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Languages associated with graphs - entropy sensitivity via Markov chains

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1 Introduction

- Let $\mathbf{\Sigma}$ be a finite alphabet and $\mathbf{\Sigma}^*$ the set of all finite words over $\mathbf{\Sigma}$.
- A language L over $\mathbf{\Sigma}$ is a subset of $\mathbf{\Sigma}^*$. Growth or entropy of L:

$$\mathsf{h}(L) = \limsup_{n\to\infty} \frac{1}{n} \log\big|\{w \in L:\ |w| = n\}\big|.$$

- All our languages are infinite. For finite, non-empty set $F \subset \mathbf{\Sigma}^*$, $F = \{\text{subwords of elements of } L\}$, let

$$L^F = \{w \in L:\ \text{no } v \in F \text{ is a subword of } w\}.$$

- Associate with infinite, directed, graphs, a class of languages L.
- **Question:** under which conditions, for a class of languages associated with infinite graphs, is

$$\mathsf{h}(L) < \mathsf{h}(L^F), \quad \text{strictly?}$$

- If this holds for any set F of forbidden words, then L is called growth sensitive.

2 Languages on graphs

- Basic object: oriented, labeled graph (X, E, l) with edges labeled by elements of a finite alphabet $\mathbf{\Sigma}$, each edge $e \in E$ is of the form $e = (x, a, y)$; multiple edges and loops are allowed.
- The label $l(\pi)$ of a path $\pi = e_1 e_2 \dots e_n$ is $l(\pi) = l(e_1)l(e_2)\dots l(e_n)$.
- Let $\Pi_{x,y}$ be the set of all paths π from x to y in X.
- With X we associate the language

$$L_{x,y} = \{l(\pi) \in \mathbf{\Sigma}^* : \pi \in \Pi_{x,y}\}, \text{ where } x, y \in X.$$

- Question: Is this language growth-sensitive? For which class of graphs?
- Answer: Yes, for uniformly connected and fully deterministic graphs.
- (X, E, l) is determinstic if for every $x \in X$ and $a \in \mathbf{\Sigma}$, there is at most one edge with initial point x and label a, and fully determinstic if there is exactly one edge with label a outgoing from x.
- Entropy of a graph: $\mathsf{h}(X) = \mathsf{h}(X, E, \ell) = \sup_{x,y\in X} \mathsf{h}(L_{x,y})$.

Example of a language associated with a graph. Let $\mathbf{\Sigma} = \{a, b\}$ be an alphabet and consider the following graph X with label alphabet $\mathbf{\Sigma}$. Then $L_{x,y}$ is the set of all labeles of paths from x to y.

$$L_{x,y} = \{b^2, abb, (ba)^i b^2, aa^j b(ab)^k, \dots\}$$

$\forall i, j, k \geq 0$.

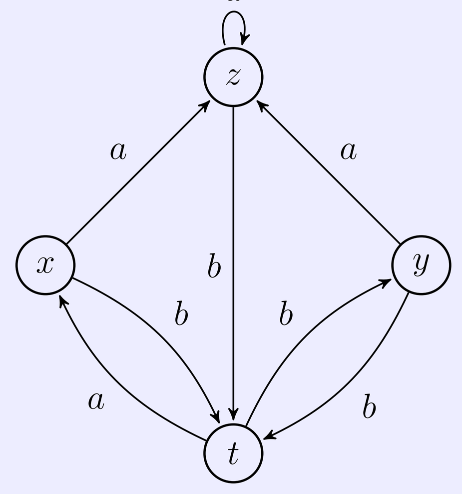

Figure: A fully deterministic graph.

3 Results

Theorem 3.1 (W. Huss, E. Sava, W. Woess '09)**.** *Suppose that (X, E, l) is uniformly connected and deterministic with label alphabet $\mathbf{\Sigma}$. Let $F \subset \mathbf{\Sigma}^+$ be a finite, non-empty set which is relatively dense in X. Then*

$$\sup_{x,y\in X} \mathsf{h}(L^F_{x,y}) < \mathsf{h}(X) \quad \textit{strictly.}$$

Corollary 3.2 (W. Huss, E. Sava, W. Woess '09)**.** *If (X, E, l) is uniformly connected and fully deterministic then $L_{x,y}$ is growth-sensitive for all $x, y \in X$.*

Proof: consider Markov chains with forbidden transitions over X.

4 Markov chains

- Equip the graph X with transition probabilites: to each edge $e = (x, a, y)$ associate $p(e)$ s.t. $\sum_{e\in E:\, e^- = x} p(e) \leq 1$ for every $x \in X$.
- Consider the Markov chain over X with one-step transition probabilities

$$p(x, y) = \sum_{a\in\Sigma:\ (x,a,y)\in E} p(x, a, y)\,.$$

- $p^{(n)}(x, y)$: the probability that the particle starting at x is at y at time n.
- Let $F \subset \mathbf{\Sigma}^+$ as sequence of forbidden transitions: we restrict the motion of the particle, and $p^{(n)}_F(x, y)$ is the pb that the particle does not use any word in F. Now consider the spectral radius

$$\rho(P) = \limsup_{n\to\infty} p^{(n)}(x, y)^{1/n}$$

and the following quantity

$$\rho_{x,y}(P_F) = \limsup_{n\to\infty} p^{(n)}_F(x, y)^{1/n}, \quad x, y \in X\,.$$

The proof of Theorem [3.1] is based on a comparison of $\rho(P)$ with $\rho(P_F)$. The correspondence $\mathsf{h}(X) = \log(\rho(P) \cdot |\Sigma|)$ between the entropy and spectral redius is needed.

5 Application to Schreier graphs

Corollary 5.1. *The word problem of the pair (G, K) with respect to any semigroup presentation ψ is growth sensitive with respect to forbidding an arbitrary non-empty subset $F \subset \mathbf{\Sigma}^*$.*

Example Let $G = \{1, t\}$ be the group of order two and $K = \{1\}$ the trivial subgroup. Let $\mathbf{\Sigma} = \{a\}$ and consider the presentation $\psi : \mathbf{\Sigma} \to G$ such that $\psi(a) = t$. Then the word problem of (G, K) w. r. t the presentation ψ is $L(G, K, \psi) = \{a^{2n} : n \geq 0\}$.

[1] T. Ceccherini-Silberstein, W. Woess, *Growth-sensitivity of context-free languages.* Theoret. Comput. Sci. **307** (2003) 103–116.
[2] T. Ceccherini-Silberstein, W. Woess, *Context-free pairs of groups I - Context-free pairs and graphs.* in preparation.
[3] W. Huss, E. Sava, W. Woess, *Entropy sensitivity of languages defined by infinite automata, via Markov chains with forbidden transitions.* preprint, 2009.

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