The Poisson boundary of lamplighter random walks

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Outline

LRW-The model

- $\mathbb{T}$ homogeneous tree, degree $q + 1 \geq 3$.
- Select a vertex $o \in \mathbb{T}_{q+1}$ as the root.
- Lamp at each vertex: states 0 (switched off) and 1 (switched on).
- A lamplighter person starts in o and performs a random walk on $\mathbb{T}$: makes random moves and / or changes the state of the lamp at current position (or nearby).
- Configuration on $\mathbb{T}$: a function $\eta : \mathbb{T} \to \{0, 1\}$ with finite support

$$supp(\eta) = \{x \in \mathbb{T} : \ \eta(x) \neq 0\}$$

- Suppose that we start the random walk in o with all lamps switched off.
- Denote by $\mathcal{O}$ the configuration which corresponds to all lamps switched off, that is,

$$\mathcal{O}(x) = 0, \text{ for all } x \in \mathbb{T}.$$

- Initial position of the random walk is the pair $(\mathcal{O}, o)$.
- At every moment of time we have to observe the pair (η, x), where η is the current configuration of the lamps and x is the current position of the lamplighter in the tree.

- Let $\mathcal{C} = \{configurations\}$.
- The process evolves on the state space $\mathcal{C} \times \mathbb{T}$ consisting of pairs (η, x).
- The lamplighter random walk $Z_n = (Y_n, X_n)$ on $\mathcal{C} \times \mathbb{T}$ is given by the transition matrix $P = (p((\eta, x), (\eta^{'}, x^{'})))$.
- $p((\eta, x), (\eta^{'}, x^{'}))$ describes the one-step transition probabilities, that is,

$$\mathbb{P}[Z_{n+1} = (\eta^{'}, x^{'}) | Z_n = (\eta, x)] = p((\eta, x), (\eta^{'}, x^{'})).$$

- Assume that the LRW Z_n is irreducible.

Basic examples of lamplighter random walks

X_n a *random walk* on the tree $\mathbb{T}$.

1. Walk or switch model: at every step the lamplighter tosses a coin:
   - if head comes up then he walks, leaving the lamps unchangend.
   - if tail comes up then he modifies the lamp at the current position, without moving.
2. Switch-walk-switch model: if the lamplighter stands at x and the actual configuration is η, then:
   - he first switches the lamp at x to a random state.
   - then he walks to some neighbour $x^{'}$.
   - at last, he switches the lamp at $x^{'}$ to a random state.

Base RW X_n

- Under the above assumptions, X_n is a random walk on $\mathbb{T}$, with one-step transition probabilities

$$p(x, x') = \sum_{\eta'} p((\eta, x), (\eta', x')).$$

- X_n is known to be transient (it visits every finite subset of $\mathbb{T}$ only finitely many times) $\Rightarrow$ escape of the random walk to "infinity".
- Attach to $\mathbb{T}$ a geometrical boundary $\partial\mathbb{T} \equiv$space of ends.
- An end (boundary point)-a way of going off to infinity.
- Ends $\mathfrak{u} \in \partial\mathbb{T}$ -represented by *geodesic rays* $\mathfrak{u} = [o = x_0, x_1, \ldots]$ starting from a root $o \in \mathbb{T}$.
- The tree $\mathbb{T}_{q+1}$ has infinitely many ends.

Convergence to the boundary

- Convergence to the boundary $\partial\mathbb{T}$ of the RW $(X_n)\equiv \exists$ a $\partial\mathbb{T}$-valued random variable X_∞ such that

$$\mathbb{P}_x[\lim_{n\to\infty} X_n = X_\infty] = 1, \text{ for all } x.$$

- If (X_n) on $\mathbb{T}$ is of nearest neighbour type and transient $\Rightarrow$ it converges to $\partial\mathbb{T}$.
- Convergence of the base random walk X_n on $\mathbb{T} \Rightarrow$ convergence of the lamplighter random walk $Z_n = (Y_n, X_n)$ on the lamplighter graph $\mathcal{C} \times \mathbb{T}$.

- Transience $\Rightarrow$ every vertex visited finitely often $\Rightarrow$ after the last visit at a vertex the state of the lamp sitting there remains unchanged $\Rightarrow \exists$ random limit configuration Y_∞, such that:

$$Y_\infty = \lim_{n\to\infty} Y_n \in \widehat{\mathcal{C}}$$

- $\widehat{\mathcal{C}}$={all configurations, finitely or infinitely supported}
- $Y_\infty(x)$ is the definite state of the lamp at x.
- $\widehat{\mathbb{T}} = \mathbb{T} \cup \partial\mathbb{T}$ is a compactification of $\mathbb{T}$.
- $\widehat{\mathcal{C}}$ is the natural compactification of $\mathcal{C}$ in the topology of pointwise convergence.
- Then $\widehat{\mathcal{C}} \times \widehat{\mathbb{T}}$ is a compactification of the state space $\mathcal{C} \times \mathbb{T}$ for the lamplighter random walk $Z_n = (Y_n, X_n)$.

Convergence of the lamplighter random walk

- $\widehat{\mathcal{C}} \times \partial\mathbb{T}$ is a boundary for the state space $\mathcal{C} \times \mathbb{T}$.
- Metric on $\mathcal{C} \times \mathbb{T}$: $d((\eta, x), (\eta', x')) = d(x, x') + |\eta \triangle \eta'|$.

Theorem (Karlsson and Woess, 2006)

If the lamplighter random walk $Z_n = (Y_n, X_n)$ has finite first moment

$$\sum_{(\eta', x') \in \mathcal{C} \times \mathbb{T}} d((\eta, x), (\eta', x')) p((\eta, x), (\eta', x')) < \infty$$

then (Z_n) converges a.s. to a limit random variable

$$Z_\infty = (Y_\infty, X_\infty) \in \widehat{\mathcal{C}} \times \partial\mathbb{T}.$$

- $Z_n = (Y_n, X_n)$ is transient$\equiv$ goes off to infinity.
- Topology of $\widehat{\mathcal{C}} \times \widehat{\mathbb{T}}$ provides the model $\widehat{\mathcal{C}} \times \partial\mathbb{T}$ at infinity for the behaviour of the LRW $Z_n = (Y_n, X_n)$.
- Is this the finest model? YES !!! (as a measure space).
- Set now $\Omega = \bigcup_{\mathfrak{u} \in \partial\mathbb{T}} \mathcal{C}_{\mathfrak{u}} \times \{\mathfrak{u}\}$.
- $\mathcal{C}_{\mathfrak{u}} = \{\zeta \in \widehat{\mathcal{C}} : \zeta$ accumulates only at $\mathfrak{u}\}$.
- $\mathcal{C}_{\mathfrak{u}}$ is dense in $\widehat{\mathcal{C}}$ and Ω is dense in $\widehat{\mathcal{C}} \times \partial\mathbb{T}$.
- Let $\nu_{(\eta,x)}$ be the distribution of Z_∞ on Ω (given the position $x \in \mathbb{T}$ and the configuration η).
- Write ν if the LRW starts in $(\mathcal{O}, o)$.
- ν is the probability measure defined for Borel sets $B \in \Omega$ by

$$\nu(B) = \mathbb{P}[Z_\infty \in B | Z_0 = (\mathcal{O}, o)].$$

- (Ω, ν) is a measure space which describes the behaviour of the LRW at infinity. How "*good*" is this space?
- Is this the Poisson boundary, that is, the finest model of a probability space at infinity of $\mathcal{C} \times \mathbb{T}$ for distinguishing the possible limitting behaviour of (Z_n)?
- There are several equivalent definitions for the Poisson boundary. See [Kaimanovich and Vershik, 1983] for RW on discrete groups.
- If every bounded harmonic function h on $\mathcal{C} \times \mathbb{T}$ w.r.t the transition matrix has a unique integral representation

$$h(\eta, x) = \int_\Omega \varphi d\nu_{(\eta,x)}, \text{for some } \varphi \in L^\infty(\Omega, \nu)$$

then (Ω, ν) is the Poisson boundary of (Z_n).

- Triviality of the Poisson boundary $\equiv$ absence of non-constant bounded harmonic functions.
- Poisson boundary=unique up to sets of measure 0.
- For proving the maximility of the measure space (Ω, ν) we use the Strip Criterion [Kaimanovich, 2000].
- Strip criterion= a purely geometrical criterion, under the assumption of space-homogeneity (there is a group of isometries of $\mathbb{T}$ which acts transitively on $\mathbb{T}$ and the transition probabilities are invariant w.r.t. the group action).

Theorem (Woess and Karlsson, 2006)

If the lamplighter random walk Z_n has finite first moment on $\mathcal{C} \times \mathbb{T}$, then the measure space (Ω, ν) is the Poisson boundary of the LRW.

- Base graphs $\mathbb{X}$ for the LRW:
  - a graph with infinitely many ends, or
  - a hyperbolic graph, or
  - an euclidean lattice (RW with drift $\neq 0$).
- Endowed with natural geometric boundaries $\partial\mathbb{X}$.
  - space of ends
  - hyperbolic boundary
  - the unit sphere

Theorem (S.,2007)

If the LRW $Z_n = (Y_n, X_n)$ on the base graph $\mathbb{X}$ has finite first moment then it converges a.s. to a $(\widehat{\mathcal{C}} \times \partial\mathbb{X})$-valued r.v. and the Poisson boundary of the LRW is the space of infinite limit configurations (i.e. the set of all boundary elements $\mathfrak{u} \in \partial\mathbb{X}$ together with all configurations which accumulate only at $\mathfrak{u}$).

1. Take a group which acts transitively on an homogeneous tree and fixes one end of the tree (this is not a discrete group). Then the Poisson boundary of LRW on this tree, in the case of drift-free case, is unknown.
2. The Poisson boundary of LRW on $\mathbb{Z}^d$, for $d \geq 3$, when there is no drift, is still unknown.