Internal Diffusion Limited Aggregation

(joint work with Wilfried Huss)

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May 24, 2011

Miniworkshop, Siegen University

Graz University of Technology

Outline

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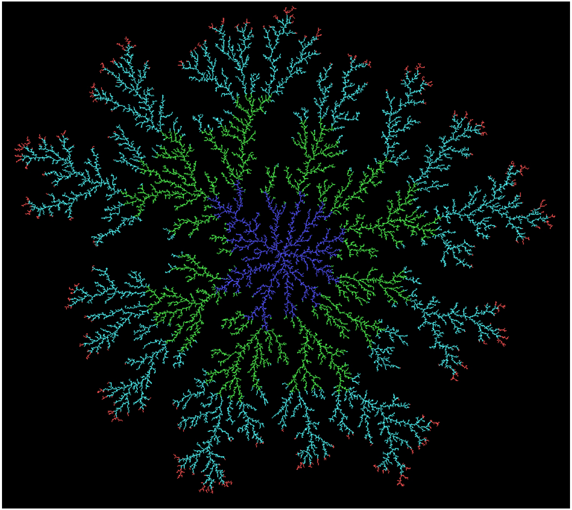
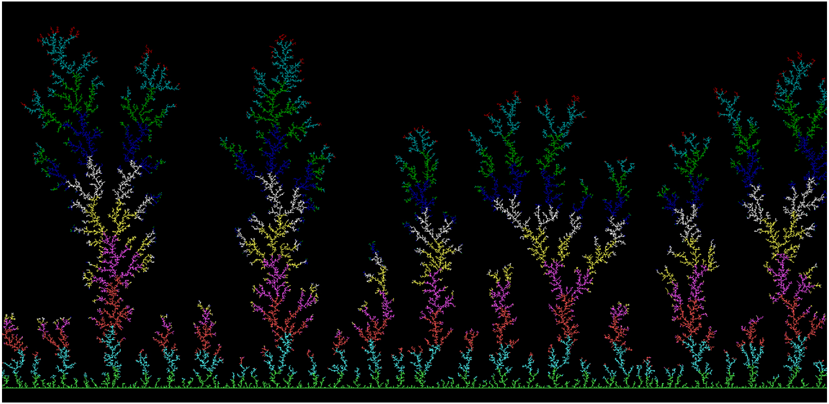
- DLA tends to build irregularities.

DLA 33000 particles, center initially occupied



Different colors = different arrival times of the random walkers.

Random walkers sticking to a straight line



Internal DLA

Random growth model, **internal version of DLA**, which contrary to DLA, tends to eliminate irregularities.

- Let G be some infinite graph, and o some fixed vertex.
- one by one, particles perform discrete-time random walks.
- each particle starts from o and moves until it reaches a site unoccupied previously, where it stops.
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$$A(n+1) = A(n) \cup \{X^n(\tau_n)\},$$

  where $X^1, X^2, \ldots$ are independent random walks starting at o, and

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- **Main question: limiting shape of $A(n)$ as $n \rightarrow \infty$?**

DLA and internal DLA : comparison

DLA: tendrils result from the fact that the particles tend to hit first the neighbourhood of extreme sites in the occupied cluster $\Rightarrow$ fractal structure.

internal DLA: particles diffusing through the interior of the occupied cluster are most likely to stop at unoccupied sites that are closest to 0 $\Rightarrow$ $A(n)$ tends to eliminate irregularities $\Rightarrow$ expected to grow like an expanding ball on a regular graph.

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Theorem (Lawler-Bramson-Griffeath '92)

For simple random walks on $\mathbb{Z}^d$, $d \geq 2$, *the limiting shape of internal DLA is a ball:* $\forall \epsilon > 0$, *with probability* 1*:*

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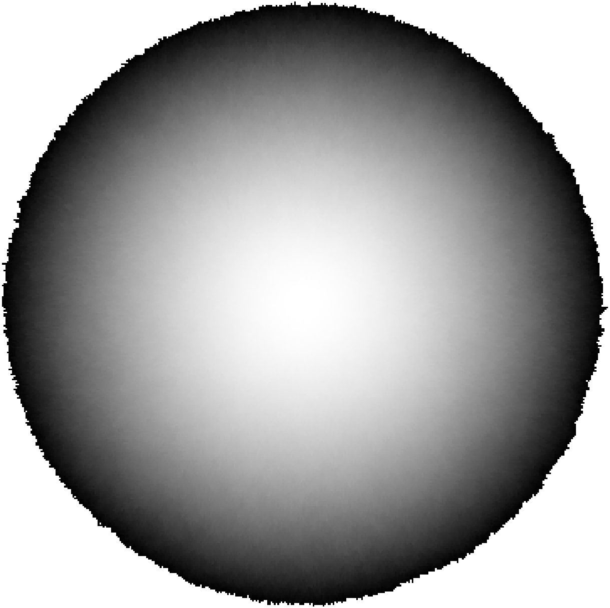
Question: what about fluctuations for internal DLA, i.e. how smooth the surface formed by internal DLA can be?

The cluster A(n), n=100000



Fluctuations on $\mathbb{Z}^d$: history

- Lawler ['95]: with probability 1

$$B_{r - r^{1/3}\log^2 r} \subset A(\pi r^2) \subset B_{r + r^{1/3}\log^4 r}$$

Can the errors be of order $o(n^\alpha)$, for $\alpha < 1/3$? Indeed there are only logarithmic fluctuations.

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- For $d \geq 3$: Jerison-Levine-Sheffield ['10] and Asselah-Gaudillière ['10]

$$B_{r-C\sqrt{\log r}} \subset A(\omega_d r^d) \subset B_{r+C\sqrt{\log^2 r}}, \text{eventually,}$$

for a constant C depending only on d. ω_d is the volume of the d-dimensional Euclidean ball of radius 1.

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- *(Huss '07) strongly reversible, uniformly irreducible random walks on non-amenable graphs.*

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Counter examples:

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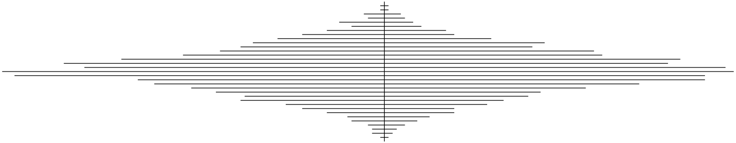
- Simple random walk on the comb



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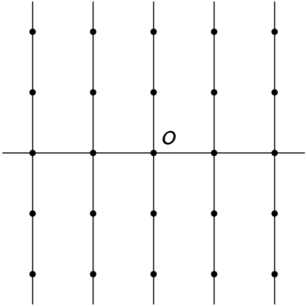

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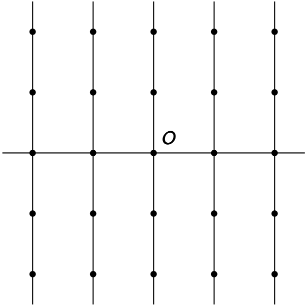

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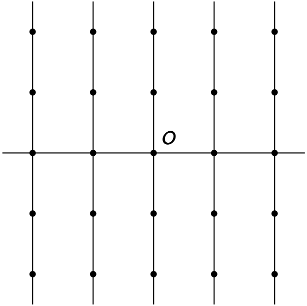

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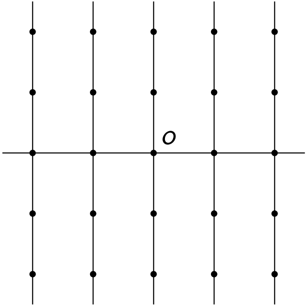

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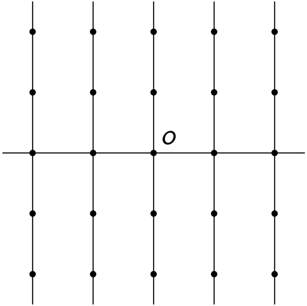

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What is the limiting shape A(n)?

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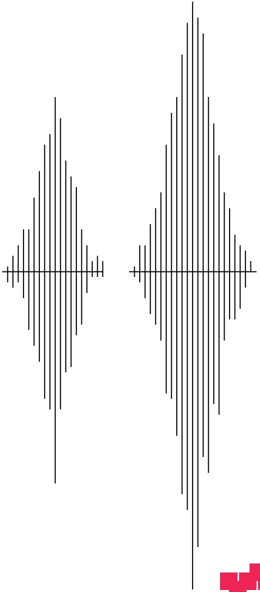
- $A(n)$ for $n = 500$ and $n = 1000$.



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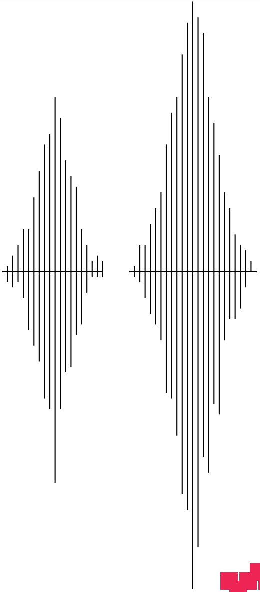

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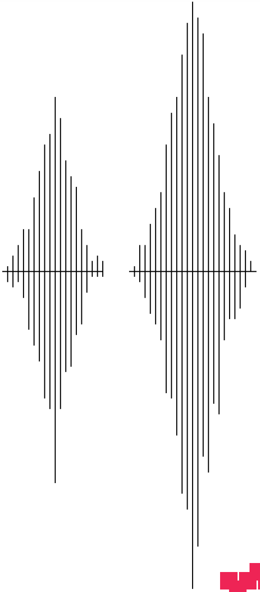

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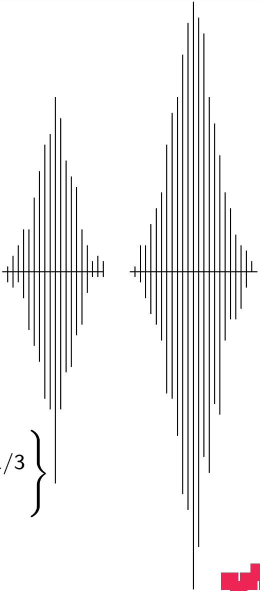
- want to prove that this is the limiting shape of internal DLA on the comb $\mathcal{C}_2$.
- unfortunatelly, up to now we can prove only an **inner bound**: with probability 1

$$\mathcal{B}_{n(1-\epsilon)} \subset A(n),$$

$$\mathcal{B}_n = \left\{ (x,y) \in \mathcal{C}_2 : \frac{|x|}{k} + \left(\frac{|y|}{l} \right)^{1/2} \leq n^{1/3} \right\}$$



Outer bound: someone in the audience [May '11]: $A(n) \subset \mathcal{B}_{n(1+\epsilon)}$?

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Internal DLA on the comb

Theorem (Huss - S. '10)

Let $A(n)$ be the internal DLA cluster after n random walks start at the origin of $\mathcal{C}_2$. Then, for all $\epsilon > 0$, we have with probability 1

$$\mathcal{B}_{n(1-\epsilon)} \subset A(n), \text{ for all sufficiently large } n.$$

Proof sketch.

- Inspired by the Lawler-Bramson-Griffeath argument.
- By Borel-Cantelli Lemma, a sufficient condition for proving the inner bound is

$$\sum_{n \geq n_0} \sum_{z \in \mathcal{B}_{n(1-\varepsilon)}} \mathbb{P}[z \notin A_n] < \infty.$$

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Fix $z \in \mathcal{B}_n$. We want an upper bound for $\mathbb{P}\big[z \notin A(n)\big]$.

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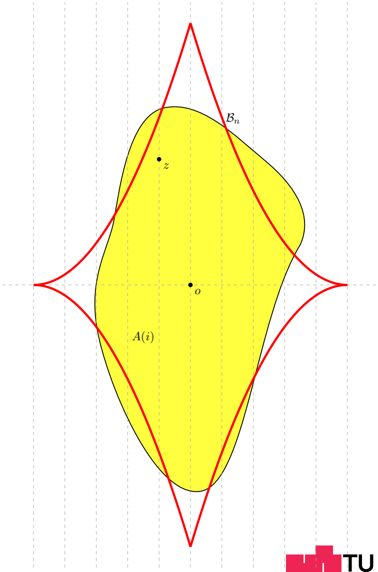

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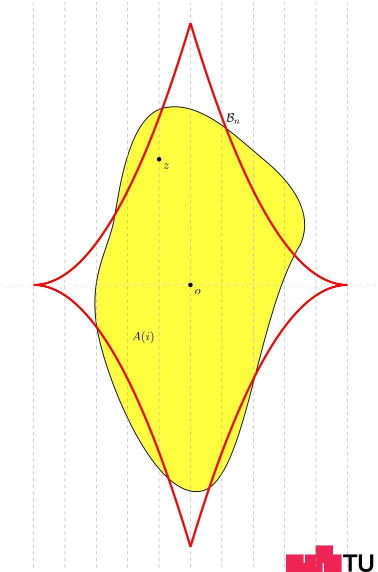

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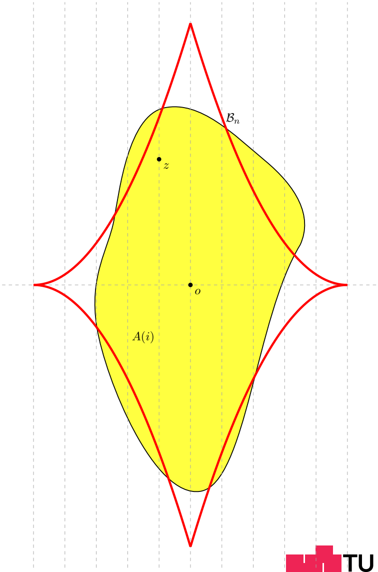

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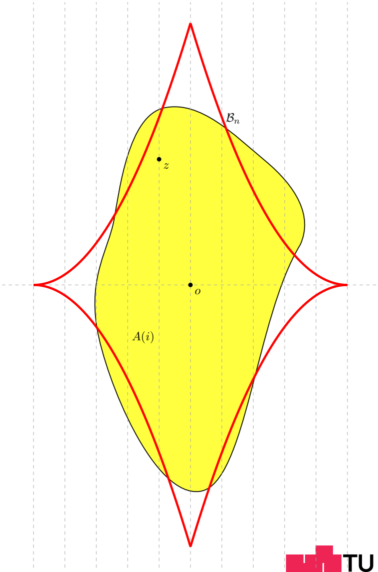

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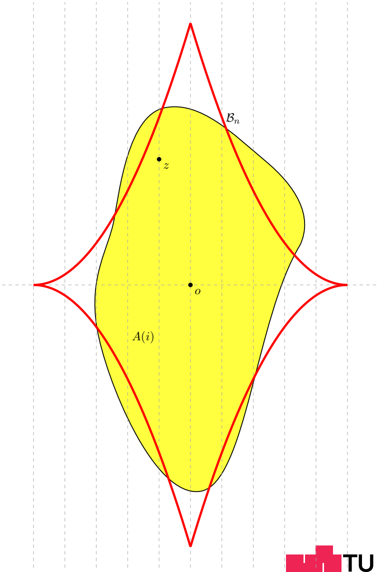

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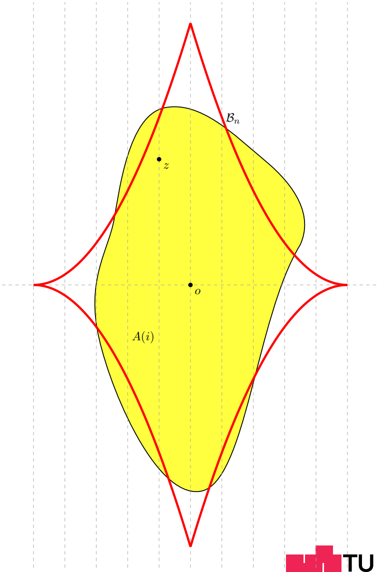

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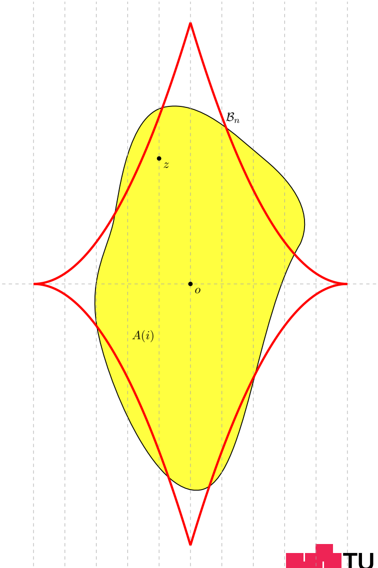

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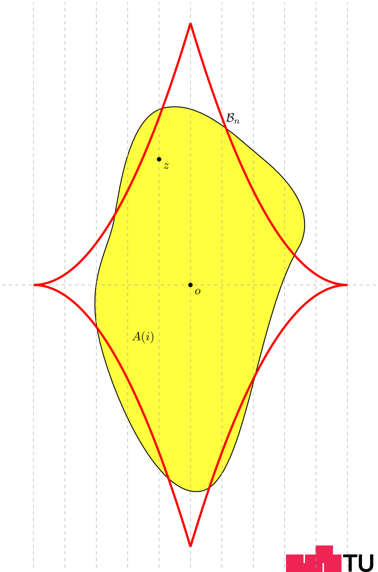
- the summands of L are dependent.
- bound L by a sum of i.i.d rv's
- only the walks that leave $A(i)$ in $\mathcal{B}_n$ contribute to L: start one new walk from every point in $\mathcal{B}_n$ where the cluster is left.



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- we show that

$$\sum_{n \geq n_\epsilon} \sum_{z \in \mathcal{B}_{n(1-\epsilon)}} \mathbb{P}[M \leq \tilde{L}] \leq 4 \sum_{n \geq n_\epsilon} n \exp\{-C_\epsilon n^{2/3}\} < \infty,$$

which proves the inner bound

$$\mathbb{P}\left[\mathcal{B}_{n(1-\epsilon)} \subset A_n, \text{ for all } n \geq n_\epsilon\right] = 1.$$

Outlook

For internal DLA on the comb lattice $\mathcal{C}$, an outer bound

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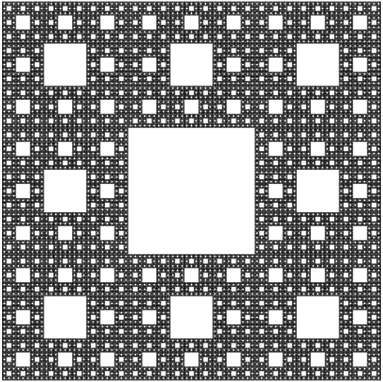
The study of the harmonic measure and the Green function stopped on sets $\mathcal{B}_n$ may help.

Sierpinski carpet

Graphical Sierpinski carpet in dimension 2: infinite graph derived from the Sierpinski carpet - a fractal created from the unit square in $\mathbb{R}^2$ by dividing it into 9 equal squares of which the one in the center is deleted. The same procedure is then repeated recursively to the remaining 8 squares.



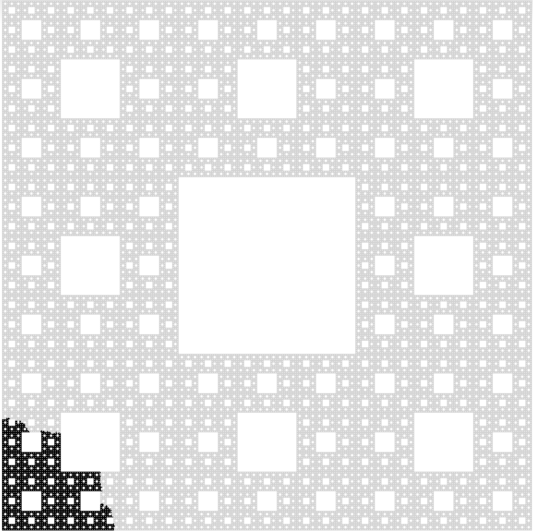


Figure: IDLA clusters on the Sierpinski carpet for 10000 particles.

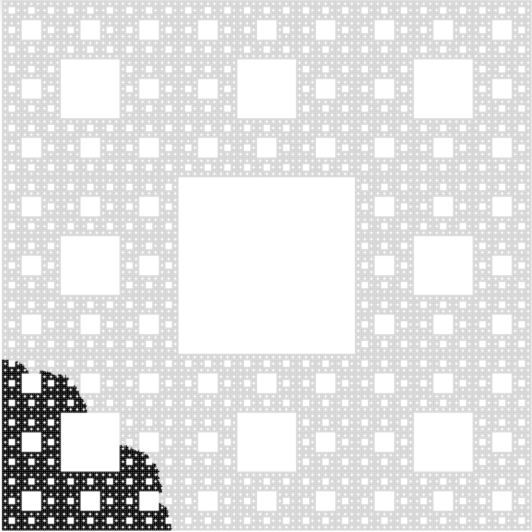


Figure: IDLA clusters on the Sierpinski carpet for 20000 particles.

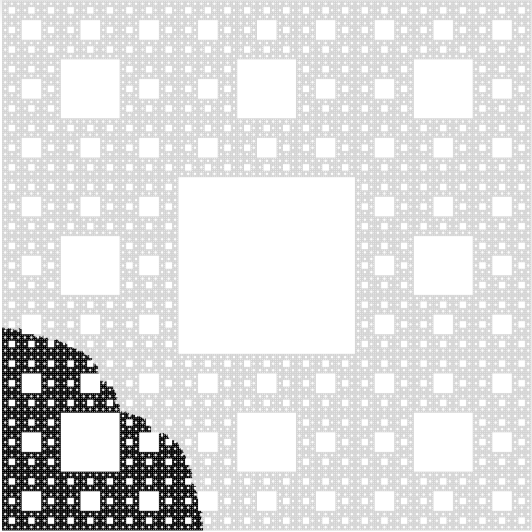


Figure: IDLA clusters on the Sierpinski carpet for 30000 particles.

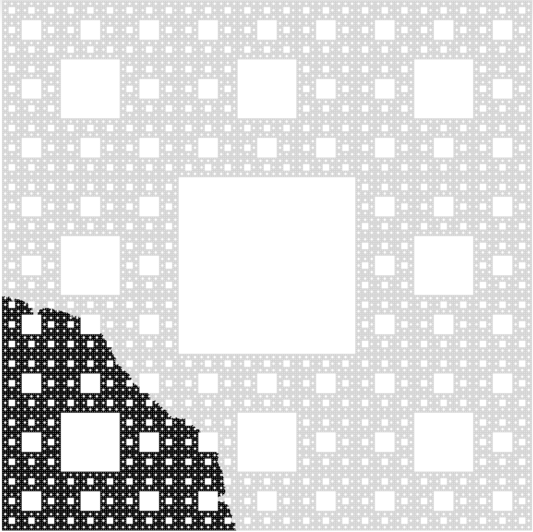


Figure: IDLA clusters on the Sierpinski carpet for 40000 particles.

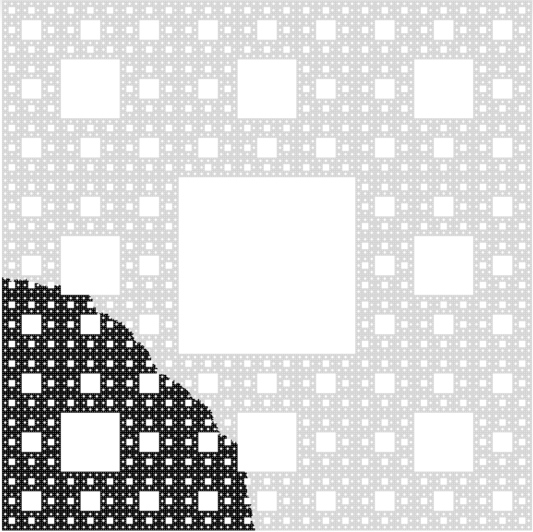


Figure: IDLA clusters on the Sierpinski carpet for 50000 particles.

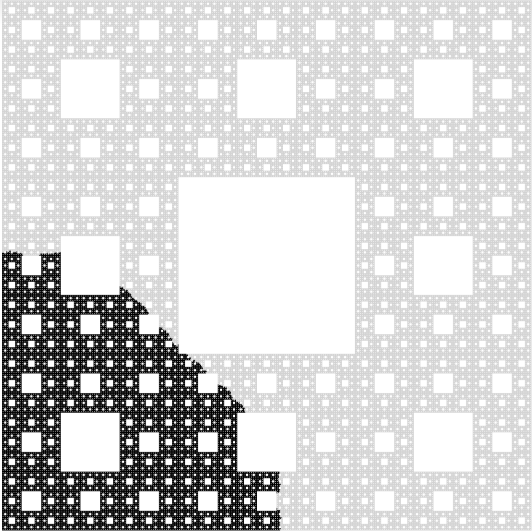


Figure: IDLA clusters on the Sierpinski carpet for 60000 particles.

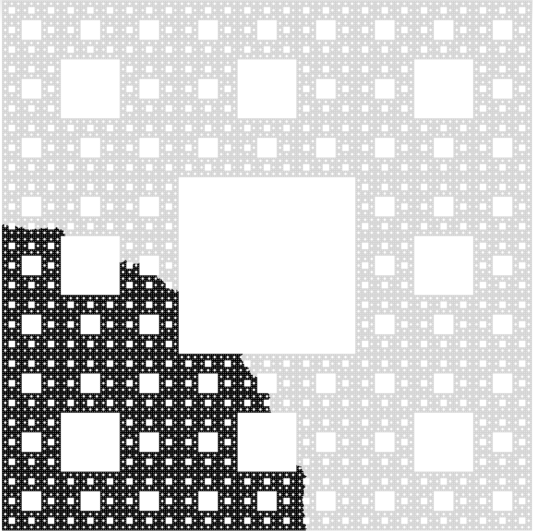


Figure: IDLA clusters on the Sierpinski carpet for 70000 particles.

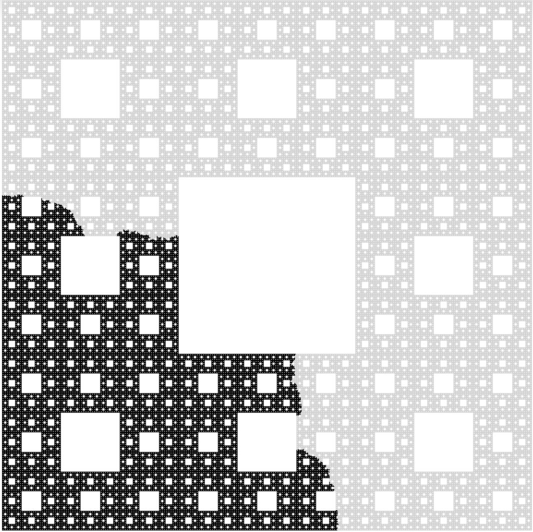


Figure: IDLA clusters on the Sierpinski carpet for 80000 particles.

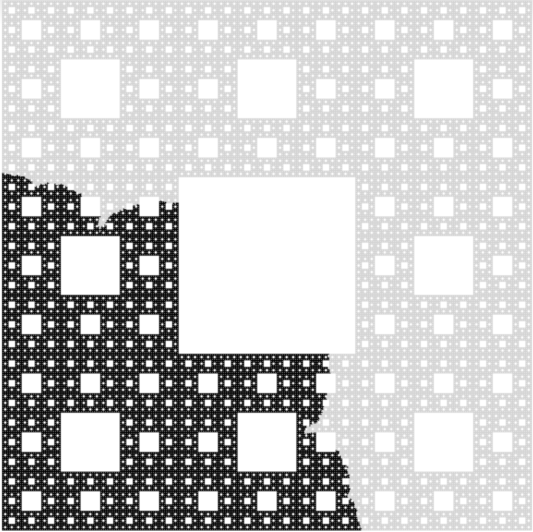


Figure: IDLA clusters on the Sierpinski carpet for 90000 particles.

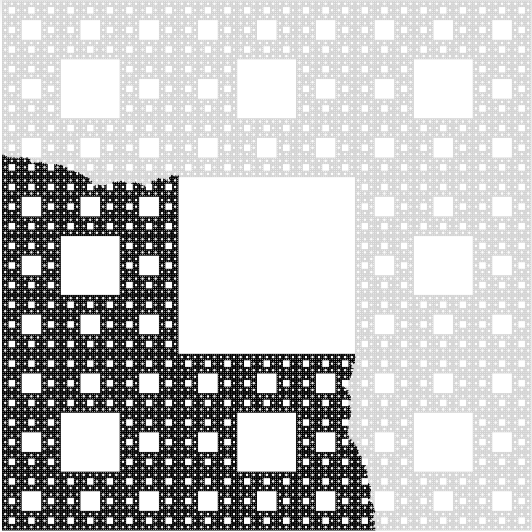


Figure: IDLA clusters on the Sierpinski carpet for 100000 particles.

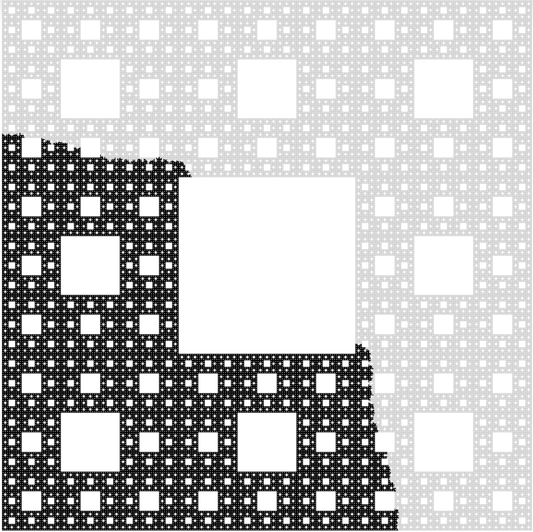


Figure: IDLA clusters on the Sierpinski carpet for 110000 particles.

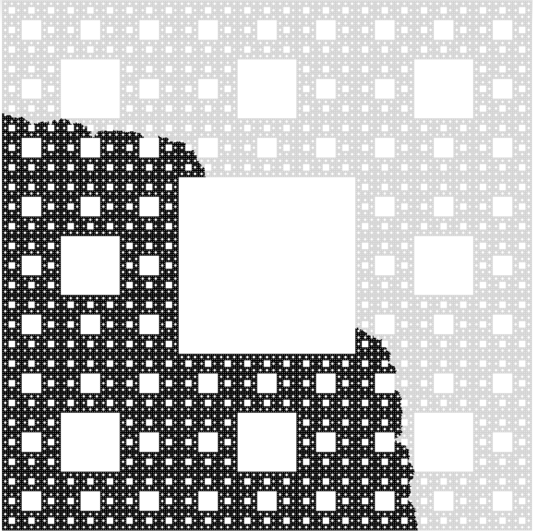


Figure: IDLA clusters on the Sierpinski carpet for 120000 particles.

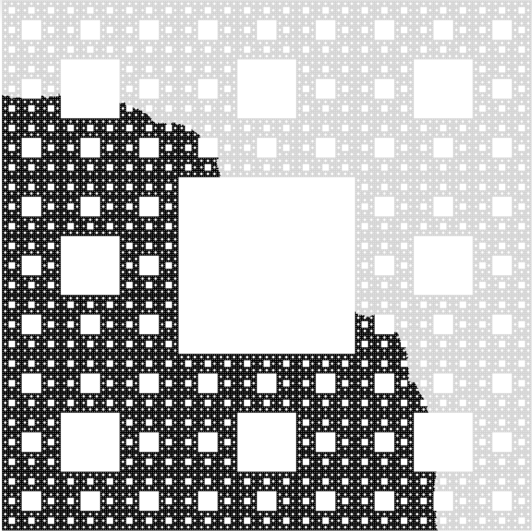


Figure: IDLA clusters on the Sierpinski carpet for 130000 particles.

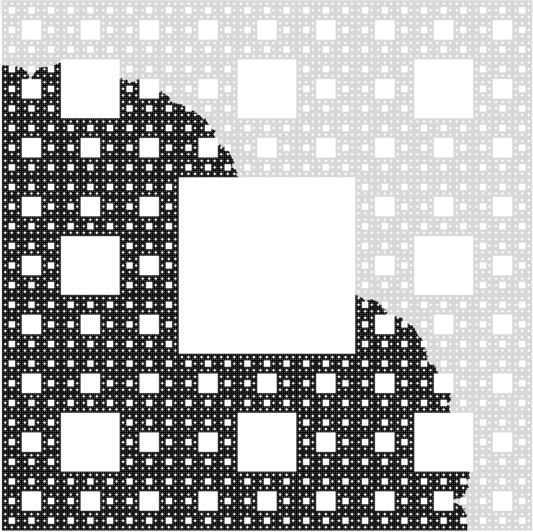


Figure: IDLA clusters on the Sierpinski carpet for 140000 particles.

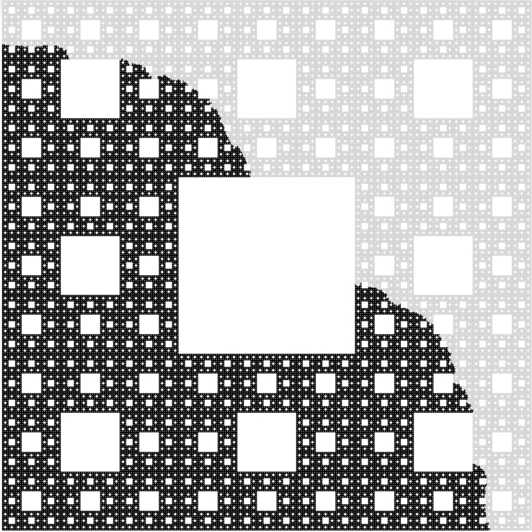


Figure: IDLA clusters on the Sierpinski carpet for 150000 particles.

Internal DLA on Sierpinski carpet

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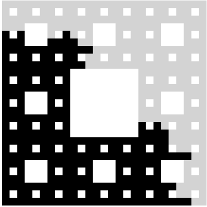
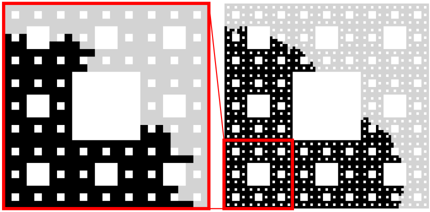
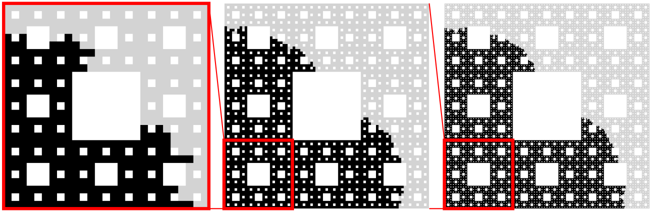


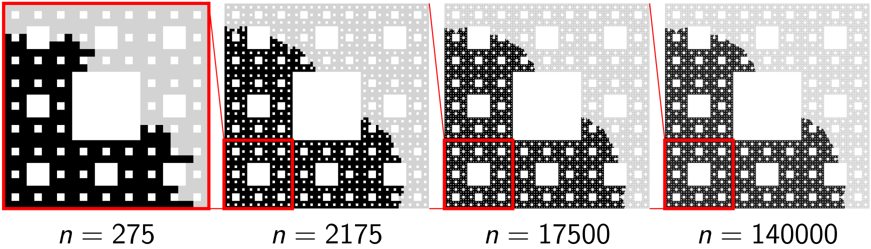
$n = 275$

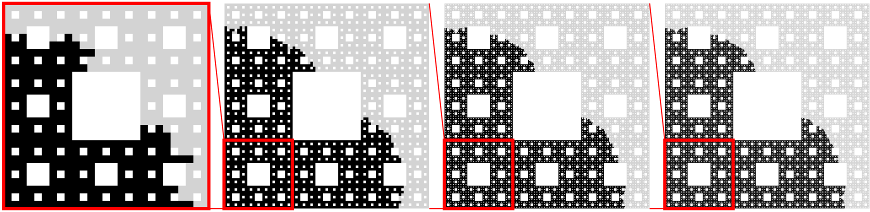


$n = 275$ $n = 2175$



$n = 275$ $n = 2175$ $n = 17500$


$n = 275$
$n = 2175$
$n = 17500$
$n = 140000$



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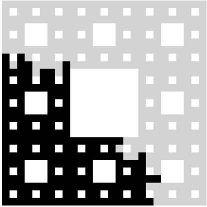
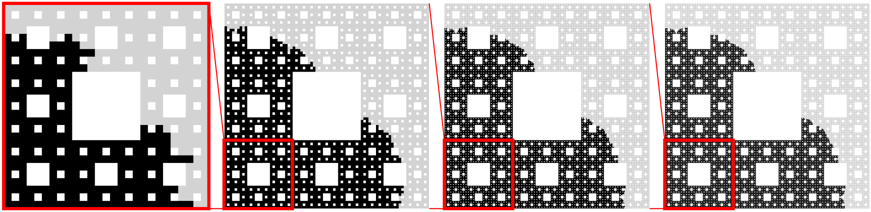
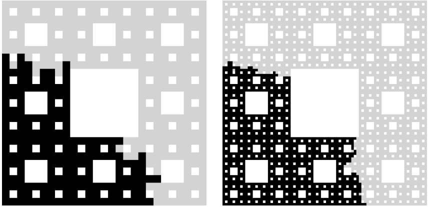


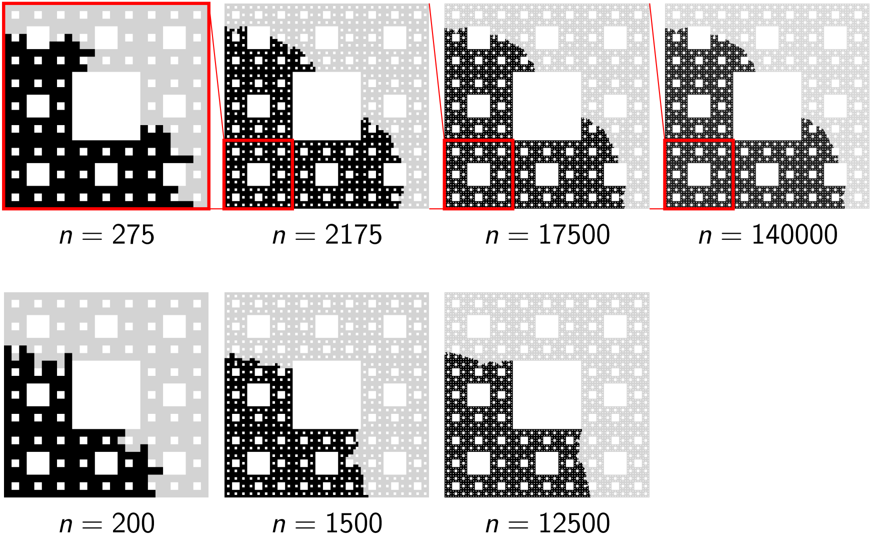
$n = 200$



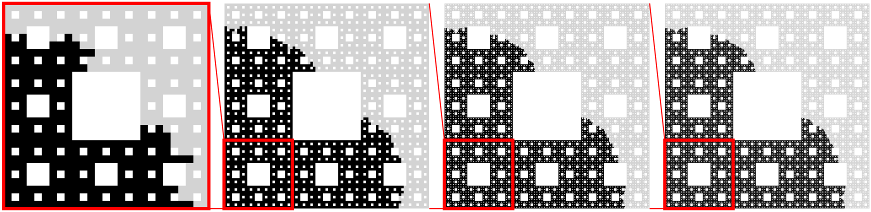
$n = 275$ $n = 2175$ $n = 17500$ $n = 140000$



$n = 200$ $n = 1500$



$n = 275$
$n = 2175$
$n = 17500$
$n = 140000$
$n = 200$
$n = 1500$
$n = 12500$



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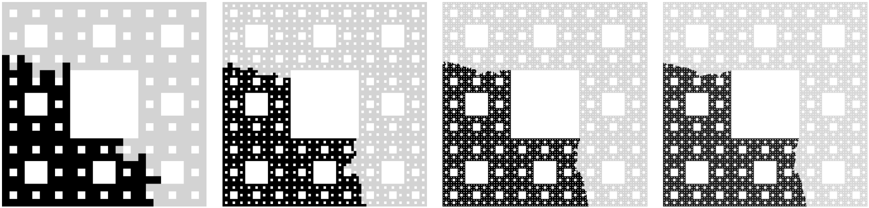

$n = 200$ $n = 1500$ $n = 12500$ $n = 98000$

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Thank your for your interest!