Internal aggregation on the comb lattice
Joint work with Wilfried Huss (2010)

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Internal DLA

- Given a random walk on a state space $G$.
- Start with $n$ particles at the origin $o \in G$.
- Each particle walks until it finds an unoccupied site, stays there.
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- $A(n)$: the resulting random set of $n$ points in $G$. 

Growth rule:

Let $A(0) = \{o\}$ and define $A(n+1) = A(n) \cup \{X_{n}(\tau_{n})\}$, where $X_{1}, X_{2}, \ldots$ are independent random walks, and $\tau_{n} = \min \{t: X_{n}(t) \notin A(n)\}$.

Main question: limiting shape of $A(n)$ as $n \to \infty$?
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Simple random walk on $\mathbb{Z}^d$

Theorem (Lawler-Bramson-Griffeath ’92)

The \textit{limiting shape is a ball}: $\forall \epsilon > 0$, with probability 1:

$$B_n(1-\epsilon) \subset A(\pi n^2) \subset B_n(1+\epsilon), \text{ for all sufficiently large } n.$$
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Theorem (Lawler ’95)

*Improvement of the previous result:* for $f(n) = n^{1/3} \log^4 n$

$$B_{n-f(n)} \subset A(\pi n^2) \subset B_{n+f(n)}, \text{ for } n \text{ big enough.}$$
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Theorem (Asselah-Gaudilliere ’10)

*New result on the order of fluctuations* $f(n) = n^{1/(d+1)} \log n$. 

The cluster $A(n)$, $n=100000$
What about other walks on $\mathbb{Z}^2$?

Modify the transition probabilities on the axes:

- Steps **toward the origin** along the $x$- and $y$-axes are **reflected** away from the origin. So

\[
P((x, 0), (x + 1, 0)) = \frac{1}{2}
\]

\[
P((x, 0), (x, 1)) = \frac{1}{4}.
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- Off the axes, same as simple random walk.
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  \]
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- Off the axes, same as simple random walk.

**Theorem (Kager-Levine ’09)**

The **limiting shape is a diamond**, that is, with probability 1

$$
\mathcal{D}_{n-4\sqrt{n \log n}} \subset \mathcal{A}(d_n) \subset \mathcal{D}_{n+20\sqrt{n \log n}},
$$

and $d_n = \#\mathcal{D}_n = 2n(n + 1) + 1$. 
Diamond aggregation: the limiting shape $D_n$

Diamond of radius $n$ is

$$D_n = \{ z \in \mathbb{Z}^2 : ||z|| \leq n \},$$

with $z = (x, y) \in \mathbb{Z}^2$ its norm is $||(x, y)|| = |x| + |y|$. 
Consider the 2-dimensional comb $C_2$.

Recall: divisible sandpile cluster $S_n$ on $C_2$ is given by $B_n - c \subset S_n \subset B_n + c$, $B_n = \{ (x, y) \in C_2 : |x| k + |y| l \geq n^{1/3} \}$, with $k = \left( \frac{3}{2} \right)^{2/3}$ and $l = \frac{1}{2} \left( \frac{3}{2} \right)^{1/3}$.
Internal DLA on the comb lattice

- Consider the 2-dimensional comb $C_2$.
- Perform internal DLA with $n$ simple random walks starting at the origin $o = (0, 0) \in C_2$. 

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- Recall: divisible sandpile cluster $S_n$ on $C_2$ is given by

$$B_{n-c} \subset S_n \subset B_{n+c},$$

$$B_n = \left\{ (x, y) \in C_2 : \frac{|x|}{k} + \left( \frac{|y|}{l} \right)^{1/2} \leq n^{1/3} \right\},$$

with $k = \left( \frac{3}{2} \right)^{2/3}$ and $l = \frac{1}{2} \left( \frac{3}{2} \right)^{1/3}$. 
The internal DLA cluster $A(n)$

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**Inner bound** [Huss-Sava ’10]:

$$B_{n(1-\varepsilon)} \subset A(n), \text{ for all } \varepsilon > 0.$$
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**Inner bound** [Huss-Sava ’10]:

$$\mathcal{B}_{n(1-\varepsilon)} \subset A(n), \text{ for all } \varepsilon > 0.$$  

**Outer bound** [Someone in the audience May ’10 ?]: $A(n) \subset \mathcal{B}_{n(1+\varepsilon)}, \text{ for all } \varepsilon > 0$. **EXERCISE!**
The inner bound

Theorem (Huss-Sava ’10)

Let $A(n)$ be the IDLA cluster after $n$ particles start at the origin of $C_2$. Then, for all $\varepsilon > 0$, we have with probability 1 that

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Proof sketch.

- Inspired by the Lawler-Bramson-Griffeath argument.
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Theorem (Huss-Sava ’10)

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$$\mathcal{B}_{n(1-\varepsilon)} \subset A(n), \text{ for all sufficiently large } n.$$ 

Proof sketch.

- Inspired by the Lawler-Bramson-Griffeath argument.
- By Borel-Cantelli Lemma, a sufficient condition for proving the inner bound is

$$\sum_{n \geq n_0} \sum_{z \in \mathcal{B}_{n(1-\varepsilon)}} \mathbb{P}[z \notin A_n] < \infty.$$
Proof sketch: the inner bound

Fix $z \in B_n$. We want an upper bound for $\mathbb{P}[z \notin A(n)]$.

Among the first $n$ particles, let

$$M = \# \text{ of particles that visit } z \text{ before leaving } B_n,$$

$$L = \# \text{ of particles that visit } z \text{ after leaving } A(i),$$

while still in $B_n$, for all $1 \leq i \leq n.$
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- If $L < M$ then $z \in A(n)$ and

$$\{z \notin A(n)\} \subset \{M = L\}.$$
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- Both $L$ and $M$ are sums of indicator RV’s.
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- Both \( L \) and \( M \) are sums of indicator RV’s.

- **Main difficulty**: the summands of \( L \) are dependent.
Let us find some independence!

- Bound $L$ by a sum of i.i.d RV’s. How?
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- Bound $L$ by a sum of i.i.d RV’s. How?
- Start one new walk from every point in $B_n$, and let

\[ \tilde{L} = \# \text{ of new walks that hit } z \text{ before leaving } B_n. \]

Since at most one particle can attach to the cluster at a given site, $L \leq \tilde{L}$. 

\[ \mathbb{P}[z/ \in A(n)] \leq \mathbb{P}[M = L] \leq \mathbb{P}[M \leq \tilde{L}]. \]

Show that $E[M] > E[\tilde{L}]$ and use Large deviation estimate for sum of i.i.d indicators to bound $P[M \leq \tilde{L}]$. 
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- Show that $\mathbb{E}[M] > \mathbb{E}[\tilde{L}]$ and use Large deviation estimate for sum of i.i.d indicators to bound $\mathbb{P}[M \leq \tilde{L}]$. 
Dirichlet problem $\Rightarrow \mathbb{E}[\mathcal{M}] > \mathbb{E}[\tilde{\mathcal{L}}]$

Let

$$f_n(z) = \frac{G_n(z, z)}{d(z)} \mathbb{E}[\mathcal{M} - \tilde{\mathcal{L}}]$$

where $G_n$ is the Green function for SRW stopped on exiting $B_n$. 

But we have $u_n$ explicitly, and $u_n > 0$ on $B_n$.

Therefore $\mathbb{E}[\mathcal{M}] > \mathbb{E}[\tilde{\mathcal{L}}]$. 
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- Compute the Laplace $\triangle f_n$. We get

$$\begin{cases} 
\triangle f_n(z) = \frac{1}{d(z)}(1 - n \cdot \delta_o(z)), & \text{for } z \in B_n \\
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- But we have $u_n$ explicitly, and $u_n > 0$ on $B_n$.

- Therefore $\mathbb{E}[M] > \mathbb{E}[\tilde{L}]$. 
Large deviation estimate

Lemma
If $N$ is a sum of finitely many independent indicator RV’s,

$$
\mathbb{P}\left[|N - \mathbb{E}N| > \lambda \mathbb{E}N\right] < 2e^{-c_{\lambda} \mathbb{E}N},
$$

$\forall \lambda > 0$, where $c_{\lambda}$ is a constant depending only on $\lambda$. 

Apply it for $\tilde{L}$ and $M$, and bound $\mathbb{P}\left[M \leq \tilde{L}\right]$. Then

$$
\mathbb{P}\left[\tilde{L} > (1 + \lambda) \mathbb{E}\tilde{L}\right] < 2e^{-c_{\lambda} \mathbb{E}\tilde{L}}
$$

$$
\mathbb{P}\left[M < (1 - \lambda) \mathbb{E}M\right] < 2e^{-c_{\lambda} \mathbb{E}M},
$$

Problem: choose $\lambda > 0$ s.t

$$
(1 + \lambda) \mathbb{E}\tilde{L} \leq (1 - \lambda) \mathbb{E}M.
$$
**Large deviation estimate**

**Lemma**

*If* $N$ *is a sum of finitely many independent indicator RV’s,*

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\forall \lambda > 0, \text{ where } c_\lambda \text{ is a constant depending only on } \lambda.

- **Apply it for** $\tilde{L}$ *and* $M,$ *and bound* $\Pr [M \leq \tilde{L}].$ *Then*

  $$
  \Pr [\tilde{L} > (1 + \lambda) \mathbb{E}\tilde{L}] < 2e^{-c_\lambda \mathbb{E}\tilde{L}}
  \quad \Pr [M < (1 - \lambda) \mathbb{E}M] < 2e^{-c_\lambda \mathbb{E}M}
  $$

- **Problem:** *choose* $\lambda > 0 \text{ s.t } (1 + \lambda) \mathbb{E}\tilde{L} \leq (1 - \lambda) \mathbb{E}M:*

  $$
  0 < \lambda \leq \frac{\mathbb{E}[M - \tilde{L}]}{\mathbb{E}[M + \tilde{L}]}
  $$
How to choose $\lambda$ 

- Set now 

$$g_n(z) = \frac{G_n(z, z)}{d(z)} \mathbb{E}[M + \tilde{L}].$$
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  \[ g_n(z) = \frac{G_n(z, z)}{d(z)} \mathbb{E}[M + \tilde{L}] \cdot \]

- Then choose $0 < \lambda \leq \frac{f_n}{g_n}$.
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- To identify the maximal subset of $B_n$ on which $f_n/g_n > 0$. 
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- To identify the maximal subset of $\mathcal{B}_n$ on which $f_n/g_n > 0$.
- We know explicitly $f_n$, we want to find $g_n$ explicitly.
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- We know explicitly $f_n$, we want to find $g_n$ explicitly.

- Like before, we solve a Dirichlet problem for $g_n$.

\[
\begin{align*}
\Delta g_n(z) &= \frac{1}{d(z)}( - 1 - n \cdot \delta_o(z)), \quad \text{for } z \in \mathcal{B}_n \\
g_n &= 0, \quad \text{on } \partial \mathcal{B}_n
\end{align*}
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- $g_n$ is not a nice function, but it doesn’t matter.
Last step of the proof

Then \( \frac{f_n}{g_n} \) is decreasing and, and for all \( \varepsilon > 0 \), \( \forall n \geq n_\varepsilon \):

\[
\min \left\{ \frac{f_n(z)}{g_n(z)} : z \in B_n(1-\varepsilon) \right\} = \frac{\varepsilon}{4 - \varepsilon},
\]

that is, \( 0 < \frac{\varepsilon}{4 - \varepsilon} \leq \frac{f_n}{g_n} \) on \( B_n(1-\varepsilon) \). Choose \( \lambda = \varepsilon/4 \).
Last step of the proof

Then $\frac{f_n}{g_n}$ is decreasing and, and for all $\varepsilon > 0$, $\forall n \geq n_\varepsilon$:

$$\min\left\{ \frac{f_n(z)}{g_n(z)} : z \in B_{n(1-\varepsilon)} \right\} = \frac{\varepsilon}{4-\varepsilon},$$

that is, $0 < \frac{\varepsilon}{4-\varepsilon} \leq \frac{f_n}{g_n}$ on $B_{n(1-\varepsilon)}$. Choose $\lambda = \varepsilon/4$.

$$\mathbb{P}[M \leq \tilde{L}] \leq \mathbb{P}[M < (1 - \lambda)\mathbb{E}[\tilde{M}]] + \mathbb{P}[\tilde{L} > (1 + \lambda)\mathbb{E}[\tilde{L}]] < 2 \exp\{-c_\lambda \mathbb{E}[M]\} + 2 \exp\{-c_\lambda \mathbb{E}[\tilde{L}]\} < 4 \exp\{-c_\lambda \mathbb{E}[\tilde{L}]\} \leq 4 \exp\left\{-c_\lambda \frac{g_n(z) - f_n(z)}{G_n(z, z)} \right\}.$$
Last step of the proof

- Then $\frac{f_n}{g_n}$ is decreasing and, and for all $\varepsilon > 0$, $\forall n \geq n_\varepsilon$:

  \[
  \min \left\{ \frac{f_n(z)}{g_n(z)} : z \in B_{n(1-\varepsilon)} \right\} = \frac{\varepsilon}{4 - \varepsilon},
  \]

  that is, $0 < \frac{\varepsilon}{4 - \varepsilon} \leq \frac{f_n}{g_n}$ on $B_{n(1-\varepsilon)}$. Choose $\lambda = \varepsilon/4$.

  \[
  \mathbb{P}[M \leq \tilde{L}] \leq \mathbb{P}[M < (1 - \lambda) \mathbb{E}[\tilde{M}]] + \mathbb{P}[\tilde{L} > (1 + \lambda) \mathbb{E}[\tilde{L}]]
  \leq 2 \exp\{-c_\lambda \mathbb{E}[M]\} + 2 \exp\{-c_\lambda \mathbb{E}[\tilde{L}]\}
  \leq 4 \exp\{-c_\lambda \mathbb{E}[\tilde{L}]\} \leq 4 \exp\{-c_\lambda \frac{g_n(z) - f_n(z)}{G_n(z, z)}\}.
  \]

- The function $g_n - f_n > O(n^{4/3})$, but we need $G_n(z, z)$, for all $z \in B_{n(1-\varepsilon)}$. 
Estimate $G_n(z, z)$ with the stopped Green function for SRW on $\mathbb{Z}$, upon exiting a finite interval. We get

$$\sum_{n \geq n_\varepsilon} \sum_{z \in B_n(1-\varepsilon)} \mathbb{P}[z \notin A_n] \leq \sum_{n \geq n_\varepsilon} \sum_{z \in B_n(1-\varepsilon)} \exp\{-c_\lambda n^{1/3}\} < \infty,$$

and this implies the inner bound

$$\mathbb{P}[B_{n(1-\varepsilon)} \subset A_n, \text{ for all sufficiently large } n] = 1.$$
Thank you for your attention!

L. Levine and Y. Peres, Scaling Limits for Internal Aggregation Models with Multiple Sources, arXiv:0712.3378.