# Lamplighter random walks (LRW)

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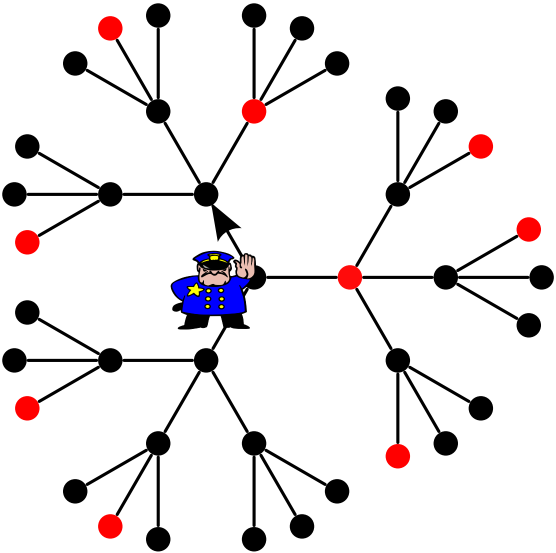
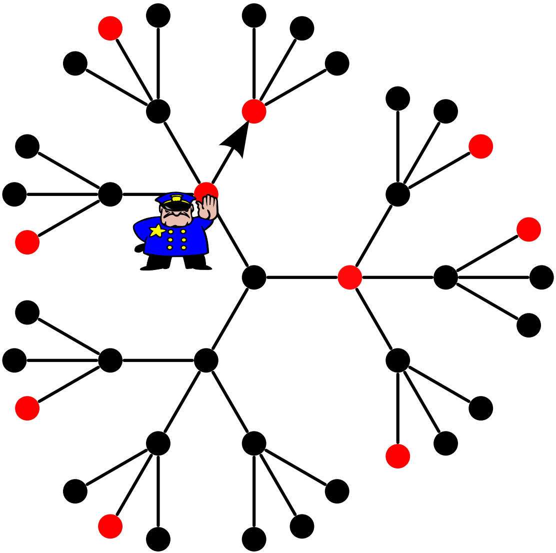
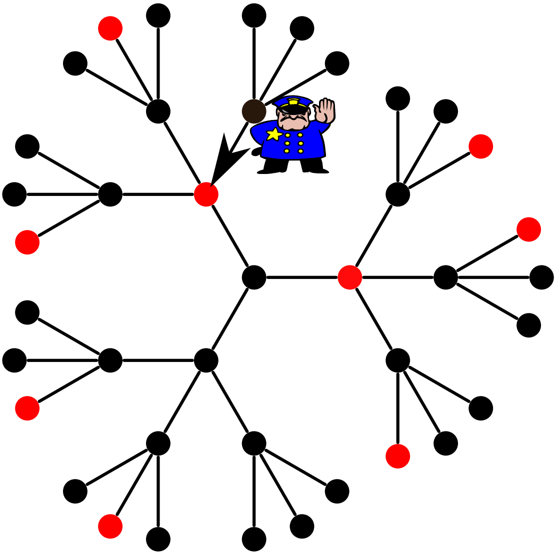
# Outline

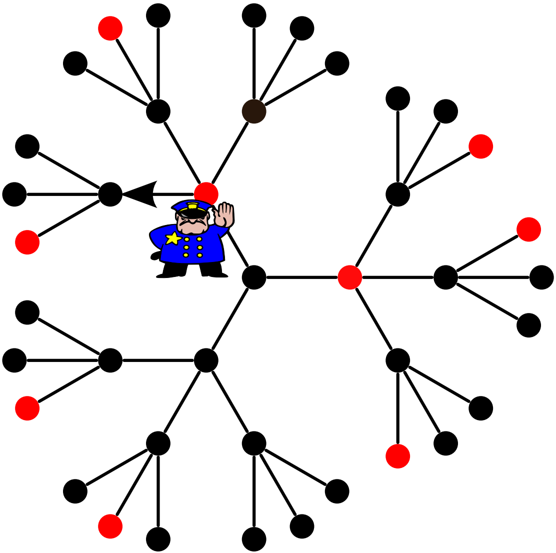
# LRW on trees

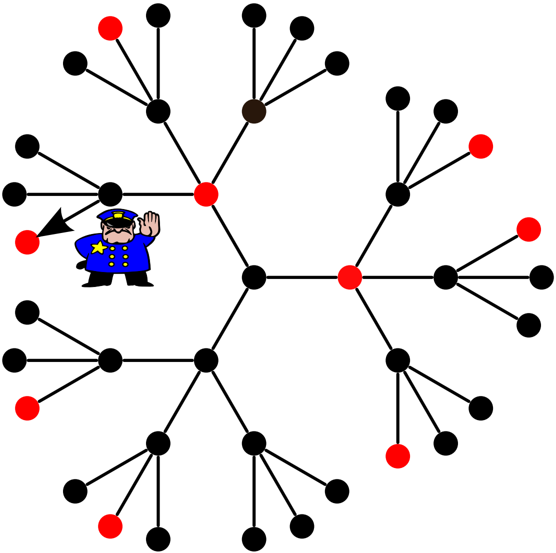
- $\mathbb{T}$ homogeneous tree, degree $\geq 3$.
- Select a vertex $o \in \mathbb{T}$ as the root.
- Lamp at each vertex: states 0 (switched off) and 1 (switched on).
- A lamplighter person starts in $o$ and performs a random walk on $\mathbb{T}$: makes random moves and / or changes the state of the lamp at current position (or nearby).
- At every moment of time we record the configuration of the lamps which are switched on and the position of the lamplighter.
- Consider an example where we start with finitely lamps switched on.











# Basic models of LRW

Start with a *random walk* on the tree $\mathbb{T}$.

1. Walk or switch model: at every step the lamplighter tosses a coin:
   - if head comes up then he walks, leaving the lamps unchangend.
   - if tail comes up then he modifies the lamp at the current position, without moving.
2. Switch-walk-switch model: if the lamplighter stands at $x$ and the actual configuration is $\eta$, then:
   - he first switches the lamp at $x$ to a random state.
   - then he walks to some neighbour $x^{'}$.
   - at last, he switches the lamp at $x^{'}$ to a random state.

# Formal definition

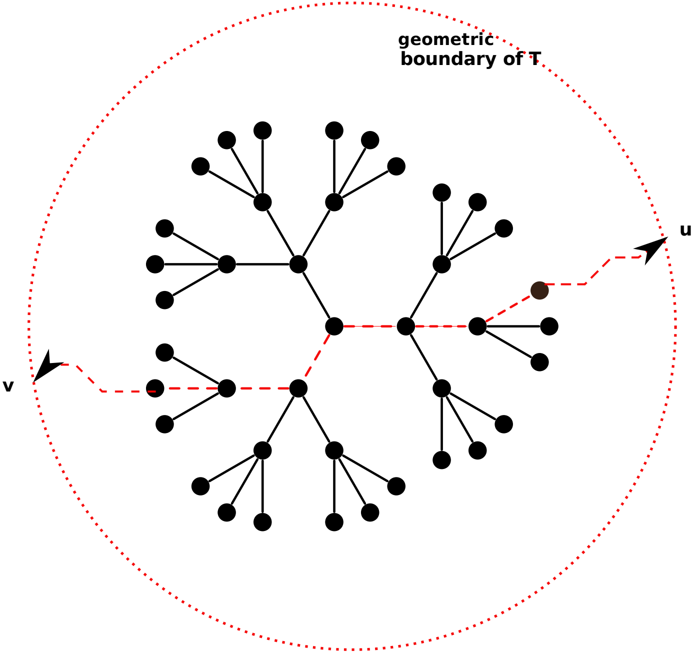
- Configuration of lamps on $\mathbb{T}$: a function $\eta : \mathbb{T} \to \{0,1\}$ with finite support $supp(\eta) = \{x \in \mathbb{T} : \ \eta(x) \neq 0\}$.
- At every moment of time we observe a pair $(\eta, x)$, with $\eta$ the configuration of the lamps and $x$ the current position of the lamplighter in the tree.
- The random process evolves on the state space $\mathcal{C} \times \mathbb{T}$ consisting of pairs $(\eta, x)$, with $\mathcal{C} = \{configurations\}$.
- The lamplighter random walk (LRW) $Z_n = (\eta_n, X_n)$ on $\mathcal{C} \times \mathbb{T}$: given by the transition matrix $P = \Big(p((\eta, x), (\eta^{'}, x^{'}))\Big)$.
- $Z_n = (\eta_n, X_n)$ projects into: $\eta_n$ random configurations, and $X_n$ random walk on $\mathbb{T}$.

# Base RW $X_n$

- The projection $X_n$ of $Z_n = (\eta_n, X_n)$ on the underlying tree is a random walk with transition pb.

$$p(x, x') = \sum_{\eta'} p\left((\eta, x), (\eta', x')\right).$$

- $X_n$ is known to be transient (it visits every finite subset of $\mathbb{T}$ only finitely many times) $\Rightarrow$ escape of the random walk to "infinity".
- Attach to $\mathbb{T}$ a geometric boundary $\partial\mathbb{T} \equiv$ space of ends.
- An end (boundary point) - a way of going off to infinity.
- Ends $\mathfrak{u} \in \partial\mathbb{T}$ -represented by *geodesic rays* $\mathfrak{u} = [o = x_0, x_1, \ldots]$ starting from a root $o \in \mathbb{T}$.
- The tree $\mathbb{T}$ has infinitely many ends.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# Convergence to the boundary

- Convergence to the boundary $\partial\mathbb{T}$ of the RW $X_n \equiv \exists$ a $\partial\mathbb{T}$-valued random variable $X_\infty$ such that

$$\mathbb{P}_x[\lim_{n\to\infty} X_n = X_\infty] = 1, \text{ for all } x.$$

- If $X_n$ on $\mathbb{T}$ is of nearest neighbour type and transient $\Rightarrow$ it converges to $\partial\mathbb{T}$.
- Convergence of the random walk $X_n$ on $\mathbb{T}$ $\Rightarrow$ convergence of the LRW $Z_n = (\eta_n, X_n)$ on the $\mathcal{C} \times \mathbb{T}$.
- Transience $\Rightarrow$ every vertex visited finitely often $\Rightarrow$ after the last visit at a vertex the state of the lamp sitting there remains unchanged $\Rightarrow$ $\exists$ random limit configuration $\eta_\infty$, such that:

$$\eta_\infty = \lim_{n\to\infty} \eta_n \in \widehat{\mathcal{C}}$$

where $\eta_\infty(x)$ is the definite state of the lamp at $x$.

# Convergence of the LRW

- $\widehat{\mathcal{C}}$={all configurations, finitely or infinitely supported}
- $\widehat{\mathbb{T}} = \mathbb{T} \cup \partial\mathbb{T}$ is a compactification of $\mathbb{T}$.
- $\widehat{\mathcal{C}} \times \widehat{\mathbb{T}}$ is a compactification and $\widehat{\mathcal{C}} \times \partial\mathbb{T}$ is a boundary for the state space $\mathcal{C} \times \mathbb{T}$.

**Theorem (Karlsson and Woess, 2006)**

*If the lamplighter random walk $Z_n = (\eta_n, X_n)$ has finite first moment*

$$\sum_{(\eta',x')\in\mathcal{C}\times\mathbb{T}} d\Big((\eta,x),(\eta',x')\Big)\, p\Big((\eta,x),(\eta',x')\Big) < \infty$$

*then $Z_n$ converges a.s. to a limit random variable*

$$Z_\infty = (\eta_\infty, X_\infty) \in \widehat{\mathcal{C}} \times \partial\mathbb{T}.$$

# The Poisson boundary

- $Z_n = (\eta_n, X_n)$ is transient$\equiv$ goes off to infinity.
- Topology of $\widehat{\mathcal{C}} \times \widehat{\mathbb{T}}$ provides the model $\widehat{\mathcal{C}} \times \partial\mathbb{T}$ at infinity for the behaviour of the LRW $Z_n = (\eta_n, X_n)$.
- Is this the finest model? YES !!! (as a measure space).
- Define $\Omega = \bigcup_{\mathfrak{u}\in\partial\mathbb{T}} \mathcal{C}_{\mathfrak{u}} \times \{\mathfrak{u}\}$ which is dense in $\widehat{\mathcal{C}} \times \partial\mathbb{T}$.
- $\mathcal{C}_{\mathfrak{u}} = \{\zeta \in \widehat{\mathcal{C}} : \zeta \text{ accumulates only at } \mathfrak{u}\}$.
- Let $\nu_{(\eta,x)}$ be the hitting distribution of $Z_n$, i.e. the distribution on $Z_\infty$ on $\Omega$; write $\nu$ if the LRW starts in $o$ with all lamps swiched off.
- $\nu$ is the probability measure defined for Borel sets $B \in \Omega$ by

$$\nu(B) = \mathbb{P}[Z_\infty \in B].$$

- $(\Omega, \nu)$ is a measure space which describes the behaviour of the LRW at infinity. How *"good"* is this space?
- Is this the Poisson boundary, that is, the finest model of a probability space at infinity of $\mathcal{C} \times \mathbb{T}$ for distinguishing the possible limitting behaviour of $Z_n$? YES, IT IS.
- There are several equivalent definitions for the Poisson boundary. See [Kaimanovich and Vershik, 1983] for RW on discrete groups.
- Poisson boundary=unique up to sets of measure 0.
- For proving the maximility of the measure space $(\Omega, \nu)$ we use the Strip Criterion [Kaimanovich, 2000] = a purely geometric criterion.

**Theorem (Woess and Karlsson, 2006)**

*If the lamplighter random walk $Z_n$ has finite first moment on $\mathcal{C} \times \mathbb{T}$, then the measure space $(\Omega, \nu)$ is the Poisson boundary of the LRW.*

- Base graphs $\mathbb{X}$ for the LRW:
  - a graph with infinitely many ends, or
  - a hyperbolic graph, or
  - an euclidean lattice (RW with drift $\neq 0$).
- Endowed with natural geometric boundaries $\partial\mathbb{X}$.
  - space of ends
  - hyperbolic boundary
  - the unit sphere

**Theorem (S., 2008)**

*If the LRW $Z_n = (\eta_n, X_n)$ on the base graph $\mathbb{X}$ has finite first moment then it converges a.s. to a $(\widehat{\mathcal{C}} \times \partial\mathbb{X})$-valued r.v. and the Poisson boundary of the LRW is the space of infinite limit configurations (i.e. the set of all boundary elements $\mathfrak{u} \in \partial\mathbb{X}$ together with all configurations which accumulate only at $\mathfrak{u}$).*

V. A. Kaimanovich, A. M. Vershik: *Random walks on discrete groups: Boundary and entropy*
Ann. Probab. 11 (1983) 457-490.

A. Karlsson, W. Woess: *The Poisson boundary of lamplighter random walks on trees*.
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E. Sava: *A note on the Poisson boundary of lamplighter random walks*

Monatshefte für Mathematik, (2009) (in print).

Thank you for your attention!