# Internal Diffusion Limited Aggregation (joint work with Wilfried Huss)

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# Outline

1 Warm-up: DLA

### 2 Internal DLA

### 3 References





Introduced in physics by Sander and Witten ['81], as a model of fractal growth. The growth rule is extremly simple:

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- DLA tends to build irregularities.



# DLA 33000 particles, center initially occupied





Different colors = different arrival times of the random walkers.

References

### Random walkers sticking to a straight line





# Internal DLA

Random growth model, **internal version of DLA**, which contrary to DLA, tends to eliminate irregularities.

- Let G be some infinite graph, and o some fixed vertex.
- one by one, particles perform discrete-time random walks.
- each particle starts from o and moves until it reaches a site unoccupied previously, where it stops.
- get a random subset of *n* occupied sites in *G*: internal DLA cluster  $A(n) \rightarrow$  the resulting random cluster of occupied sites after the *n*th particle stops.



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 $A(n+1) = A(n) \cup \{X^n(\tau_n)\},\$ 

where  $X^1, X^2, \ldots$  are independent random walks starting at o, and

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#### • Main question: limiting shape of A(n) as $n \to \infty$ ?

# DLA and internal DLA : comparison

**DLA**: tendrils result from the fact that the particles tend to hit first the neighbourhood of extreme sites in the occupied cluster  $\Rightarrow$  fractal structure.

**internal DLA**: particles diffusing through the interior of the occupied cluster are most likely to stop at unoccupied sites that are closest to  $0 \Rightarrow A(n)$  tends to eliminate irregularities  $\Rightarrow$  expected to grow like an expanding ball on a regular graph.



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Theorem (Lawler-Bramson-Griffeath '92)

For simple random walks on  $\mathbb{Z}^d$ ,  $d \ge 2$ , the limiting shape of internal DLA is a ball:  $\forall \epsilon > 0$ , with probability 1:

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**Question:** what about fluctuations for internal DLA, i.e. how smooth the surface formed by internal DLA can be?



# The cluster A(n), n=100000





■ Lawler ['95]: with probability 1

$$B_{r-r^{1/3}\log^2 r} \subset A(\pi r^2) \subset B_{r+r^{1/3}\log^4 r}$$

Can the errors be of order  $o(n^{\alpha})$ , for  $\alpha < 1/3$ ? Indeed there are only logarithmic fluctuations.



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■ For *d* ≥ 3: Jerison-Levine-Sheffield ['10] and Asselah-Gaudillière ['10]

$$B_{r-C\sqrt{\log r}} \subset A(\omega_d r^d) \subset B_{r+C\sqrt{\log^2 r}},$$
 eventually,

for a constant *C* depending only on *d*.  $\omega_d$  is the volume of the *d*-dimensional Euclidean ball of radius 1.



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- (Blachère-Brofferio '06) symmetric random walks on Cayley graphs of finitely generated groups with exponential growth.
- (Huss '07) strongly reversible, uniformly irreducible random walks on non-amenable graphs.



Does this theorem hold for IDLA in general?



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#### **Counter examples:**

**•** Random walk with drift in  $\mathbb{Z}^2$ 





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■ Simple random walk on the comb





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- consider simple random walk on  $C_2$ .

$$p(x,y) = rac{1}{d(x)}, ext{ for all } x \in \mathcal{C}_2,$$

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What is the limiting shape A(n)?









• A(n) for n = 500 and n = 1000.

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- A(n) for n = 500 and n = 1000.
- the set in the figure grows like n<sup>2/3</sup> in the vertical direction and like n<sup>1/3</sup> in the horizontal direction.
- want to prove that this is the limiting shape of internal DLA on the comb C<sub>2</sub>.
- unfortunatelly, up to now we can prove only an inner bound: with probability 1

$$\mathcal{B}_{n(1-\epsilon)} \subset A(n),$$

$$\mathcal{B}_n = \left\{ (x, y) \in \mathcal{C}_2 : \frac{|x|}{k} + \left(\frac{|y|}{l}\right)^{1/2} \le n^{1/3} \right\}$$

**Outer bound**: someone in the audience [May '11]:  $A(n) \subset \mathcal{B}_{n(1+\epsilon)}$  ?



Theorem (Huss - S. '10)

Let A(n) be the internal DLA cluster after n random walks start at the origin of  $C_2$ . Then, for all  $\epsilon > 0$ , we have with probability 1

 $\mathcal{B}_{n(1-\epsilon)} \subset A(n)$ , for all sufficiently large n.

Proof sketch.

- Inspired by the Lawler-Bramson-Griffeath argument.
- By Borel-Cantelli Lemma, a sufficient condition for proving the inner bound is

$$\sum_{n\geq n_0}\sum_{z\in \mathcal{B}_{n(1-\varepsilon)}}\mathbb{P}[z\notin A_n]<\infty.$$



#### Fix $z \in \mathcal{B}_n$ . We want an upper bound for $\mathbb{P}[z \notin A(n)]$ . $\blacksquare$ *n* random walks start at $o \Rightarrow A(n)$ .



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- the summands of L are dependent.
- bound *L* by a sum of i.i.d rv's
- only the walks that leave *A*(*i*) in *B<sub>n</sub>* contribute to *L*: start one new walk from every point in *B<sub>n</sub>* where the cluster is left.



#### Proof sketch

• enlarge the index set to all of  $\mathcal{B}_n$ 



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- $\tilde{L} = \#$  of new walks that hit z before leaving  $\mathcal{B}_n$ . Then

$$L \leq \tilde{L},$$

and

$$\mathbb{P}[z \notin A(n)] \leq \mathbb{P}[M = L] \leq \mathbb{P}[M \leq \tilde{L}].$$



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$$\mathbb{P}[z \notin A(n)] \leq \mathbb{P}[M = L] \leq \mathbb{P}[M \leq \tilde{L}].$$

we show that

$$\sum_{n\geq n_{\epsilon}}\sum_{z\in\mathcal{B}_{n(1-\epsilon)}}\mathbb{P}[M\leq\tilde{L}]\leq 4\sum_{n\geq n_{\epsilon}}n\exp\{-C_{\epsilon}n^{2/3}\}<\infty,$$

which proves the inner bound

$$\mathbb{P}\big[\mathcal{B}_{n(1-\epsilon)}\subset A_n, \text{ for all } n\geq n_\epsilon\big]=1.$$



### Outlook

#### For internal DLA on the comb lattice $\ensuremath{\mathcal{C}}$ , an outer bound

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In all previous proofs, internal DLA clusters A(n) grow uniformly, and this makes easy the study of random walks. In our case, this is violated, since the sets  $\mathcal{B}_n$  grow with rate  $n^{1/3}$  in the x-direction and with rate  $n^{2/3}$  in the y-direction.



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The study of the harmonic measure and the Green function stopped on sets  $\mathcal{B}_n$  may help.



# Sierpinski carpet

Graphical Sierpinski carpet in dimension 2: infinite graph derived from the Sierpinski carpet - a fractal created from the unit square in  $\mathbb{R}^2$  by dividing it into 9 equal squares of which the one in the center is deleted. The same procedure is then repeated recursively to the remaining 8 squares.





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Figure: IDLA clusters on the Sierpinski carpet for 10000 particles.



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Figure: IDLA clusters on the Sierpinski carpet for 20000 particles.



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Figure: IDLA clusters on the Sierpinski carpet for 30000 particles.



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Figure: IDLA clusters on the Sierpinski carpet for 40000 particles.



Figure: IDLA clusters on the Sierpinski carpet for 50000 particles.



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Figure: IDLA clusters on the Sierpinski carpet for 60000 particles.



Figure: IDLA clusters on the Sierpinski carpet for 70000 particles.



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Figure: IDLA clusters on the Sierpinski carpet for 80000 particles.



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Figure: IDLA clusters on the Sierpinski carpet for 90000 particles.



Figure: IDLA clusters on the Sierpinski carpet for 100000 particles.



Figure: IDLA clusters on the Sierpinski carpet for 110000 particles.



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Figure: IDLA clusters on the Sierpinski carpet for 120000 particles.



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Figure: IDLA clusters on the Sierpinski carpet for 130000 particles.



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Figure: IDLA clusters on the Sierpinski carpet for 140000 particles.



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Figure: IDLA clusters on the Sierpinski carpet for 150000 particles.


#### **Problems:**

the internal DLA cluster does not seem to have an unique scaling limit.



- the internal DLA cluster does not seem to have an unique scaling limit.
- simulations suggest that may be a whole family of scaling limits.



- the internal DLA cluster does not seem to have an unique scaling limit.
- simulations sugest that may be a whole family of scaling limits.
- this scaling limits seem to have a fractal boundary.



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*n* = 275



















*n* = 200



















## References

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# Thank your for your interest!

