Internal aggregation on the comb lattice

Joint work with Wilfried Huss (2010)

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  Growth rule: Let $A(0) = \{o\}$ and define

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where $X^1, X^2, \ldots$ are independent random walks, and

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- Main question: limiting shape of $A(n)$ as $n \to \infty$?

Simple random walk on $\mathbb{Z}^d$

Theorem (Lawler-Bramson-Griffeath '92)

The limiting shape is a ball: $\forall \epsilon > 0$, *with probability* 1*:*

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Theorem (Lawler '95)

Improvement of the previous result: for $f(n) = n^{1/3} \log^4 n$

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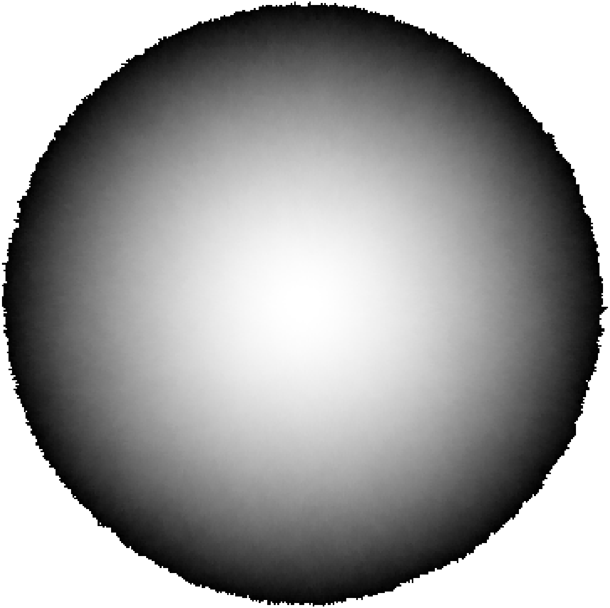
Theorem (Asselah-Gaudilliere '10)

New result on the order of fluctuations $f(n) = n^{1/(d+1)} \log n$.

The cluster A(n), n=100000



What about other walks on $\mathbb{Z}^2$?

Modify the transition probabilities on the axes:

- Steps toward the origin along the x- and y-axes are reflected away from the origin. So

$$\mathbb{P}\big((x,0),(x+1,0)\big) = \frac{1}{2}$$
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Theorem (Kager-Levine '09)

The limiting shape is a diamond, that is, with probability 1

$$\mathcal{D}_{n-4\sqrt{n\log n}} \subset A(d_n) \subset \mathcal{D}_{n+20\sqrt{n\log n}},$$

and $d_n = \#\mathcal{D}_n = 2n(n+1)+1.$

Diamond aggregation: the limiting shape $\mathcal{D}_n$

Diamond of radius n is

$$\mathcal{D}_n = \{z \in \mathbb{Z}^2 : ||z|| \leq n\},$$

with $z = (x, y) \in \mathbb{Z}^2$ its norm is $||(x, y)|| = |x| + |y|$.

Internal DLA on the comb lattice

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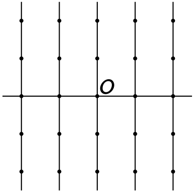

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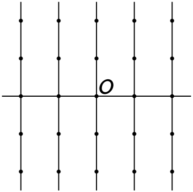

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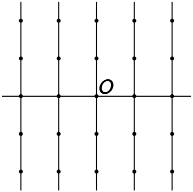

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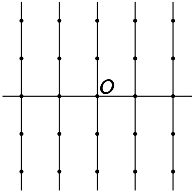

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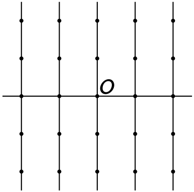
- Recall: divisible sandpile cluster S_n on $\mathcal{C}_2$ is given by



$$\mathcal{B}_{n-c} \subset S_n \subset \mathcal{B}_{n+c},$$

$$\mathcal{B}_n = \left\{ (x,y) \in \mathcal{C}_2 : \frac{|x|}{k} + \left(\frac{|y|}{l} \right)^{1/2} \leq n^{1/3} \right\},$$

with $k = \left(\frac{3}{2}\right)^{2/3}$ and $l = \frac{1}{2}\left(\frac{3}{2}\right)^{1/3}$.

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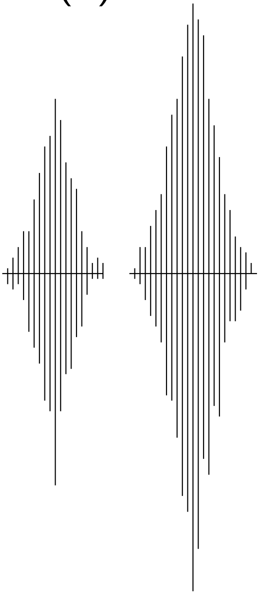

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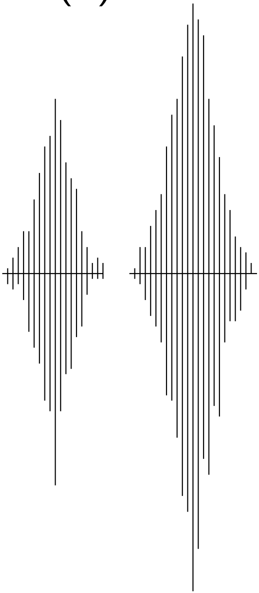

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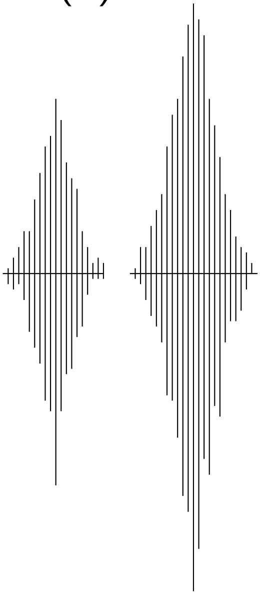

Inner bound [Huss-Sava '10]:

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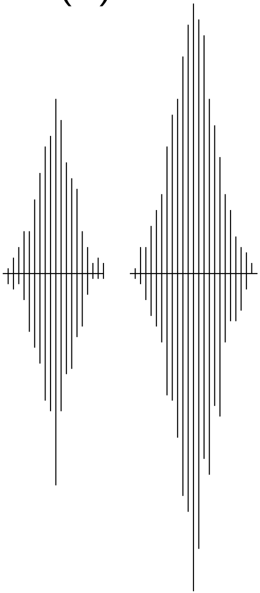

Inner bound [Huss-Sava '10]:

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Outer bound [Someone in the audience May '10 ?]: $A(n) \subset \mathcal{B}_{n(1+\varepsilon)}$, for all $\varepsilon > 0$. EXERCISE!

The inner bound

Theorem (Huss-Sava '10)

Let $A(n)$ *be the IDLA cluster after* n *particles start at the origin of* $\mathcal{C}_2$*. Then, for all* $\varepsilon > 0$*, we have with probability* 1 *that*

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Proof sketch.

- Inspired by the Lawler-Bramson-Griffeath argument.
- By Borel-Cantelli Lemma, a sufficient condition for proving the inner bound is

$$\sum_{n \geq n_0} \sum_{z \in \mathcal{B}_{n(1-\varepsilon)}} \mathbb{P}[z \notin A_n] < \infty.$$

Proof sketch: the inner bound

Fix $z \in \mathcal{B}_n$. We want an upper bound for $\mathbb{P}\big[z \notin A(n)\big]$.

- Among the first n particles, let

$$M = \#\text{ of particles that visit } z \text{ before leaving } \mathcal{B}_n,$$
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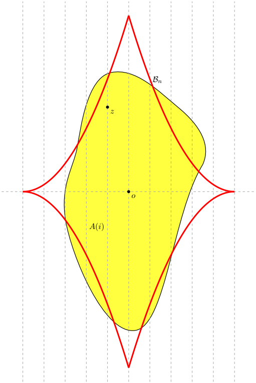

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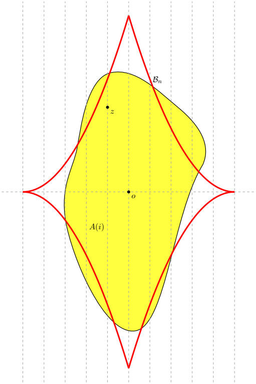

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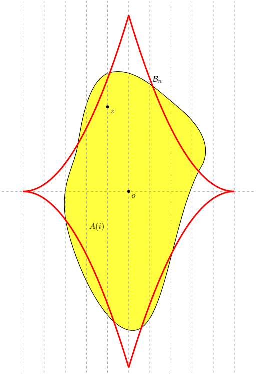

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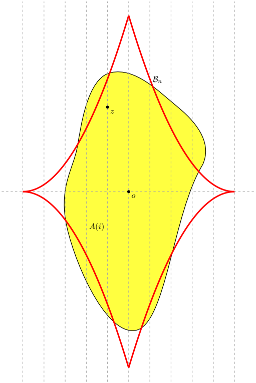
- Main difficulty: the summands of L are dependent.



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- Show that $\mathbb{E}[M] > \mathbb{E}[\tilde{L}]$ and use Large deviation estimate for sum of i.i.d indicators to bound $\mathbb{P}[M \leq \tilde{L}]$.

Dirichlet problem $\Rightarrow \mathbb{E}[M] > \mathbb{E}[\tilde{L}]$

- Let
$$f_n(z) = \frac{G_n(z,z)}{d(z)}\mathbb{E}[M - \tilde{L}]$$
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- Compute the Laplace $\triangle f_n$. We get
$$\begin{cases} \triangle f_n(z) & = \frac{1}{d(z)}\big(1 - n \cdot \delta_o(z)\big), \text{ for } z \in \mathcal{B}_n \\ f_n & = 0, \text{ on } \partial\mathcal{B}_n \end{cases}$$

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- Therefore $\mathbb{E}[M] > \mathbb{E}[\tilde{L}]$.

Large deviation estimate

Lemma

If N is a sum of finitely many independent indicator RV's,

$$\mathbb{P}\big[|N-\mathbb{E}N|>\lambda\mathbb{E}N\big]<2e^{-c_\lambda \mathbb{E}N},$$

$\forall\lambda>0$, *where c_λ is a constant depending only on λ.*

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- Apply it for $\tilde{L}$ and M, and bound $\mathbb{P}[M \leq \tilde{L}]$. Then

$$\mathbb{P}[\tilde{L} > (1+\lambda)\mathbb{E}\tilde{L}] < 2e^{-c_\lambda \mathbb{E}\tilde{L}}$$
$$\mathbb{P}[M < (1-\lambda)\mathbb{E}M] < 2e^{-c_\lambda \mathbb{E}M}$$

- Problem: choose $\lambda > 0$ s.t $(1+\lambda)\mathbb{E}\tilde{L} \leq (1-\lambda)\mathbb{E}M$:

$$0 < \lambda \leq \frac{\mathbb{E}[M - \tilde{L}]}{\mathbb{E}[M + \tilde{L}]}.$$

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- Is this Dirichlet problem explicitely solvable?

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$$g_n(z) = \frac{G_n(z,z)}{d(z)} \mathbb{E}[M + \tilde{L}].$$
- Then choose $0 < \lambda \leq \dfrac{f_n}{g_n}$.
- To identifiy the maximal subset of $\mathcal{B}_n$ on which $f_n/g_n > 0$.
- We know explicitely f_n, we want to find g_n explicitely.
- Like before, we solve a Dirichlet problem for g_n.
$$\begin{cases} \triangle g_n(z) & = \frac{1}{d(z)}\big(-1 - n \cdot \delta_o(z)\big), \text{ for } z \in \mathcal{B}_n \\ g_n & = 0, \text{ on } \partial \mathcal{B}_n \end{cases}$$
- Is this Dirichlet problem explicitely solvable?
- With a lot of luck is solvable, and we have g_n.

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- g_n is not a nice function, but it doesn't matter.

Last step of the proof

- Then $\frac{f_n}{g_n}$ is decreasing and, and for all $\varepsilon > 0$, $\forall n \geq n_\varepsilon$:

$$\min\left\{\frac{f_n(z)}{g_n(z)} : z \in \mathcal{B}_{n(1-\varepsilon)}\right\} = \frac{\varepsilon}{4-\varepsilon},$$

that is, $0 < \frac{\varepsilon}{4-\varepsilon} \leq \frac{f_n}{g_n}$ on $\mathcal{B}_{n(1-\varepsilon)}$. Choose $\lambda = \varepsilon/4$.

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$$\begin{aligned}
\mathbb{P}[M \leq \tilde{L}] &\leq \mathbb{P}\big[M < \big(1-\lambda\big)\mathbb{E}[\tilde{M}]\big] + \mathbb{P}\big[\tilde{L} > \big(1+\lambda\big)\mathbb{E}[\tilde{L}]\big] \\
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- The function $g_n - f_n > \mathcal{O}(n^{4/3})$, but we need $G_n(z,z)$, for all $z \in \mathcal{B}_{n(1-\varepsilon)}$.

- Estimate $G_n(z,z)$ with the stopped Green function for SRW on $\mathbb{Z}$, upon exiting a finite interval. We get

$$\sum_{n \geq n_\varepsilon} \sum_{z \in \mathcal{B}_{n(1-\varepsilon)}} \mathbb{P}[z \notin A_n] \leq \sum_{n \geq n_\varepsilon} \sum_{z \in \mathcal{B}_{n(1-\varepsilon)}} \exp\{-c_\lambda n^{1/3}\} < \infty,$$

and this implies the inner bound

$$\mathbb{P}\Big[\mathcal{B}_{n(1-\varepsilon)} \subset A_n, \quad \text{for all sufficiently large } n\Big] = 1.$$

Thank you for your attention!

- G. Lawler, M. Bramson and D. Griffeath, Internal Diffusion Limited Aggregation, *Ann. Probab.* 20, no. 4 (1992), 2117–2140.
- L. Levine and Y. Peres, Scaling Limits for Internal Aggregation Models with Multiple Sources, `arXiv:0712.3378`.