FACTORIZATIONS RELATED TO SOME NUMERICAL TRIANGLES

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ABSTRACT. An asymmetric combinatorial matrix with two free parameters is factored in 4 different ways. The results are obtained by a combination of guessing and applying combinatorial identities.

1. INTRODUCTION

In a series of papers Trzaska [5, 6, 7] considered the numerical triangles

$$A_1(n,k)=\binom{n+k}{2k} \quad \text{and} \quad A_2(n,k)=\binom{n+k}{2k+1}.$$

It is interesting to note that the explicit forms never appear in the above mentioned papers, but the results are easy to prove by induction, once one has guessed to correct formulæ.

In another series of papers, Ferri, Faccio, and D'Amico [1, 2] considered the triangles

$$B_1(n,k)=A_1(n,k)=\binom{n+k}{2k} \quad \text{and} \quad B_2(n,k)=\binom{n+k+1}{2k}=B_1(n+1,k),$$

and, again, the explicit forms were never mentioned.

In the present paper, we consider the triangle (generalizing all the mentioned triangles)

$$A(n,k)=\binom{n+k+a}{2k+b}$$

for integer parameters $a \ge b \ge 0$; in particular, we want to study LU-decompositions of matrices related to the numbers $A(n,k)$. In order to make this a meaningful project, we should avoid entries 0, which would lead to uninteresting results. It is better to use the numbers

$$A(n+k,k)=\binom{n+2k+a}{2k+b},$$

and these numbers are never zero for integers $n,k \ge 0$ and $a \ge b \ge 0$, which we will assume. Notice that there is no symmetry $n \longleftrightarrow k$. We will thus consider in four sections the matrices $M=(\binom{n+2k+a}{2k+b})_{n,k}$, $M=(\binom{n+2k+a}{2k+b})_{k,n}$, $M=(1/\binom{n+2k+a}{2k+b})_{n,k}$, $M=(1/\binom{n+2k+a}{2k+b})_{k,n}$ and compute the LU-decomposition $M=LU$. This will be done by guessing and later proving the results. We use the generic form $M=LU$, but there is no danger of confusion, as the notion will be restricted to the relevant section.

Key words and phrases. LU-decomposition; identities; guessing.

The author was supported by an incentive grant of the National Research Foundation of South Africa.

Decompositions of that type have appeared previously in this journal, see, e.g., [4, 3]. In earlier instances, both, L and U had attractive coefficients that appeared in closed form. Here, the situation is a bit more complicated: In two instances, U has nice coefficients, but the coefficients of L are only given as a summation, and in the other two (transposed) instances, it is the reversed scenario.

2. Factorization of the original matrix

We start with the matrix M that has entries $M_{n,k} = \binom{n+2k+a}{2k+b}$ for $n,k \ge 0$.

Theorem 1.

$$U_{k,n} = \frac{(2n+a)!n!2^k}{(2n+b)!(n-k)!(k+a-b)!},$$
$$U^{-1}_{k,n} = \frac{(n+a-b)!(2k+b)!(-1)^{n-k}}{(n-k)!k!(2k+a)!2^n}.$$

We prove that the matrices are indeed inverse. Assume that $k \le l$ and compute

$$\begin{aligned}
(U \cdot U^{-1})_{k,l} &= \sum_{k\le j\le l} U_{k,j}U^{-1}_{j,l}\\
&= \sum_{k\le j\le l} \frac{(2j+a)!j!2^k}{(2j+b)!(j-k)!(k+a-b)!}\frac{(l+a-b)!(2j+b)!(-1)^{l-j}}{(l-j)!j!(2j+a)!2^l}\\
&= \frac{2^{k-l}(l+a-b)!}{(k+a-b)!}\sum_{k\le j\le l}\frac{(-1)^{l-j}}{(j-k)!(l-j)!}\\
&= \frac{2^{k-l}(l+a-b)!(-1)^{l-k}}{(k+a-b)!(l-k)!}\sum_{k\le j\le l}\binom{l-k}{j-k}(-1)^{k-j}\\
&= \frac{2^{k-l}(l+a-b)!(-1)^{l-k}}{(k+a-b)!(l-k)!}[\![k=l]\!] = [\![k=l]\!].
\end{aligned}$$

Values of $(U \cdot U^{-1})_{k,l}$ for $k < l$ are of course also 0. We use Iverson's notation: $[\![P]\!]$ is 1 if P is true, and 0 otherwise.

Now we define $L = MU^{-1}$ and express the coefficients as convolutions:

$$\begin{aligned}
L_{n,k} &= \sum_{0\le j\le k} M_{n,j}U^{-1}_{j,k}\\
&= \sum_{0\le j\le k}\binom{n+2j+a}{2j+b}\frac{(k+a-b)!(2j+b)!(-1)^{k-j}}{(k-j)!j!(2j+a)!2^k}\\
&= \frac{(k+a-b)!n!(-1)^k}{(n+a-b)!k!2^k}\sum_{0\le j\le k}(-1)^j\binom{k}{j}\binom{n+2j+a}{n}.
\end{aligned}$$

In order to see that this is indeed a lower triangular matrix, notice that

$$\binom{n+2j+a}{n} = \sum_{0\le h\le n} \lambda(n,h) j^h,$$

for some coefficients $\lambda(n,h)$, and that

$$\sum_{0\le j\le k} (-1)^j \binom{k}{j} j^h = 0$$

for $h<k$. Therefore for $n\le k$, there are no contributions to the sum. If $n=k$, then

$$\begin{aligned}
L_{n,n} &= \frac{(n+a-b)!(-1)^n}{(n+a-b)!2^n} \sum_{0\le j\le n} (-1)^j \binom{n}{j}\binom{n+2j+a}{n} \\
&= \frac{(n+a-b)!(-1)^n}{(n+a-b)!2^n} \sum_{0\le j\le n} (-1)^j \binom{n}{j}\Big[\frac{2^n}{n!} j^n + \text{lower powers of } j\Big] \\
&= \frac{(n+a-b)!}{(n+a-b)!n!} \sum_{0\le j\le n} (-1)^{n-j} \binom{n}{j} j^n \\
&= \frac{(n+a-b)!}{(n+a-b)!n!} n! = 1,
\end{aligned}$$

as it should. So L is a lower triangular matrix with coefficients 1 in the main diagonal, and U is an upper triangular matrix. Since the LU-decomposition is unique, we have found the matrices that are involved.

3. Factorization of the transposed matrix

We continue with the matrix M that has entries $M_{k,n} = \binom{n+2k+a}{2k+b}$ for $n,k\ge 0$.

Theorem 2.

$$\begin{aligned}
L_{n,k} &= \frac{(2n+a)!n!(2k+b)!}{(2n+b)!(n-k)!k!(2k+a)!}, \\
L^{-1}_{n,k} &= \frac{(-1)^{n-k}(2n+a)!n!(2k+b)!}{(2n+b)!(n-k)!k!(2k+a)!}.
\end{aligned}$$

We check that the matrices are inverse:

$$\begin{aligned}
(L\cdot L^{-1})_{n,l} &= \sum_{l\le j\le n} \frac{(2n+a)!n!(2j+b)!}{(2n+b)!(n-j)!j!(2j+a)!} \frac{(-1)^{j-l}(2j+a)!j!(2l+b)!}{(2j+b)!(j-l)!l!(2l+a)!} \\
&= \frac{(2n+a)!n!(2l+b)!}{(2n+b)!l!(2l+a)!(n-l)!} \sum_{l\le j\le n} (-1)^{j-l} \binom{n-l}{j-l} \\
&= \frac{(2n+a)!n!(2l+b)!}{(2n+b)!l!(2l+a)!(n-l)!} [\![n=l]\!] = [\![n=l]\!].
\end{aligned}$$

And now we define $U = L^{-1}M$; we get these entries:

$$\begin{aligned}U_{n,k} &= \sum_j L^{-1}_{n,j} M_{j,k}\\ &= \sum_j \frac{(-1)^{n-j}(2n+a)!n!(2j+b)!}{(2n+b)!(n-j)!j!(2j+a)!}\binom{k+2j+a}{2j+b}\\ &= \frac{(2n+a)!k!}{(k+a-b)!(2n+b)!}\sum_j (-1)^{n-j}\binom{n}{j}\binom{k+2j+a}{k}\end{aligned}$$

As before, we see that there only nonzero contributions when $k \ge n$.

The diagonal elements simplify:

$$U_{n,n} = \frac{(2n+a)!n!2^n}{(2n+b)!(n+a-b)!}.$$

4. Factorization of the reciprocal matrix

Now we consider the matrix M that has entries $M_{n,k} = 1/\binom{n+2k+a}{2k+b}$ for $n,k \ge 0$.

Theorem 3.

$$U_{k,n} = \frac{(-1)^k(2n+b)!n!(k+a-b)!\Gamma(\frac{k+a}{2})}{(2n+k+a)!(n-k)!\Gamma(\frac{3k+a}{2})}$$

$$U^{-1}_{k,n} = \frac{2(n+2k+a-1)!(-1)^k\Gamma(\frac{3n+2+a}{2})}{(n+a-b)!k!(n-k)!(2k+b)!\Gamma(\frac{n+a}{2})}$$

We prove that the two matrices are inverse to each other:

$$\begin{aligned}(U\cdot U^{-1})_{k,l} &= \sum_j U_{k,j}U^{-1}_{j,l}\\ &= \sum_j \frac{(-1)^k(2j+b)!j!(k+a-b)!\Gamma(\frac{k+a}{2})}{(2j+k+a)!(j-k)!\Gamma(\frac{3k+a}{2})}\,\frac{2(l+2j+a-1)!(-1)^j\Gamma(\frac{3l+2+a}{2})}{(l+a-b)!j!(l-j)!(2j+b)!\Gamma(\frac{l+a}{2})}\\ &= \frac{2(k+a-b)!\Gamma(\frac{k+a}{2})\Gamma(\frac{3l+2+a}{2})}{\Gamma(\frac{3k+a}{2})\Gamma(\frac{l+a}{2})(l+a-b)!(l-k)}\sum_{k\le j\le l}(-1)^{j-k}\binom{l-k}{j-k}\binom{l+2j+a}{l-k-1}.\end{aligned}$$

For $l > k$, the inner sum evaluates to 0. For $k = l$, we are left to compute

$$\frac{2(3k+a-1)!\Gamma(\frac{3k+2+a}{2})}{(3k+a)!\Gamma(\frac{3k+a}{2})} = \frac{2\frac{3k+a}{2}}{3k+a} = 1,$$

as it should. Therefore we define $L := MU^{-1}$, with coefficients

$$\begin{aligned}L_{n,k} &= \sum_{0\le j\le k} M_{n,j}U^{-1}_{j,k}\\ &= \frac{2(n+a-b)!\Gamma(\frac{3k+2+a}{2})}{k!(k+a-b)!\Gamma(\frac{k+a}{2})}\sum_{0\le j\le k}\frac{(k+2j+a-1)!(-1)^j}{(n+2j+a)!}\binom{k}{j}.\end{aligned}$$

If $k > n$, we may write this as

$$L_{n,k} = \frac{2(n+a-b)!(k-n-1)!\Gamma(\frac{3k+2+a}{2})}{k!(k+a-b)!\Gamma(\frac{k+a}{2})}\sum_{0\le j\le k}(-1)^j\binom{k}{j}\binom{k+2j+a-1}{k-n-1},$$

which evaluates to 0. If $n = k$, we are left with

$$\begin{aligned}L_{n,n} &= \frac{2\Gamma(\frac{3n+2+a}{2})}{n!\Gamma(\frac{n+a}{2})}\sum_{0\le j\le n}\frac{(-1)^j}{n+2j+a}\binom{n}{j}\\ &= \frac{2\Gamma(\frac{3n+2+a}{2})}{n!\Gamma(\frac{n+a}{2})}\frac{\Gamma(\frac{n+a}{2})\Gamma(n+1)}{2\Gamma(\frac{3n+2+a}{2})} = 1.\end{aligned}$$

So, L is a lower triangular matrix with ones in the main diagonal, and we have indeed found the LU-decomposition of M.

5. Factorization of the reciprocal transposed matrix

Our last section deals with the matrix M that has entries $M_{k,n} = 1/\binom{n+2k+a}{2k+b}$ for $n, k \ge 0$.

Theorem 4.

$$\begin{aligned}L_{n,k} &= \frac{(2n+b)!n!(3k+a)!}{(2n+k+a)!k!(n-k)!(2k+b)!}\\ L^{-1}_{n,k} &= \frac{(-1)^{n-k}(2n+b)!n!(n+2k+a-1)!}{k!(n-k)!(3n+a-1)!(2k+b)!}.\end{aligned}$$

We check that the matrices are inverse:

$$\begin{aligned}(L\cdot L^{-1})_{n,l} &= \sum_{l\le j\le n} L_{n,j}L^{-1}_{j,l}\\ &= \sum_{l\le j\le n}\frac{(2n+b)!n!(3j+a)!}{(2n+j+a)!j!(n-j)!(2j+b)!}\frac{(-1)^{j-l}(2j+b)!j!(j+2l+a-1)!}{l!(j-l)!(3j+a-1)!(2l+b)!}\\ &= \frac{(2n+b)!n!(2l-2n-1)!}{l!(2l+b)!(n-l)!}\sum_{l\le j\le n}(-1)^{j-l}(3j+a)\binom{n-l}{l-j}\binom{j+2l+a-1}{j+2n+a}\end{aligned}$$

For $l < n$, the sum evaluates to 0 (computer algebra systems are capable to evaluate this), and we are left to consider $(L\cdot L^{-1})_{n,n}$, which is straightforwardly seen to be 1.

And, as before, we define $U = L^{-1}M$ and get these entries:

$$
\begin{aligned}
U_{n,k} &= \sum_j L^{-1}_{n,j} M_{j,k} \\
&= \sum_j \frac{(-1)^{n-j}(2n+b)!n!(n+2j+a-1)!}{j!(n-j)!(3n+a-1)!(2j+b)!} \frac{(2j+b)!(k+a-b)!}{(k+2j+a)!} \\
&= \frac{(2n+b)!(k+a-b)!}{(3n+a-1)!} \sum_{0\le j\le n} (-1)^{n-j} \frac{(n+2j+a-1)!}{(k+2j+a)!} \binom{n}{j}.
\end{aligned}
$$

If $n > k$ then the sum evaluates to 0, so U is indeed an upper triangular matrix. The diagonal elements can be computed:

$$
\begin{aligned}
U_{n,n} &= \frac{(2n+b)!(n+a-b)!}{(3n+a-1)!} \sum_{0\le j\le n} (-1)^{n-j} \frac{1}{n+2j+a} \binom{n}{j} \\
&= \frac{(2n+b)!(n+a-b)!}{(3n+a-1)!} \frac{(-1)^n \Gamma(\frac{n+a}{2}) n!}{2\Gamma(\frac{3n+a+2}{2})} \\
&= \frac{(-1)^n (2n+b)!(n+a-b)!\Gamma(\frac{n+a}{2})}{(3n+a)!n!\Gamma(\frac{3n+a}{2})}.
\end{aligned}
$$

6. Generalizations

We briefly mention that the results may be generalized. For instance, instead of working with $\binom{n+2k+a}{2k+b}$, we may work with $\binom{n+dk+a}{dk+b}$, where d is a positive integer. We just cite the theorems here and leave the (analogous) proofs to the interested reader.

Theorem 5 (Generalization of Theorem 1).

$$
\begin{aligned}
U_{k,n} &= \frac{(dn+a)!n!d^k}{(dn+b)!(n-k)!(k+a-b)!}, \\
U^{-1}_{k,n} &= \frac{(n+a-b)!(dk+b)!(-1)^{n-k}}{(n-k)!k!(dk+a)!d^n}.
\end{aligned}
$$

Theorem 6 (Generalization of Theorem 2).

$$
\begin{aligned}
L_{n,k} &= \frac{(dn+a)!n!(dk+b)!}{(dn+b)!(n-k)!k!(dk+a)!}, \\
L^{-1}_{n,k} &= \frac{(-1)^{n-k}(dn+a)!n!(dk+b)!}{(dn+b)!(n-k)!k!(dk+a)!}.
\end{aligned}
$$

Theorem 7 (Generalization of Theorem 3).

$$
U_{k,n} = \frac{(-1)^k (dn+b)!n!(k+a-b)!\Gamma(\frac{k+a}{d})}{(dn+k+a)!(n-k)!\Gamma(\frac{(d+1)k+a}{d})}
$$

$$U_{k,n}^{-1} = \frac{d(n+dk+a-1)!(-1)^k\Gamma(\frac{(d+1)n+d+a}{d})}{(n+a-b)!k!(n-k)!(dk+b)!\Gamma(\frac{n+a}{d})}$$

Theorem 8 (Generalization of Theorem 4)**.**

$$L_{n,k} = \frac{(dn+b)!n!((d+1)k+a)!}{(dn+k+a)!k!(n-k)!(dk+b)!}$$

$$L_{n,k}^{-1} = \frac{(-1)^{n-k}(dn+b)!n!(n+dk+a-1)!}{k!(n-k)!((d+1)n+a-1)!(dk+b)!}.$$

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