The Poisson boundary of lamplighter random walks on general graphs

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The lamplighter graph
Let \( X \) be an infinite, locally finite, connected graph and \( u \in X \) (can be viewed as the root). A lamplighter graph is obtained by defining the lamplighters to be

\[ \lambda \Gamma = \{ (x,y) \mid x,y \in \Gamma \} \]

A lamplighter graph is a random walk on \( X \) that makes lamplighters move and / or changes the state of the lamp at current position (or nearby).

Result
The theorem regarding the convergence of lamplighter random walks \( Z_n \) on general base graphs \( X \) is a generalization of Karlsson and Woess [7]. Thus 2.2 (Lamplighter random walks on trees).

Theorem 1. Let \( Z_n = (Y_n, X_n) \) be a random walk with law \( \mu \) on the group \( G \) and \( \Gamma = \mathbb{Z} \times X \) such that \( \text{supp}(\mu) \) generates \( G \). If \( \Omega \) is defined as in (1) and \( \mu \) has finite first moment, then there exists an \( \Omega \)-valued random variable \( Z_\infty = (Y_\infty, X_\infty) \) such that \( Z_n \to Z_\infty \) almost surely, for every starting point. Moreover the distribution of \( Z_\infty \) is a continuous measure on \( \Omega \).

The Poisson boundary
Let \( \nu \) be the distribution of \( Z_\infty \) on \( \Omega \) (given the initial position \( u \in X \) and the initial configuration \( 0 \)). The measure \( \nu \) is the harmonic measure for the random walk \( Z_n \) with law \( \mu \).

\( (\Omega, \nu) \) is a measure space which describes the behaviour of the LRW at infinity. How “good” is this space? In this the Poisson boundary, that is, the finest model of a probability space at infinity of \( C \times X \) for distinguishing the possible limiting behaviour of \( Z_n \). There are several equivalent definitions for the Poisson boundary. See [Kaimanovich and Vershik, 1988] for RW on discrete groups.

Application of the half-space method
We consider some specific base graphs \( X \). In all these examples the Basic assumptions are fulfilled and we have the convergence of both the base RW \( X_n \) and the reversed RW \( X_n \) to the boundary \( \Omega X \) (boundary which will be specified below).

1. Graphs with infinitely many ends
   - \( \Omega X = \{ \text{space of ends} \} \)
   - For the graph \( X \) with infinitely many ends, construct its structure tree \( T \), which is quasi-isometric with \( X \). The ends of the base graph \( X \) towards the base random walk converge, are in bijection with the ends of the structure tree \( T \). Therefore, instead of working with the graph \( X \), we can work with its structure tree. For this, since a tree, the half-space method can be easily applied.
   - It is an easy exercise to construct the strip in the structure tree and then to lift it up to a bigger strip, as a subset of the lamplighter graph. Theorem 3 holds.

2. Hyperbolic graphs in the sense of Gromov
   - \( \Omega X = \{ \text{the hyperbolic boundary} \} \)
   - Define \( \omega(u,v) \) the union of all geodesics between \( u, v \), for all \( u, v \in \Omega X \).
   - For every \( x \in \{u,v\} \) the half-spaces are \( V_x(u,v) \) the horoball with centre \( u \) and passing through \( x \) and \( V_x(v,u) \) the horoball with centre \( v \) and passing through \( x \).
   - For the Poisson boundary, apply the Theorem 3.

3. Euclidean lattices
   - \( X = \mathbb{Z}^d, d \geq 3 \) and the boundary \( \Omega X \) is the unit sphere \( S_{d-1} \) in \( \mathbb{R}^d \).
   - For \( u \in S_{d-1} \) define the strip \( \omega(u,v) = \mathbb{R}^d \), fulfills the conditions required in Proposition 2.
   - For every \( x \in S_{d-1} \), let \( \mathbb{R}^d \) the chord joining them, and for every \( x \in \mathbb{R}^d \) let \( H_x \) be the hyperplane passing through \( x \) and orthogonal to \( \mathbb{R}^d \).
   - \( H_x \) splits \( \mathbb{R}^d \) into two half-spaces. The configuration can be chosen appropriately, and Theorem 3 holds.

References

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