Lamplighter Random Walks

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  - randomizes the lamp at x
  - picks uniformly some neighbour x' of x
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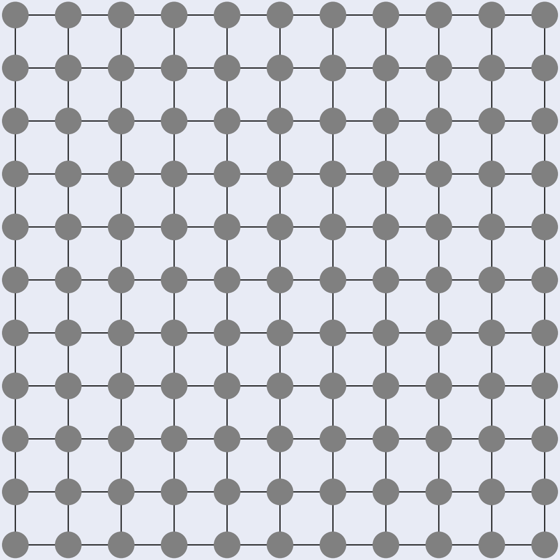
- Labelling configuration of lamps together with the position of the walker have a structure of graph known as lamplighter graph or wreath product, denoted by

$$G^{\diamond} = \{0, 1\} \wr G.$$

Lamplighter Graphs



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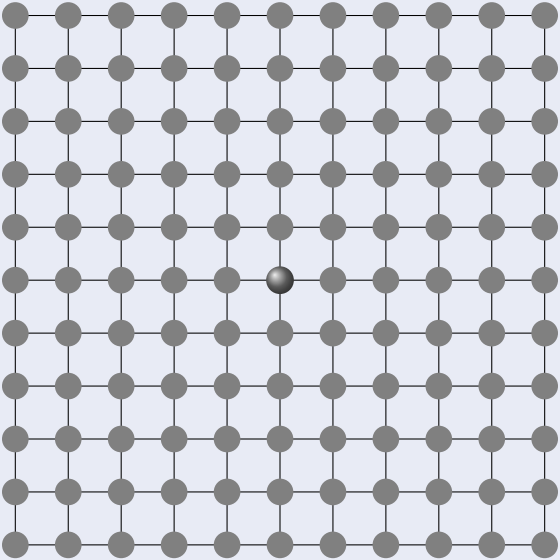

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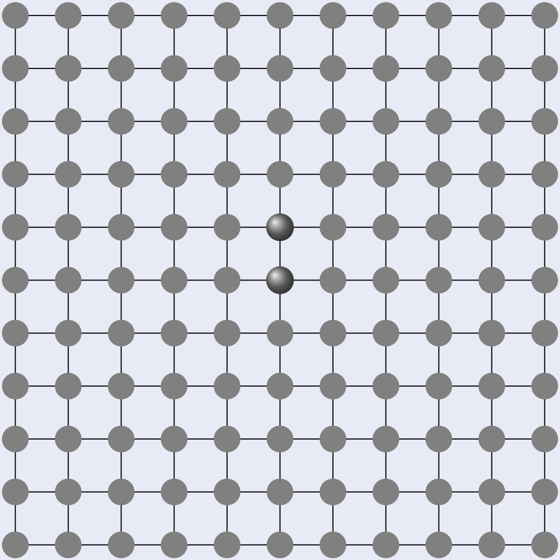

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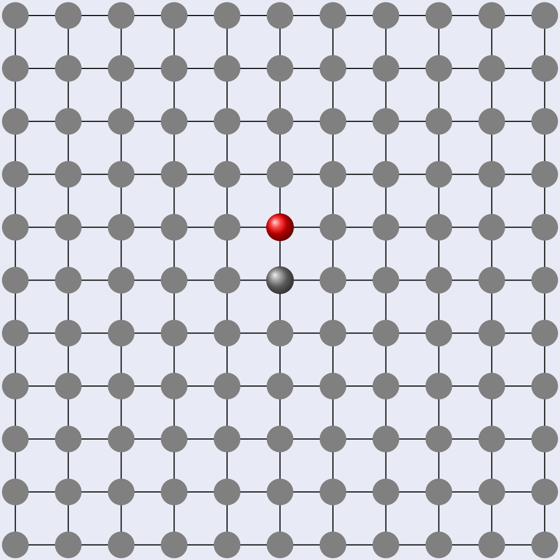

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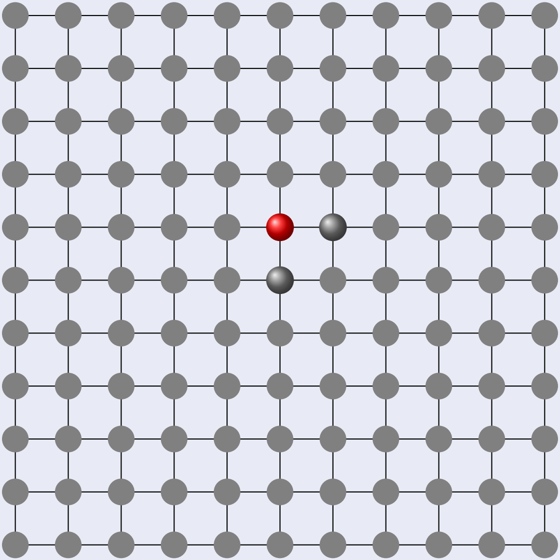

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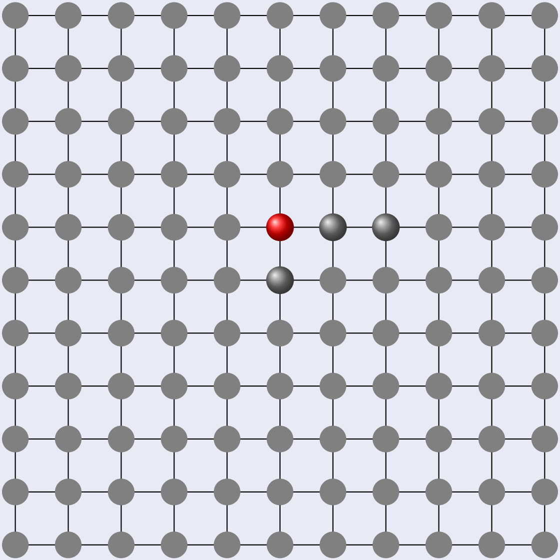

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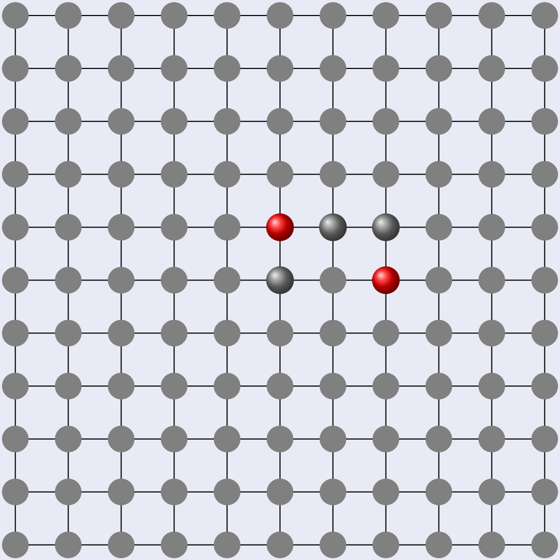

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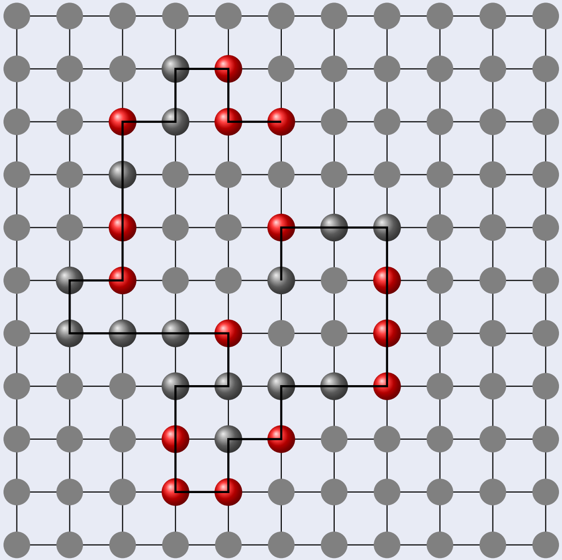

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- for $x \in \mathsf{G}$, $\eta(x) =$ the status of the lamp sitting at x
- lamplighter graph $\mathsf{G}^\diamond = \{0, 1\}^G \times \mathsf{G}$ has vertices of the form (η, x).
- neighbourhood relation in $\mathsf{G}^\diamond$: $(\eta, x) \sim (\eta', x')$ if either

$$x \sim x' \text{ and } \eta = \eta',$$

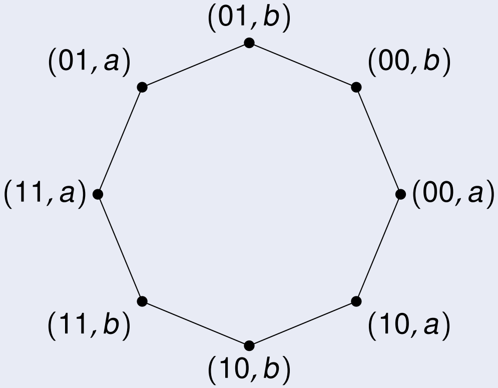
which corresponds to the lamplighter moving between x and x' without changing the lamp, or

$$\eta(y) = \eta'(y), \text{ for all } y \neq x, \text{ and } x = x'$$

which corresponds to the lamplighter not moving, but he might have changed the lamp at x.

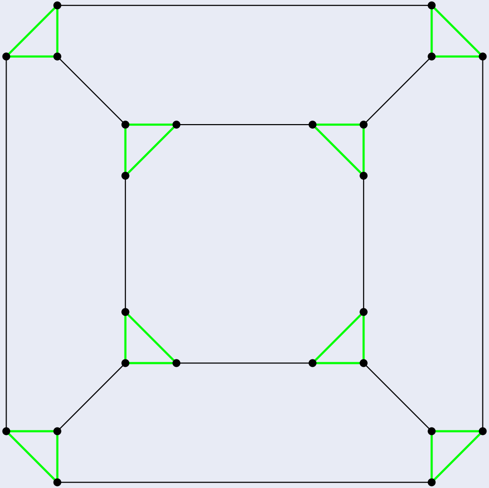
Example

$\mathbb{Z}_2^{\diamond} = \{0, 1\} \wr \mathbb{Z}_2$ has 8 vertices and it looks like below. If $\mathbb{Z}_2 = \{a, b\}$, the set of all lamp configurations on two vertices is $\{00, 01, 10, 11\}$.



Example

$\mathbb{Z}_3^{\diamond} = \{0, 1\} \wr \mathbb{Z}_3$ has 24 vertices and it looks like this:



Random Walks

Start with a random walk on G, construct one on $G^\diamond$. Basic models:

1. Walk or Switch: at each step the lamplighter tosses a coin
   - if head comes up then he walks, leaving the lamps unchanged.
   - if tail comes up, then he modifies the lamp at the current position, without moving.
2. Switch-Walk-Switch: if the lamplighter stands at x, then
   - he first randomizes the lamp at x
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We shall focus on Switch-Walk-Switch model. For this, let

- μ a Simple Random Walk on G: $\mu(x,y) = \frac{1}{d(x)}$
- ν a measure causing the lamplighter to randomize the lamp

$$\nu(0,0) = \nu(0,1) = \frac{1}{2} \text{ and } \nu(1,1) = \nu(1,0) = \frac{1}{2}.$$

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Interesting questions:

- speed or rate of escape of LRW: $\lim \frac{d(Z_n, Z_0)}{n}$ and $\lim \frac{d(X_n, X_0)}{n}$.
- long-term behaviour of the return probabilities $\mu^{\diamond^{(n)}}\big((\eta, x), (\eta, x)\big)$.
- convergence to the boundary, Poisson and Martin boundary.
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Important: the behaviour of LRW depends strongly on the structure of the base graph G and the random walk X_n on it.

Return Probabilities

For $\mathbb{Z}^d$, there are good estimates:

- Revelle ['03], studied LRW on $\mathbb{Z}$ and obtained very good asymptotics:

$$\mu^{\diamond(n)} \approx c_1 n^{1/6} \exp -c_2 n^{1/3}.$$

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For general graphs G and LRW on them:

$$\mu^{\diamond(n)}\big((\eta, x), (\eta, x)\big) = \mathbb{E}[2^{-R_n} \mathbf{1}_{\{X_n = x\}}],$$

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Question: How does R_n behave for random walks on graphs (or groups) with exponential growth, for instance on trees? Good estimates for R_n are needed in order to get asymptotics for $\mu^{\diamond(n)}$, on other base graphs, different from $\mathbb{Z}^d$.

Poisson Boundary

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In all cases, it is proven the a.s. convergence of LRW paths to boundaries associated with $G^\diamond$, and the Poisson boundary is the space of limit configurations of lamps with are switched on.

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- Erschler ['01]: the rate of escape of $G^\diamond$ is zero iff the random walk on G is recurrent.
- Gilch ['08]: the LRW on $G^\diamond$ escapes faster to infinity than the original random walk on G.

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Thank your for your interest!