The Poisson boundary of lamplighter random walks on general graphs

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Lamplighter graphs and random walks

- Let X be an infinite, locally finite, connected graph and $o \in X$ (can be viewed as the root).
- Lamp at each vertex: states 0 (switched off) and 1 (switched on). We identify the set $\{0, 1, \}$ with the finite group $\mathbb{Z}/2\mathbb{Z}$, such that 0 corresponds to the group identity and write $\mathbb{Z}_2 := \mathbb{Z}/2\mathbb{Z}$.
- A lamplighter person starts in o and performs a random walk on X: makes random moves and / or changes the state of the lamp at current position (or nearby).
- Configuration on X: a function $\eta : X \to \mathbb{Z}_2$. Write $\hat{\mathcal{C}} = \{\eta : X \to \mathbb{Z}_2\}$ for the set of all configurations and define $\mathcal{C} \subset \hat{\mathcal{C}}$ as the set of all finitely supported configurations, where a configuration is said to have finite support if the set $supp(\eta) = \{x \in X : \eta(x) \neq 0\}$ is finite.
- Denote by $\mathbf{0}$ the configuration which corresponds to "all lamps switched off", i.e. $\mathbf{0}(x) = 0$ for all $x \in X$. Also, $\mathcal{C}$ becomes a group with point-wise addition modulo 2.
- The process described above evolves on the state space $G = \mathbb{Z}_2 \wr X = \mathcal{C} \times X$ (called the lamplighter graph). It is a Markov chain $Z_n = (Y_n, X_n)$ on G, where X_n is the random position of the lamplighter and Y_n the random configuration of the lamps at time n. We shall call Z_n the lamplighter random walk on G. We can also view Z_n as a random walk on a group. For this, let Γ be a finitely generated group which acts transitively on X by graph automorphisms, such that X is the Cayley graph of Γ. Consider now the *wreath product*

$$\mathcal{G} = \left(\sum_{x \in \Gamma} \mathbb{Z}_r\right) \rtimes \Gamma = \mathbb{Z}_r \wr \Gamma$$

of Γ with $\mathbb{Z}_r$. Every $x \in \Gamma$ acts on $\mathcal{C}$ by the translation T_x defined as $(T_x\eta)(y) = \eta(x^{-1}y), \forall y \in \Gamma$. A group operation on $\mathcal{G}$ is given by

$$(\eta, x)(\eta', x') = (\eta \oplus T_x\eta', xx'),$$

where $x, x' \in \Gamma, \eta, \eta' \in \mathcal{C}$. We shall call $\mathcal{G}$ together with this operation the *lamplighter group* over Γ. The lamplighter graph $G = \mathbb{Z}_r \wr X$ becomes a Cayley graph of $\mathcal{G}$.

- Let μ the law of the lamplighter random walk $Z_n = (Y_n, X_n)$ on $\mathcal{G}$ and $\tilde{\mu}$ the law of the base random walk X_n on Γ. Suppose that the lamplighter random walk has finite first moment ($\sum_{(\eta,x)\in G} d_G\left((\mathbf{0}, o), (\eta, x)\right) \mu\left((\eta, x)\right) < \infty$), where d_G a metric on G. .

Convergence to the geometric boundary

We introduce a class of boundaries for the base graph X and then, to every boundary element of X, we shall attach a configuration defined in a natural way.
Consider an extended space $\widehat{X} = X \cup \partial X$ (not necessarily compact) with ideal *boundary* ∂X, the set of points at infinity. We require that this space has "good" properties: it is a Hausdorff space with countable base of topology, the inclusion $X \hookrightarrow \widehat{X}$ is a homeomorphism, and X is open and dense in $\widehat{X}$. Suppose that the action of Γ on X extends to an action on $\widehat{X} = X \cup \partial X$ by homeomorphisms.
Basic assumptions: the law $\tilde{\mu}$ of the base random walk X_n has finite first moment, the RW X_n converges to a random element of ∂X , and the boundary ∂X of X has the property: whenever $(x_n), (y_n)$ are sequences in X such that

$$x_n \to \xi \in \partial X \text{ and } \frac{d_X(x_n, y_n)}{d_X(x_n, x_0)} \to 0 \text{ then } y_n \to \xi. \qquad \textbf{(CP)}$$

The space $\partial G = (\hat{\mathcal{C}} \times \widehat{X}) \setminus (\mathcal{C} \times X)$ is a natural boundary at infinity for the lamplighter graph G. Let us write $\widehat{G} = \hat{\mathcal{C}} \times \widehat{X}$. Define the set

$$\Omega = \bigcup_{\mathfrak{u} \in \partial X} \mathcal{C}_{\mathfrak{u}} \times \{\mathfrak{u}\}, \qquad (1)$$

where a finitely or infinitely supported configuration ζ is in $\mathcal{C}_{\mathfrak{u}}$ if and only if $supp(\zeta)$ is finite or else accumulates only at $\mathfrak{u}$.

Result

The result regarding the convergence of lamplighter random walks Z_n on general base graphs X is a generalization of Karlsson and Woess [7, Thm 2.2] (Lamplighter random walks on trees).

Theorem 1. *Let $Z_n = (Y_n, X_n)$ be a random walk with law μ on the group $\mathcal{G} = \mathbb{Z}_r \wr \Gamma \equiv \mathbb{Z}_r \wr X$ such that $supp(\mu)$ generates $\mathcal{G}$. If Ω is defined as in (1) and μ has finite first moment, then there exists an Ω-valued random variable $Z_\infty = (Y_\infty, X_\infty)$ such that $Z_n \to Z_\infty$ almost surely, for every starting point. Moreover the distribution of Z_∞ is a continuous measure on Ω.*

The Poisson boundary

Let ν be the distribution of Z_∞ on Ω (given the initial position $o \in X$ and the initial configuration $\mathbf{0}$). The measure ν is the *harmonic measure* for the random walk Z_n with law μ.
(Ω, ν) is a measure space which describes the behaviour of the LRW at infinity. How "*good*" is this space? Is this the Poisson boundary, that is, the finest model of a probability space at infinity of $\mathcal{C} \times X$ for distinguishing the possible limitting behaviour of (Z_n)? There are several equivalent definitions for the Poisson boundary. See [Kaimanovich and Vershik, 1983] for RW on discrete groups.
For proving the maximility of the measure space (Ω, ν) we use the Strip Criterion [Kaimanovich, 2000].

Proposition 2 (Strip Criterion). *Let μ a probability measure on $\mathcal{G}$ with finite first moment, and let (B, λ) and $(\widehat{B}, \widehat{\lambda})$ be a μ- and a $\widehat{\mu}$-boundary, respectively. Suppose that there exists a measurable $\mathcal{G}$-equivariant map S assigning to every pair of points $(b, \widehat{b}) \in B \times \widehat{B}$ a non-empty "strip" $S(b, \widehat{b}) \subset G$, such that, for the ball $B(id, n)$ of radius n in the metric of $\mathcal{G}$,*

$$\frac{1}{n} \log |S(b, \widehat{b}) \cap B(id, n)| \to 0 \text{ as } n \to \infty$$

for $\sigma \times \widehat{\sigma}$- almost every $(b, \widehat{b}) \in B \times \widehat{B}$. Then (B, λ) and $(\widehat{B}, \widehat{\lambda})$ are the Poisson boundaries of the random walks with law μ and $\widehat{\mu}$, respectively.

The half-space method

It is a general method for constructing the strip S as a subset of the lamplighter graph, with the properties required in Proposition 2. We shall apply it to some special cases.

- Suppose that the Basic assumptions hold.
- Suppose to have the extended space $\widehat{X} = X \cup \partial X$ as above and, in addition:
  1. The random walks X_n and $\widehat{X}_n$ on the graph X converge, a.s. to ∂X, with the hitting distributions μ_∞ and $\widehat{\mu}_\infty$, respectively.
  2. For $\mu_\infty \times \widehat{\mu}_\infty$-almost every pair $(\mathfrak{u} \times \mathfrak{v}) \in \partial X \times \partial X$, one has a strip $\mathfrak{s}(\mathfrak{u}, \mathfrak{v})$ which satisfies the properties from the Proposition 2. That is, it is a subset of X, it is Γ-equivariant, and it has subexponential growth
  $$\frac{1}{n} \log |\mathfrak{s}(\mathfrak{u}, \mathfrak{v}) \cap B(o, n)| \to 0, \text{ as } n \to \infty, \qquad (2)$$
  where $B(o, n) = \{x \in X : d_X(o, x) \leq n\}$.
  3. For every $x \in \mathfrak{s}(\mathfrak{u}, \mathfrak{v})$, there is a canonical way (which shall be specified in several examples) of partitioning the space $\widehat{X} = X \cup \partial X$ into Γ-equivariant half-spaces $V_x(\mathfrak{u})$ and $V_x(\mathfrak{v})$, such that $V_x(\mathfrak{u})$ is a neighbourhood of $\mathfrak{u}$ and $V_x(\mathfrak{v})$ is a neighbourhood of $\mathfrak{v}$.

We state here the main result.

Theorem 3. *Let $Z_n = (Y_n, X_n)$ a random walk with law μ on $\mathcal{G} = \mathbb{Z}_2 \wr \Gamma \equiv \mathbb{Z}_r \wr X$, such that $supp(\mu)$ generates $\mathcal{G}$. Suppose that μ has finite first moment and Ω is defined as in (1). If the above conditions are satisfied, then the measure space (Ω, ν) is the Poisson boundary of Z_n, where ν is the limit distribution on Ω of Z_n starting at $id = (\mathbf{0}, o)$.*

See [8] for the general proof of this theorem.

Application of the half-space method

We consider some specific base graphs X. In all these examples the Basic assumptions are fullfilled and we have the convergence of both the base RW X_n and the reversed RW $\widehat{X}_n$ to the boundary ∂X (boundary which will be specified below).

1. Graphs with infinitely many ends
   - $\partial X \equiv$ space of ends.
   - For the graph X with infinitely many ends, construct its structure tree $\mathcal{T}$, which is quasi-isometric with X. The ends of the base graphs X, towards the base random walk converge, are in bijection with the ends of the structure tree $\mathcal{T}$. Therefore, instead of working with the graph X, we can work with its structure tree. For this, since is a tree, the half-space method can be easily applied.
   - It is an easy exercise to construct the strip in the structure tree and then to lift it up to a bigger strip, as a subset of the lamplighter graph. Theorem 3 holds.
2. Hyperbolic graphs in the sense on Gromov
   - $\partial X \equiv$ the hyperbolic boundary.
   - Define $\mathfrak{s}(\mathfrak{u}, \mathfrak{v})$= the union of all geodesics between $\mathfrak{u}, \mathfrak{v}$, for all $\mathfrak{u}, \mathfrak{v} \in \partial X$.
   - For every $x \in \mathfrak{s}(\mathfrak{u}, \mathfrak{v})$ the half-spaces are: $V_x(\mathfrak{u})$ the horoball with centre in $\mathfrak{u}$ and passing through x and $V_x(\mathfrak{v})$ the horoball with centre in $\mathfrak{v}$ and passing through x.
   - For the Poisson boundary, apply the Theorem 3.
3. Euclidean lattices
   - $X = \mathbb{Z}^d$, $d \geq 3$ and the boundary ∂X is the unit sphere S_{d-1} in $\mathbb{R}^d$.
   - For $\mathfrak{u}, \mathfrak{v} \in S_{d-1}$ define the strip $\mathfrak{s}(\mathfrak{u}, \mathfrak{v}) = \mathbb{Z}^d$, which fullfills the conditions required in Proposition 2.
   - For every $\mathfrak{u}, \mathfrak{v} \in S_{d-1}$, let $\overline{\mathfrak{u}\mathfrak{v}}$ the chord joining them, and for every $x \in \overline{\mathfrak{u}\mathfrak{v}}$ let H_x be the hyperplane passing through x and orthogonal on $\overline{\mathfrak{u}\mathfrak{v}}$.
   - H_x splits $\mathbb{Z}^d$ into two half-spaces. The configuration can be chosen appropriately, and Theorem 3 holds.

References

[1] Brofferio, S., and Woess, W.: *Positive harmonic functions for semi-isotropic random walks on trees, lamplighter groups, and DL-graphs*, Potential Analysis **24** (2006) 245-265.

[2] Cartwright, D. I., and Soardi, P. M.: *Convergence to ends for random walks on the automorphism group of a tree*, Proc. Amer. Math. Soc. **107** (1989) 817-823.

[3] Gromov, M.: *Hyperbolic groups.* In Essay in Group Theory (S.M.Gersten, ed) 75-263, Springer, New York, (1987)

[4] Kaimanovich, V.A. : *The Poisson formula for groups with hyperbolic properties*, Annals of Math. **152** (2000), 659-692.

[5] Kaimanovich, V.A. and Vershik, A.M.: *Random walks on discrete groups: Boundary and entropy*, Ann. Probab. **11** (1983), 457-490.

[6] Kaimanovich, V. A. and Woess, W.: *Boundary and entropy of space homogeneous Markov Chains*, Ann. Probab. **30** (2002) 323-363.

[7] Karlsson, A. and Woess, W.: *The Poisson boundary of lamplighter random walks on trees*, Geometriae Dedicata **124** (2007) 95-107.

[8] Sava, E.: *The Poisson boundary of lamplighter random walks*, submitted.

[9] Woess, W.: *Random Walks on Infinite Graphs and Groups*, Cambridge Tracts in Mathematics **138**, Camb. Univ. Press, (2000).

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