0(z) + zf_1(z)
\end{aligned}$$

and also

$$\begin{aligned}
g_0(z,u) &= zu + zug_1(z,u) + zug_2(z,u),\\
g_1(z,u) &= zg_0(z,u) + zg_2(z,u),\\
g_2(z,u) &= zg_0(z,u) + zg_1(z,u)
\end{aligned}$$

and

$$\begin{aligned}
h_0(z,u) &= zu + zuh_1(z,u) + zuh_2(z,u),\\
h_1(z,u) &= zuh_0(z,u) + zuh_2(z,u),\\
h_2(z,u) &= zh_0(z,u) + zh_1(z,u).
\end{aligned}$$

These systems are readily solved, and we only give the final answer:

$$\begin{aligned}
&\frac{\partial}{\partial u}\Big(\frac{1}{z}f_0(z)f_0(z) + \frac{1}{z}f_1(z)g_1(z,u) + \frac{1}{z}f_2(z)h_2(z,u)\Big)\Big|_{u=1}\\
&\qquad\qquad = \frac{1}{18}\frac{1}{(1-2z)^3} - \frac{5}{54}\frac{1}{(1-2z)^2} + \frac{1}{9}\frac{1}{(1+z)^3} + \frac{4}{27}\frac{1}{(1+z)^2}.
\end{aligned}$$

The coefficient of z^n in this is routinely found to be

$$\Big(\frac{n^2}{36} - \frac{n}{108} - \frac{1}{27}\Big)2^n - \Big(\frac{n^2}{18} + \frac{n}{54} - \frac{1}{27}\Big)(-1)^n.$$

This formula was derived in [3] with significantly more effort.

7. Conclusion

The methods described here allow for all kinds of more refined results; we don't have space here to present more calculations.

The restriction to ternary words and first/last letter being 0 is superficial and could easily be replaced by larger alphabets and different conditions on the letters.

Calculations were done by Maple; they are easy to reproduce by the reader using any computer algebra system.

References

[1] P. Flajolet and R. Sedgewick. *Analytic Combinatorics*. Cambridge University Press, Cambridge, 2009.

[2] U. Freiberg, C. Heuberger, and H. Prodinger. Application of Smirnov words to waiting time distributions of runs. *submitted*, xx:xx pages, 2017.

[3] T. Koshy and R. Grimaldi. Ternary words and Jacobsthal numbers. *The Fibonacci Quarterly*, 55:129–136, 2017.

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