Entropy sensitivity of languages associated with infinite graphs

(joint work with Wilfried Huss, Wolfgang Woess)

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Outline

Introduction

- $\mathbf{\Sigma}$ finite alphabet.
- $\mathbf{\Sigma}^*$ the set of all finite words over $\mathbf{\Sigma}$.
- A language L over $\mathbf{\Sigma}$ is a subset of $\mathbf{\Sigma}^*$.
- Growth or entropy of L is

$$\mathrm{h}(L) = \limsup_{n\to\infty} \frac{1}{n} \log\big|\{w \in L : \, |w| = n\}\big|.$$

- All our languages are infinite.
- For finite $F \subset \mathbf{\Sigma}^*$, $F = \{$subwords of elements of $L\}$

$$L^F = \{w \in L : \text{ no } v \in F \text{ is a subword of } w\}.$$

- We associate with infinite, directed, graphs, a class of languages L.

Introduction

- Question: is $\mathrm{h}(L^F) < \mathrm{h}(L)$ strictly?
- If YES, under which conditions (on the graph)?
- If $\mathrm{h}(L^F) < \mathrm{h}(L)$, for every F of forbidden words, then L is called growth sensitive or entropy sensitive.
- Group theory:
  Grigorchuk and De la Harpe ('97): *On problems related to growth, entropy, and spectrum in group theory*
  Ceccherini-Silberstein and Scarabotti ('04): *Random walks, entropy and hopfianity of free groups*
- Symbolic dynamics: Lind and Marcus ('95): *An introduction to symbolic dynamics and coding* (topological entropy of a sofic system)

Introduction

- Ceccherini-Silberstein and Woess ('03,'09):
  Growth and ergodicity of context-free languages
  and
  Context-free pairs of groups. 1–Context-free pairs and graphs
- Basic object: oriented, labeled graph (X, E, l) with edges labeled by elements of a finite alphabet $\mathbf{\Sigma}$.
- each edge $e \in E$ is of the form $e = (x, a, y)$, multiple edges and loops are allowed.
- A path of length n in X is a sequence $\pi = e_1 e_2 \dots e_n$ of edges such that $e_i^+ = e_{i+1}^-$.
- For $x, y \in X$, π is a path from x to y if $e_1^- = x$ and $e_n^+ = y$.
- The label $l(\pi)$ is $l(\pi) = l(e_1)l(e_2) \dots l(e_n) \in \mathbf{\Sigma}^*$.

Languages and graphs

- Let $\Pi_{x,y}$ be the set of all paths π from x to y in X.
- With X we associate the language

$$L_{x,y} = \{\ell(\pi) \in \mathbf{\Sigma}^* : \pi \in \Pi_{x,y}\}, \text{ where } x, y \in X.$$

- Question: Is this language growth-sensitive? For which class of graphs X?
- Answer: Yes, for uniformly connected and fully deterministic graphs X.
- (X, E, l) is determinstic if for every $x \in X$ and $a \in \mathbf{\Sigma}$, there is at most one edge with initial point x and label a.
- (X, E, l) is fully determinstic if there is exactly one edge with label a going out from x.

Let $\mathbf{\Sigma} = \{a, b\}$ and consider the following graph.

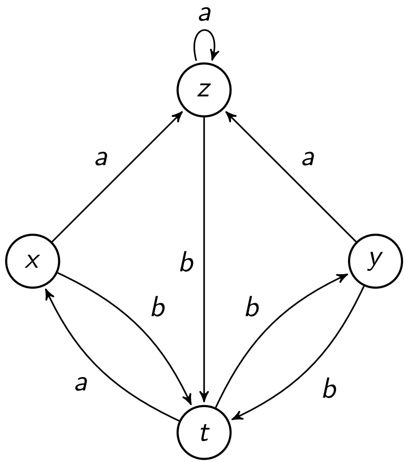

Figure: Fully deterministic graph

$L_{x,y}$ is the set of all labeles of paths from x to y.

Assumptions on the graph

- X strongly connected
- X uniformly connected = strongly connected + not too big circles.
- We write

$$\mathsf{h}(X) = \mathsf{h}(X, E, \ell) = \sup_{x,y \in X} \mathsf{h}(L_{x,y})$$

for the entropy of our oriented, labelled graph.
- For a strongly connected graph, $\mathsf{h}(L_{x,y}) = \mathsf{h}(X)$ for all $x, y \in X$.
- Assume that the set of forbidden subwords $F \subset \mathbf{\Sigma}^*$ is relatively dense in X.

Theorem (W. Huss, E. Sava, W. Woess '09)

Suppose that (X, E, ℓ) is uniformly connected and deterministic with label alphabet $\mathbf{\Sigma}$. Let $F \subset \mathbf{\Sigma}^+$ be a finite, non-empty set which is relatively dense in X. Then

$$\sup_{x,y \in X} \mathrm{h}(L^F_{x,y}) < \mathrm{h}(X) \quad \textit{strictly.}$$

Theorem (W. Huss, E. Sava, W. Woess '09)

If (X, E, ℓ) is uniformly connected and fully deterministic then $L_{x,y}$ is growth-sensitive for all $x, y \in X$.

Markov chains

- Equip the graph X with transition probabilites: to each edge $e = (x, a, y)$ we associate $p(e) \geq \alpha > 0$ s.t.

$$\sum_{e \in E:\, e^- = x} p(e) \leq 1 \quad \text{for every } x \in X\,.$$

- Consider the Markov chain over X with one-step transition probabilities

$$p(x, y) = \sum_{a \in \boldsymbol{\Sigma}:\, (x,a,y) \in E} p(x, a, y)\,.$$

- In each step we record the edges and their labels.
- $p^{(n)}(x, y)$: the probability that the particle starting at x is at y at time n, i.e. the (x, y)-element of the n^{th}-power P^n of $P = \big(p(x, y)\big)_{x,y \in X}$.

- Consider the spectral radius of the Markov chain P:

$$\rho(P) = \limsup_{n\to\infty} p^{(n)}(x,y)^{1/n}$$

- $\rho(P)$ is related to the entropy of X (if P is the SRW on X):

$$\mathrm{h}(X) = \mathrm{h}(L_{x,y}) = \log\big(\rho(P)\cdot|\mathbf{\Sigma}|\big).$$

- Let now $F \subset \mathbf{\Sigma}^*$: interpret F as a sequence of forbidden transitions, i.e. we restrict the motion of the particle such that at no time, it is allowed to traverse any path π with $l(\pi) \in F$ in k succesive steps, with $k = |\pi|$.
- $p_F^{(n)}(x,y)$: the probability that the particle starting in x is at position y after n steps, without having made any sequence of forbidden transitions.

- Consider

$$\rho_{x,y}(P_F) = \limsup_{n\to\infty} p_F^{(n)}(x,y)^{1/n}, \quad x,y \in X\,.$$

- Relation between $\rho_{x,y}(P_F)$ and the entropy $\mathrm{h}(L^F_{x,y})$:

$$\mathrm{h}(L^F_{x,y}) = \log\big(\rho_{x,y}(P_F)\cdot|\mathbf{\Sigma}|\big).$$

- Recall that $\mathrm{h}(X) = \log\big(\rho(P)\cdot|\mathbf{\Sigma}|\big)$
- How do we prove that

$$\sup_{x,y\in X} \mathrm{h}(L^F_{x,y}) < \mathrm{h}(X) \quad \text{strictly?}$$

- We just have to compare

$$\sup_{x,y\in X} \rho_{x,y}(P_F) \quad \text{with} \quad \rho(P).$$

Theorem (W. Huss, E. Sava, W. Woess, '09)

Suppose that (X, E, ℓ) *is strongly connected with label alphabet* $\mathbf{\Sigma}$ *and equipped with transition probabilities* $p(e) \geq \alpha > 0$, $e \in E$. *Let* $F \subset \mathbf{\Sigma}^+$ *be a finite, non-empty set which is relatively dense in* X. *Then*

$$\sup_{x,y \in X} \rho_{x,y}(P_F) < \rho(P) \quad \textit{strictly.}$$

Proof.

We shall proceed in two steps:

1. Step 1: P stochastic and $\rho(P) = 1$
2. Step 2: general case, when $\rho(P) < 1$, then we reduce this case to the previous one.

□

Step 1: P stochastic and $\rho(P) = 1$

- Show that there exists $k \in \mathbb{N}$ and $\varepsilon_0 > 0$ s.t. the matrix $Q = (p_F^{(k)}(x,y))_{x,y \in X}$ is strictly substochastic with all rows bounded by $1 - \varepsilon_0$, i.e

$$\sum_{y \in X} p_F^{(k)}(x,y) \leq 1 - \varepsilon_0 \quad \text{for all } x \in X.$$

- Consider $Q^n = \left(q^{(n)}(x,y)\right)_{x,y \in X}$: $q^{(n)}(x,y)$ is the probability that the MC starting at x is in y at time nk, and does not make any forbidden sequence of transitions in intervals $[(j-1)k\,,\,jk]$.

$$p_F^{(nk)}(x,y) \leq q^{(n)}(x,y)\,.$$

- Therefore, for every $x \in X$ and $i = 0, \dots, k-1$,

$$\sum_{y \in X} p_F^{(nk+i)}(x,y) \le \sum_{z \in X} q^{(n)}(x,z) \underbrace{\sum_{y \in X} p_F^{(i)}(z,y)}_{\le 1}$$

$$\le (1-\varepsilon_0)^n \,,$$

- Since $p_F^{(nk+i)}(x,y)$ is a subsequence of $p^{(n)}(x,y)$, we conclude that

$$\limsup_{n \to \infty} p_F^{(nk+i)}(x,y)^{1/(nk+i)} \le (1-\varepsilon_0)^{1/k} \,,$$

so that $\rho_{x,y}(P_F) \le (1-\varepsilon_0)^{1/k} = 1 - \varepsilon$, $\varepsilon > 0$.

Step 2: General case

- For P, there exists a strictly positive function h

$$Ph = \rho(P) \cdot h$$

- Consider now the h-transform of $p(e)$ of P:

$$p^h(e) = p^h(x,a,y) = \frac{p(x,a,y)h(y)}{\rho(P)h(x)}$$

- The associated transition matrix P^h (the h-process):

$$p^h(x,y) = \sum_{a:(x,a,y)\in E} p^h(x,a,y)\,.$$

- Then $\rho(P^h) = 1$ and using uniform connectedness, we show that there is a constant $\bar{\alpha} > 0$ such that $p^h(e) \geq \bar{\alpha}$ for each $e = (x,a,y) \in E$.

- With P^h we are now in the situation of Step 1, and we get $\rho_{x,y}(P^h_F) \leq 1 - \varepsilon$, for all $x, y \in X$.
- Show that $\rho_{x,y}(P^h_F) = \rho_{x,y}(P_F)/\rho(P)$, which will conclude the proof.

Proof of the entropy sensitivity.

- Use the previous result:

$$\sup_{x,y \in X} \rho_{x,y}(P_F) < \rho(P) \quad \text{strictly.}$$

- Equip the edges of X with $p(x, a, y) = 1/|\mathbf{\Sigma}|$.

$$\mathrm{h}(X) = \limsup_{n\to\infty} \frac{1}{n} \log\big(p^n(x, y)|\mathbf{\Sigma}|^n\big) = \log\big(\rho(P) \cdot |\mathbf{\Sigma}|\big).$$

- Analogously $\mathrm{h}(L^F_{x,y}) = \log\big(\rho_{x,y}(P_F) \cdot |\mathbf{\Sigma}|\big)$.
- It follows that $\sup_{x,y\in X} \mathrm{h}(L^F_{x,y}) < \mathrm{h}(X)$ strictly.

□

Schreier graphs

- G be a finitely generated group and K a subgroup.
- $\boldsymbol{\Sigma}$ be a finite alphabet and $\psi : \boldsymbol{\Sigma} \to G$ be such that the set $\psi(\boldsymbol{\Sigma})$ generates G as a semigroup.
- Extend ψ to a monoid homomorphism from $\boldsymbol{\Sigma}^*$ to G by $\psi(w) = \psi(a_1) \cdots \psi(a_n)$, if $w = a_1 \dots a_n$ with $a_i \in \boldsymbol{\Sigma}$ (and $\psi(\epsilon) = 1_G$)
- ψ is called a semigroup presentation of G.
- The Schreier graph $X = X(G, K, \psi)$ has vertex set

$$X = \{Kg : g \in G\},$$

the set of all right K-cosets in G, and the set of all labelled, directed edges E is given by

$$E = \{e = (x, a, y) : \ x = Kg, y = Kg\psi(a)\},$$

where $g \in G$, $a \in \boldsymbol{\Sigma}$.
- X is fully deterministic and uniformly connected.

- The word problem of (G, K) with respect to ψ is the language

$$L(G, K, \psi) = \{w \in \mathbf{\Sigma}^* : \psi(w) \in K\}.$$

- Consider the "root" vertex $o = K$ of the Schreier graph, then $L(G, K, \psi) = L_{o,o}$.

Corollary

The word problem of the pair (G, K) with respect to any semigroup presentation ψ is growth sensitive, with respect to forbidding an arbitrary non-empty subset $F \subset \mathbf{\Sigma}^*$.

Thank you for your attention!

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