Project description

Random processes on groups: Regularity and Phase Transitions

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1 Regularity and phase transitions

A population may explode if its birth rate is above 1 and otherwise becomes extinct. The nature of the population therefore depends critically on the balance of creating and annihilating particles.

Another famous example of a critical phenomenon is given by percolation models that may describe the following situation. A porous rock with randomly microscopic pores has water spilled on top. If there are very few holes no water will percolate through the rock, but if there are many holes water will flow. Interestingly, there is a critical degree of porosity that separates exactly these two behaviors.

A block of iron in a magnetic field become magnetized. If the magnetic field is turned of, there are two possibilities, either the iron remains magnetized or the magnetization vanished. In fact, block remains magnetized if and only if the temperature is below the Curie temperature of $700°C$. This model is known under the Ising model and is perhaps the most studied model in physics.

The theory and study of phase transitions is also guided by the not yet precise mathematical concept of universality. This concept refers to the fact that many essential properties of the system at the critical point depend on relatively few parameters (properties) of the system. In particular, this fact justifies oversimplified mathematical models since even simple mathematical models can capture some of the qualitative and even quantitative features of the original system. For example, the fact that a population may explode or become extinct depend only on the mean rate and not on the precise mechanism of reproduction.

The mathematical theory of phase transitions or critical phenomena is currently undergoing an intense development. As well as having applications, these models are mathematically fascinating. Deep theorems have been proved, but many problems of central importance remain unsolved and to be explored.

While most of these studies connect probability theory with statistical physics our project will concentrate on the connection of probability theory, algebra, and geometry. Our project will deal with several stochastic processes, namely random walks, branching random walks, percolation and Ising model, on infinite groups and study the concept of phase transitions and universality (or regularity).

1.1 Random walks

Let us first present some general background on the study of random walks on groups. Random walks are one important link between the following "different" fields of mathematics that may appear quite isolated at the first glance: probability theory, algebra, and geometry. There is a contribution of Anatoly V. Vershik in a special Springer volume about the future of mathematics in the 21st century that points out the prospects and challenges that are comprised in the interplay of probability theory and algebra. Here random walk theory, a branch of probability theory, plays a major role. For example, one fundamental property of a group, von Neumann's amenability property, is equivalent, as Harry Kesten showed, to subexponential decay of the return probability of a random walk.

In general, there are two points of view to look at the relation between probability theory on the onehand and algebra and geometry on the otherhand. The probabilistic viewpoint concerns all questions regarding the impact of the underlying structure on the behavior of the corresponding random walk. Typically, one is interested in transience/recurrence, spectral radius, asymptotic behavior of the transition probabilities, rate of escape, central limit theorems, harmonic functions and convergence to a boundary at infinity. On the other side, random walks are a useful tool to describe the structure that underlies the random walk. In particular, algebraic and geometric properties can be classified due to the behavior of the corresponding random walks. The project falls exactly into this field at the confluent of different branches of mathematics. Based on the interplay between probability theory (random walks) and structure theory (algebra, geometry), we will mainly study the following questions. In order to formulate these questions we need to introduce some mathematical notations. We consider a finitely generated group G and denote by S some symmetric finite generating set. The random walk X_n will be driven by a measure μ whose support contains S. Many basic questions turn around the long time behavior of the random walk X_n. In particular, we will be interested whether $\lim_{n\to\infty} |X_n|/n > 0$ or equal to 0; the existence of this limit is guaranteed by a sub-additive argument. The Shannon entropy of a probability measure μ is defined by

$$H(\mu) = -\sum_x \mu(x) \log \mu(x)$$

and describes the uncertainty of a probability measure or random variable. This concept turned out to be very important and is the basis of information theory, a branch of applied mathematics, electrical engineering, bioinformatics, and computer science involving the quantification of information. Avez lifted this concept to random walks on groups and defined the asymptotic entropy of the walk driven by μ as

$$h(\mu) = \lim_{n\to\infty} \frac{H(\mu^n)}{n}.$$

Avez used this entropy to prove that a Cayley graph satisfies the Liouville property (that means the existence of non-constant bounded harmonic functions) if the asymptotic entropy of the random walk is zero. Kaimanovich, Vershik and Derriennic then proved that also the inverse is true. Moreover, Kaimanovich and Vershik proved, using the Poisson boundary, that positivity of the asymptotic entropy is equivalent to positivity of the rate of escape.

1.1.1 Universality of positive rate of escape

Random walks on non-amenable groups are known to have positive rate of escape or in other words non-amenable Cayley graphs do not have the Liouville property. However, there are amenable groups that are Liouville and amenable groups that are not Liouville. We say a property is group invariant if the property does not depend on the choice of the generating set. A central open question that we are going to discuss is whether the Liouville property is invariant (universal) on the choice of the generating set. This intriguing problem attracted a lot of attention in the recent years and already partially results could be seen as a success.

1.1.2 Regularity of asymptotic entropy and rate of escape

As mentioned above the Liouville property is strongly connected with the asymptotic entropy and rate of escape of random walks on groups. Continuity of the rate of escape and the asymptotic entropy, proved by Erschler and Kaimanovich, is known on hyperbolic groups under the general condition of having a finite first moment.

Moreover, analyticity of the rate of escape also follows in certain cases where explicit formulæ for the rate of escape are known, see works of Mairesse and Mathéus and Gilch. Mairesse and Mathéus show that the rate of escape for some random walks on the Braid group is continuous but not(!) differentiable. Analyticity of the rate of escape and of the asymptotic entropy on free groups was proven by Ledrappier and more recently, Ledrappier proves Lipschitz continuity for the rate of escape and asymptotic entropy for random walks on Gromov hyperbolic groups.

Haïssinky, Mathieu and Mueller proved that the rate of escape of a random walk on surface groups, an important class of hyperbolic groups, is analytic under exponential moment conditions. One fist objective of our project is to prove an analogous statement for the asymptotic entropy on surface groups and then to generalize them to general hyperbolic groups. Other objectives are to advance the research of the asymptotic entropy on amenable groups where the asymptotic entropy is positive. The study is therefore related to the question of universality of the Liouville property.

1.2 Phase transitions for branching random walks

Classification of groups in terms of the behavior of random processes attracted a lot of attention. In particular, a consequence of Gromov's f amous theorem on groups of polynomial growth is that a finitely generated group admits a recurrent random walk if and only if it contains a finite-index subgroup isomorphic with the one-dimensional or two-dimensional lattice. Kesten's criterion for amenability says that a finitely-generated group is amenable if and only if the spectral radius ρ for any (or some) symmetric random walk is equal to 1. This phenomenon is also underlined by a phase transition of branching random walks (BRW). If the mean number of offspring m of the branching is less or equal than $1/\rho$ then the BRW is transient, but if $m > 1\rho$ then the process is transient. In the transient regime Hueter and Lalley studied the Hausdorff dimension of the limit set of the BRW on homogeneous trees and obtained that it is regular for $m < 1/\rho$ but there is a discontinuity at the critical case $m = 1/\rho$. In other words, the behavior changes dramatically at the critical value $m = 1/\rho$ but is regular outside the critical value. This fact was observed in several other physical models and is conjectured to be universal. Note that this conjecture is not yet formulated precisely. In a sequel work, Candellero, Gilch and Mueller generalized this work for BRW on groups with infinitely many ends. Our objective is therefore to complete the picture and treat the one-ended non-amenable groups as well.

1.3 Ising model and Schreier graphs

Geometric Group Theory is a relatively recent area of mathematical research, largely developed over the last twenty years through interaction between various approaches: geometric, algebraic, probabilistic - to the study of infinite groups. It has its roots in low-dimensional topology, combinatorial group theory, metric geometry; and its main ideas and principles have been outlined in the foundational work of M. Gromov. Our project represents a bridge connecting the algebraic part of the self-similar groups with the combinatorics of the Schreier graphs, the Statistics and behavior of the Ising and dimer model and the complex dynamics. Moreover Schreier graphs usually approximate fractal objects, the Julia sets of complex maps, so that our study can be related to the analysis on fractals.

One can study the Ising model on the sequence of Schreier graphs and ask which properties of the corresponding group can be deduced from the physical properties of the model. If one considers the Ising model on an infinite graph X, obtained as a limit of a sequence of finite graphs $X_n \to X$ (thermodynamic limit) and the partition functions ϕ_n on the finite graphs X_n as a function of the absolute temperature T , then the singularities of the function $\lim_{n\to\infty} \frac{\log \phi_n}{|X_n|}$ correspond to the critical temperature giving rise to the phenomenon of the phase transition. The absence of the phase transition is equivalent to the condition of uniqueness for the Gibbs measure on the limit graph. Self-similar graphs often have no phase transition. Indeed, there exists a sufficient condition implying the absence of the phase transition in terms of the order of ramification of the graph. By definition, the order of ramification R at a point P measures the smallest number of significant edges which one must cut in order to isolate an arbitrarily large bounded subset surrounding P. It has been shown that if R is uniformly bounded, for each vertex of the graph, then phase transition does not occur. It is clear that the Sierpinski gasket and the Schreier graph of the Hanoi Towers group do satisfy this sufficient condition. It is also true for Grigorchuk group and the Basilica group. So it is natural to ask under which conditions the Schreier graphs associated with the action of a group satisfies this condition. It is known that if the group is generated by a bounded automaton, then the associated graph is finitely ramified.

More generally, one can ask what algebraic properties of the associated group influence the presence or absence of the phase transition for the Ising model. One should mention here a result of Jonasson and Steif that gives a characterization of non-amenability for a group G in terms of the existence of the phase transition in the Cayley graph, but in the presence of a non-zero external magnetic field H. There is no such a characterization when $H = 0$. This study is related to phase transition in percolation model. In particular, the Bernoulli percolation and the Ising model are two examples of a more general model called FK-percolation.

2 Brief summary

The phase transition is a phenomenon observed in mathematics and natural sciences in many different contexts. It deals with a sudden change in the properties of a large structure caused by altering a critical parameter. The phase transition in random discrete structures (e.g. random walks, random graph processes, branching random walks, Ising/Potts model, percolation) has captured the attention of many scientists in recent years.

The goal of the project is to solve several open questions and problems in the above topic. Both institutes already have a strong expertise in these fields ; notably Mathieu, Pittet and Woess. Since moreover five participants, D'Angeli, Gilch, Huss, Mueller and Sava-Huss, will write their habiliation during the project, we are confident that several hard problems can be solved in collaboration. In both institutes there will be weakly seminars that are open for advances Master students and Phd students in order to prepare the ground for creative research and interesting teaching at the same time.