From fractals in external DLA to internal DLA on fractals

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Abstract

We present an unified approach on the behavior of two random growth models (*external DLA* and *internal DLA*) on infinite graphs, the second one being an internal counterpart of the first one. Even though the two models look pretty similar, their behavior is completely different: while external DLA tends to build irregularities and fractal-like structures, internal DLA tends to fill up gaps and to produce regular clusters. We will also consider the aforementioned models on fractal graphs like Sierpinski gasket and carpet, and present some recent results and possible questions to investigate.

Contents

1 Introduction

We consider two aggregation models initially introduced in physics, and rigorously studied in mathematics over the last three decades, models for which we present a survey on the existing results and we state several open problems. The models under consideration are *external diffussion limited aggregation* (shortly *external DLA*) and *internal diffussion limited aggregation* (shortly *internal DLA*). In the mathematical fractal community, these two models started to gain interest, nevertheless they

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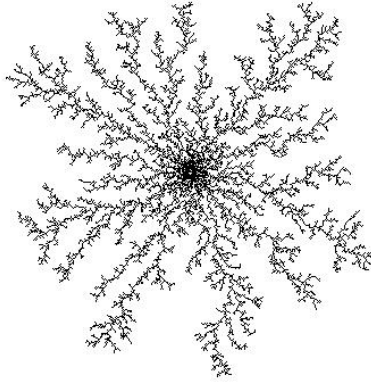
Figure 1: : External DLA cluster with center initially occupied

are understood on other state spaces, and in particular on d-dimensional integer lattices $\mathbb{Z}^d$. A few recent results concerning external DLA on the m-dimensional pre-Sierpinski carpet (Sierpinski carpet graph), for $m \geq 3$ are available. For the internal DLA on the Sierpinski gasket graph, there are also some limit shape results, but other than these two examples, there is not much known about the two growth models on other fractal graphs where according to simulations which we present towards to end of the paper, interesting behavior may be observed. With the current overview, we would like to draw the attention on the beauty of these models.

For the rest of the paper, G will be an infinite and locally finite graph, the reference state space, which will be replaced with concrete graphs as needed. We denote by $o \in \mathsf{G}$ a fixed vertex, the *origin of the graph* G.

External DLA was initially introduced in physics by WITTEN AND SANDER [WS83], as an example to create ordering out of chaos due to a simple rule. Mathematically, this ordering is far away from being understood, and new methods and ideas are needed in order to move forward in this direction. External DLA is a model of random fractal growth which exhibits self-organized criticality and complex-pattern formation, and which produces scale-invariant objects whose Hausdorff dimension is independent of short-range details. Moreover external DLA has no upper critical dimension as shown in [WS83]; it is a model which builds a sequence of random growing sets $(\mathcal{E}_n)_{n \geq 0}$, starting with one particle $\mathcal{E}_0 = \{o\}$ at the origin of G. At each time step, a new particle starts a simple random walk from "infinity" (far away) and walks until it hits the outer boundary of the existing cluster, where it stops and settles. In this way, one builds a family $(\mathcal{E}_n)_{n \geq 0}$ of growing clusters; the set $\mathcal{E}_n$ consists of exactly $n+1$ particles and it is called *external DLA cluster*. In spite of these very simple growth rules, only a few rigorous mathematical results about external DLA are available, results which will be surveyed below. A typical structure produced on a two-dimensional lattice is shown in Figure 1. External DLA was found to well represent growth processes in nature such as growth of bacterial colonies, electrodeposition, crystal growth.

Internal DLA is an attempt of a model which eliminates irregularities and fills gaps, as opposed to external DLA. It was proposed by MEAKIN AND DEUTCH [MD86] as a model of industrial chemical processes such as electropolishing, corrosion and etching. DIACONIS AND FULTON in [DF91] identified internal DLA as a special case of a "smash sum" operation on subsets of $\mathbb{Z}^2$. Internal DLA is a random growth model which builds a sequence of random growing clusters $(\mathcal{I}_n)_{n \geq 0}$ based on particles performing random walks, where all the particles start from the same fixed point o. Typically, one starts with $\mathcal{I}_0 = \{o\}$, and for each n, we let $\mathcal{I}_{n+1}$ be $\mathcal{I}_n$ plus the first point

where a random walk started at o exits $\mathcal{I}_n$. There are several modifications in this model, where one can start the random walks uniformly at random in the already existing cluster, or one can start with an initial configuration of particles on the state space G. Results in this directions will be surveyed in the following. As in external DLA, understanding the shape of the limiting cluster $\mathcal{I}_n$, the *internal DLA cluster* with $n+1$ particles, is the main question in this model. Also, of fundamental significance as mentioned in the initial paper [MD86], is to know how smooth a surface formed by diffusion limited processes may be. These problems are well understood mathematically on many state spaces, and there are very precise results. While the limiting object does not show any fractal structure (with the exception of the Sierpinski carpet graph), we may consider the state spaces to be fractal graphs. On the Sierpinski gasket graphs there are some recent results for the limit shape, but on other fractal graphs there is nothing known for the moment.

The crucial difference between the above two models is that the dynamics of the external model roughens the cluster while the dynamics of the internal model makes the cluster smoother.

Structure of the paper. After fixing the notation and the basic notions in Section 2, we focus on the external DLA model in Section 3, in which we survey the available results on the growth of arms in this model, number of holes, variations of the standard model. The results will not be stated in the chronological order of publication, but according to the state space they evolve. Finally, in Section 4 we survey the results for the limit shapes of the internal DLA cluster, and we include several questions through the whole work.

2 Preliminaries

Graphs. Let G be an infinite, locally finite graph (i.e. every vertex has finite degree denoted by $\deg(x)$). The neighborhood relation will be denoted by " $\sim$ ", and by $x \sim y$ we mean that (x, y) is an edge in G. Let $o \in \mathsf{G}$ be a fixed distinguished vertex, which will be called the *origin* or the *root*. For $x, y \in \mathsf{G}$, the distance $d(x, y)$ represents the minimal number of edges on the path connecting x with y. For a subgraph A of G, we denote by ∂A the outer boundary of A:

$$\partial A = \{y \in \mathsf{G} : y \notin A, \ \exists x \in A : \ x \sim y\}.$$

For $x \in \mathsf{G}$ and $n \geq 0$, we write $\mathcal{B}_n(x) = \{y \in G : \ d(x, y) \leq n\}$ for the ball of radius n and center x in G. If the center of the ball is o, we write only $\mathcal{B}_n$.

Random Walks. Let $(S_n)_{n \geq 0}$ be a random walk on G, and denote by $\mathbb{P}_x$ the probability measure of the random walk started at x. We do not fix yet the transition probabilities for the random walk, since those will change from case to case, and we will mention them as needed. For a subset $A \subset \mathsf{G}$, let $T(A)$ be the hitting time of A, defined as

$$T(A) = \min\{n \geq 0 : \ S_n \in A\}.$$

For a set $\{x\}$ with a single vertex, we write $T(x)$ instead of $T(\{x\})$. The *heat kernel* of the random walk S_n is defined to be

$$p_n(x, y) = \mathbb{P}_x[S_n = y],$$

and the *Green function* $G(x, y)$ is defined as

$$G(x, y) = \sum_{n \geq 0} p_n(x, y),$$

which is well defined and finite precisely when the random walk is transient. For a subset $A \subset \mathsf{G}$, the *killed or the stopped Green function* $G_A(x, y)$ is defined as

$$G_A(x, y) = \sum_{n \geq 0} \mathbb{P}_x[S_n = y, T(A) > n].$$

The *hitting distribution* $H_A(x, y)$ is then

$$H_A(x, y) = \mathbb{P}_x[S_{T(A)} = y], \quad \text{for } y \in A$$

and $S_{T(A)}$ is the hitting position of A. If the random walk (S_n) starts at o, we write

$$h_A(x) = \mathbb{P}_o[S_{T(A)} = y], \quad \text{for } y \in A \tag{1}$$

for the probability of the random walk starting at o to first hit A in y, that is h_A is the harmonic measure (from o) of the set A, and $\sum_{y \in A} h_A(y) = 1$.

3 External DLA

We define here formally the external DLA model, by explaining what does it mean to release a particle at infinity. Several variants of external DLA have been considered, but we refer here to the original, simplest model, which can be defined as follows on any space where the notion of random walk or diffusion exists. If the Poisson boundary consists of one point and the random walk is recurrent (for instance the case of simple random walks on $\mathbb{Z}$ and $\mathbb{Z}^2$), external DLA can be defined so that the law of the location of a new particle is the harmonic measure of the existing aggregate with pole at infinity. If the random walk is transient (such as the case of simple random walks on $\mathbb{Z}^d$, with $d \geq 3$, or on regular trees $\mathbb{T}_d$ of degree $d \geq 2$, n-dimensional Sierpinski carpet graph, for $n \geq 3$), one can consider the harmonic measure with a pole far away from the aggregate, let the pole go to infinity and take limits (i.e., conditioning the random walk coming from infinity to hit the cluster). That is, in defining rigorously external DLA, we have to distinguish the cases when the random walk (S_n) on the infinite graph G is recurrent or transient; the Poisson boundary of the random walk plays also a role in this case.

We shall write $\mu_A(y)$ for the *harmonic measure from infinity*, that is, for the probability to start a random walk at infinity and to hit the finite subset $A \subset \mathsf{G}$ in point y. Depending on whether the graph G is transient or recurrent, this measure can take different forms, and we cannot define it globally on any general graph here. This will be made precise in the concrete cases below.

Definition 3.1. *Let G be an infinite graph, and (S_n) a random walk on it. External DLA on G is a Markov chain $(\mathcal{E}_n)_{n \geq 0}$ on finite subsets of G, which evolves in the following way. Start with a single vertex $o \in \mathsf{G}$, that is $\mathcal{E}_0 = \{o\}$. Given the process $\mathcal{E}_n$ at time n, let y_{n+1} be a random vertex in $\partial \mathcal{E}_n$ chosen according to the harmonic measure (from infinity) of $\partial \mathcal{E}_n$. That is,*

$$\mathbb{P}[y_{n+1} = y | \mathcal{E}_n] = \mu_{\partial \mathcal{E}_n}(y), \quad \text{for } y \in \partial \mathcal{E}_n,$$

and we set $\mathcal{E}_{n+1} = \mathcal{E}_n \cup \{y_{n+1}\}$.

Definition 3.2. *The* cluster at infinity $\mathcal{E}_\infty$ *for the external DLA process $(\mathcal{E}_n)$ on G is defined as*

$$\mathcal{E}_\infty = \bigcup_{n=1}^{\infty} \mathcal{E}_n.$$

It is immediate that the external DLA cluster $\mathcal{E}_n$ at time n contains exactly $n + 1$ particles. This model is hard to study. The difficulty comes from the fact that the dynamics is neither monotone nor local (meaning that if big tentacles surround a vertex x, than x will never be added to the cluster). By non-monotonicity we mean that there is no coupling between the external DLA starting from a cluster C and another from a cluster $D \subset C$ such that, at each step, the inclusion of the clusters remains valid almost surely. Understanding the shape of $\mathcal{E}_n$, as $n \to \infty$ and the fractal nature of this object are problems one would be typically interested in. While mathematically this is out of reach for the time being, there are other partial results concerning the growth of arms in external DLA, the number of holes.

3.1 Integer lattices $\mathbb{Z}^d$

In this subsection the state space for the external DLA process is $\mathsf{G} = \mathbb{Z}^d$, $d \geq 1$. Even for $\mathbb{Z}^d$, there are no results that prove the fractal nature of the limiting object, or results that prove the 0 density in the long run. The first rigorous results go back to Kesten [Kes87a, Kes90], who gives estimates on the growth of arms in external DLA. Since for $d = 1$, the behavior of standard external DLA is trivial, we consider $d \geq 2$, and let $(S_n)_{n \geq 0}$ be a simple random walk on $\mathbb{Z}^d$.

For $d = 2$, for any finite nonempty $A \subset \mathbb{Z}^2$, we have $T(A) < \infty$ with probability one, and we define the *harmonic measure (from infinity) of A*

$$\mu_A(y) = \lim_{|x| \to \infty} H_A(x, y), \tag{2}$$

where $|x|$ denotes the Euclidean norm of x. The limit $\lim_{|x| \to \infty}$ corresponds to "releasing the particle at infinity". In this case, (S_n) is recurrent, so that by [Spi76, Theorem 14.1] the limit in (2) exists and $\sum_{y \in A} \mu_A(y) = 1$.

For $d \geq 3$, since the random walk (S_n) is transient, the limit $\lim_{|x| \to \infty} H_A(x, y)$ in (2) is identically zero (cf [Spi76, Proposition 25.3]). So in order to obtain a nontrivial limit similar to the one in (2), we have to condition on $T(A)$ being finite. This conditioning gives the factor of the capacity of the set A in the denominator. In this case, we define *harmonic measure (from infinity)* of a finite subset $A \subset G$ as

$$\mu_A(y) = \lim_{d(o,x) \to \infty} \frac{H_A(x, y)}{\sum_{z \in A} H_A(x, z)} = \lim_{d(o,x) \to \infty} \mathbb{P}_x[S_{T(A)} = y | T(A) < \infty], \quad \text{for } y \in A, \tag{3}$$

which is proportional to the so-called *equilibrium measure* associated to the set A. The limit in (3) exists again by [Spi76, Proposition 26.2] for $d = 3$ (the same proof works also for $d > 3$) and satisfies $\sum_{y \in A} \mu_A(y) = 1$. Therefore, we have a valid definition for external DLA, and we let $r(\mathcal{E}_n)$ to be the radius of $\mathcal{E}_n$, defined as

$$r(\mathcal{E}_n) = \max\{|x| : x \in \mathcal{E}_n\}, \tag{4}$$

Theorem 3.3. *[Kes87b, Theorem] and [Kes87a, Corollary] There exists constants $C(d) < \infty$ such that with probability 1*

$$\limsup_{n \to \infty} n^{-2/3} r(\mathcal{E}_n) \leq C(2), \quad \text{if } d = 2$$

$$\limsup_{n \to \infty} n^{-2/d} r(\mathcal{E}_n) \leq C(d), \quad \text{if } d \geq 3.$$

The proof uses classical estimates for the harmonic measure (from infinity) as defined in (2) and (3) and for the hitting probabilities. Simulations actually indicate that for $d = 2$, $\mathbb{E}[r(\mathcal{E}_n)] \approx n^{10/17}$ but as far as the lower bound is concerned, nothing has been proven beyond $\sqrt{n}$ in the 35 years since the model has been introduced. It would be very interesting to prove even a logarithmic correction, i.e. to prove that $\mathbb{E}[r(\mathcal{E}_n)] \geq \sqrt{n} \log(n)$. On $\mathbb{Z}^d$, a lower bound on the number $N(n)$ of vertices in $\mathcal{B}_n$ which are occupied by the cluster E_∞ is known.

Theorem 3.4. *[Kes90, Theorem 2] There exist constants $C(d)$ such that with probability* 1

$$N(n) \geq C(d) n^{d-1}, \quad \text{for infinitely many } n.$$

Another non-trivial result on $\mathbb{Z}^2$ concerns the number of holes $\mathcal{H}_n$ in the external DLA cluster $\mathcal{E}_n$. A hole of $\mathcal{E}_n$ is a finite connected component of $\mathbb{Z}^2 \setminus \mathcal{E}_n$.

Theorem 3.5. *[EW99] For any finite connected subset e of $\mathbb{Z}^2$ we have*

$$\mathbb{P}[\mathcal{H}_n \text{ converges to infinity as } n \to \infty | \mathcal{E}_0 = e] = 1.$$

Theorem 3.3 has been improved in [BY17], where upper bounds on the growth rate of arms in external DLA cluster are given on a big class of transient graphs such as: transitive of polynomial growth of degree ≥ 4, $\mathbb{Z}^3$, transitive and of exponential growth, non-amenable, n-dimensional Sierpinski gasket graphs ($n \geq 3$). In particular, on $\mathbb{Z}^3$, the factor $n^{-2/3}$ from Theorem 3.3 has been improved to $n^{-1/2}/\log(n)$. On the class of transient graphs G considered in [BY17], the harmonic measure (from infinity) μ_A of a set A is defined as in (3).

A directed version of external DLA has been recently introduced on $\mathbb{Z}^2$ in [Mar17]. In a series of three papers [AABK16, AAK13, Ami15], a one-dimensional external DLA model based on random walks with long jumps (that depend on a parameter α) is proposed, which tries to capture the fractal nature of the standard DLA. Depending on the values of α, the random walk (S_n) with long range on $\mathbb{Z}$ may be recurrent or transient, and for the precise definition of harmonic measure from infinity we refer to these three papers. The main results of [AABK16, AAK13] can be summarized into the following theorem.

Theorem 3.6. *Let (S_n) be a symmetric random walk on $\mathbb{Z}$ that satisfies $\mathbb{P}[|S_1 - S_0| = k] \sim ck^{-1-\alpha}$. Let $d(\mathcal{E}_n)$ be the diameter of the external DLA cluster $\mathcal{E}_n$. Then almost surely:*

(a) If $\alpha > 3$, then $n - 1 \leq d(\mathcal{E}_n) \leq Cn + o(n)$, where C depends only on the random walk.

(b) If $2 < \alpha \leq 3$, then $d(\mathcal{E}_n) = n^{\beta + o(1)}$, where $\beta = \frac{2}{\alpha - 1}$.

(c) If $1 < \alpha < 2$, then $d(\mathcal{E}_n) = n^{2 + o(1)}$.

(d) If $\frac{1}{3} < \alpha < 1$, then $n^{\beta + o(1)} \leq d(\mathcal{E}_n) \leq n^{\beta' + o(1)}$, where $\beta = \max(2, \alpha^{-1})$ and $\beta' = \frac{2}{\alpha(2-\alpha)}$.

(e) If $0 < \alpha < \frac{1}{3}$, then $d(\mathcal{E}_n) = n^{\beta + o(1)}$, where $\beta = \alpha^{-1}$.

The last one [Ami15] from the series of three papers mentioned above deals with the cluster at infinity E_∞, and it is shown that for random walks (S_n) with finite third moment on $\mathbb{Z}$, E_∞ has renewal structure and positive density, while for random walks with finite variance the renewal structure no longer exists and E_∞ has 0 density.

Theorem 3.7. *[Ami15, Theorem 1] Assume that the step distribution ξ of the random walk (S_n) on $\mathbb{Z}$ satisfies $\mathbb{P}[\xi > n] \le Cn^{-\alpha}$ for any n and some $\alpha > 3$. There exists some $B > 0$ such that a.s. $\mathcal{E}_\infty$ has density B. Further, B is the limit density of $\mathcal{E}_n$:*

$$B = \lim_{m_1 \to \infty, m_2 \to \infty} \frac{|\mathcal{E}_\infty \cap [-m_1, m_2]|}{m_1 + m_2} = \lim_{n \to \infty} \frac{n}{d(\mathcal{E}_n)}.$$

Theorem 3.8. *[Ami15, Theorem 2] Assume that there exist $2 < \alpha < 3$ and constants $c_1, c_2 > 0$ so that ξ satisfies $c_1 n^{-\alpha} \le \mathbb{P}[\xi > n] \le c_2 n^{-\alpha}$ for all n then a.s.*

$$|\mathcal{E}_\infty \cap [-n, n]| = n^{\frac{\alpha-1}{2} + o(1)}.$$

In particular, $\mathcal{E}_\infty$ has 0 density in the sense that $\lim_{n \to \infty} \frac{|\mathcal{E}_\infty \cap [-n,n]|}{n} = 0$.

The results mentioned above are the only ones available for external DLA on $\mathbb{Z}^d$, and the limit shape and the density problem still resist a mathematical proof. There are many open problems and questions in this direction; see [BY17] for more details.

Conjecture 3.9. *On $\mathbb{Z}^d$, the rate of growth of the radius of the external DLA cluster $\mathcal{E}_n$ started at $\mathcal{E}_0 = \{0\}$ is of order $n^{1/d}$:*

$$\limsup_{n \to \infty} n^{-1/d} \mathbb{E}[r(\mathcal{E}_n)] = 0.$$

Question 3.10. *What is the distribution of the number of ends of the cluster at infinity $\mathcal{E}_\infty$ on $\mathbb{Z}^d$?*

3.2 Trees $\mathbb{T}_d$

One reason that makes the lattice case $\mathbb{Z}^d$ hard to investigate is that there is no simple way to describe the harmonic measure (from infinity) for the boundary of an external DLA cluster on $\mathbb{Z}^d$. On other state spaces, such as trees, which have no loops, the model is more tractable and the harmonic measure (from infinity) can be understood. In [BPP97], an adjusted version of external DLA on d-regular trees $\mathbb{T}_d$, were the fingering phenomenon occurs, was introduced. The dynamics of their model is as follows: the initial cluster $\mathcal{E}_0$ contains only the root. Vertices are then added one by one from among those neighboring the current subtree. The choice of which vertices to add is random, with vertices in generation n (i.e. distance n from the root) chosen with probabilities proportional to α^{-n} where $\alpha > 0$ is a fixed parameter. Then $\mathcal{E}_n$ is the subtree at step n and let $r(\mathcal{E}_n) = \max\{d(o, x) : x \in \mathcal{E}_n\}$ denote the maximum height of a vertex in $\mathcal{E}_n$, which is similar to the radius in (4). For this model, for a finite subtree $A \subset \mathbb{T}_d$ with boundary ∂A, its harmonic measure $\mu_{\partial A}^\alpha$ (from infinity) on ∂A, with parameter $\alpha > 0$ can be computed as

$$\mu_{\partial A}^\alpha(y) = \frac{\alpha^{d(o,y)}}{\sum_{x \in \partial A} \alpha^{-d(o,y)}}, \quad \text{for } y \in \partial A,$$

see Definition on page 4 in [BPP97]. In the latter paper, the case $\alpha < 1$ is studied. The external DLA cluster $\mathcal{E}_n$ is the position at time n of the Markov chain defined in Definition 3.1, where $\mathcal{E}_{n+1}$ is obtained from $\mathcal{E}_n$ by adding a new vertex according to the harmonic measure defined in the previous equation. For $\alpha \ge 1$ it is easy to see that $\mathcal{E}_\infty$ is almost surely the entire tree. For $\alpha = 1$, one has the uniform measure $\mu_{\partial A}^1$ on $\partial \mathcal{E}_n$ (this corresponds to the Eden model). From the external DLA perspective, the case $\alpha < 1$ is the interesting one, where one obtains the so-called fingering phenomenon. For this external DLA model, in [BPP97] obtained a strong law and a central limit theorem for the height $r(\mathcal{E}_n)$ of the DLA cluster.
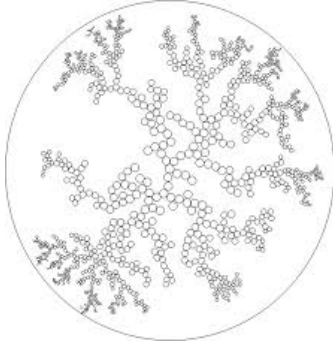
Figure 2: : (by Ronen Eldan) external DLA with 1000 particles, viewed on the Poincaré disc model.

Theorem 3.11. *Let $\mathbb{T}_d$ be a d-regular tree and $0 < \alpha < 1$. There exists constants $r_0(\alpha, d) \in (0, 1)$ and $\sigma^2 = \sigma^2(\alpha, d) > 0$ such that*

(a) $\lim_{n \to \infty} \frac{r(\mathcal{E}_n)}{n} = r_0(\alpha, d)$ *a.s.*

(b) $\frac{r(\mathcal{E}_n) - n r_0(\alpha, d)}{\sqrt{n}} \xrightarrow{\mathcal{D}} N(0, \sigma^2)$ *as $n \to \infty$.*

The model considered here can be also interpreted as a model of first passage percolation on $\mathbb{T}_d$.

3.3 Hyperbolic plane $\mathbb{H}^2$

In [Eld15], external DLA on the hyperbolic plane $\mathbb{H}^2$ is considered, and it is shown that the cluster at infinity E_∞ almost surely admits a positive upper density. In the hyperbolic setting the behavior of the aggregate is simpler to analyze than the Euclidean one; the rate of decay of the hyperbolic potential plays an important role in understanding the external DLA. See Figure 2 for a picture of external DLA model with 1000 particles, viewed on the Poincaré disc model. In his construction, particles are metric balls of radius 1, $\mathcal{E}_0 = \{p_0\}$, where p_0 is a fixed point in $\mathbb{H}^2$, and recursively $\mathcal{E}_{n+1} = \mathcal{E}_n \cup \{y_{n+1}\}$, where $\{y_{n+1}\}$ is added to the aggregate $\mathcal{E}_n$ according to a (harmonic) measure $\mu_{\partial \mathcal{E}_n}(y)$ with pole at infinity that has to be carefully constructed on $\mathbb{H}^2$, such that external DLA makes sense in this setting. For details on this construction, we refer to [Eld15]; the main result of his paper reads as following.

Theorem 3.12. *[Eld15, Theorem 1.1] The external DLA cluster at infinity E_∞ almost surely has an upper density greater than c, where $c > 0$ is an universal constant.*

We would like to point out the fact that the behavior of external DLA on the hyperbolic plane and on the regular tree $\mathbb{T}_d$ as considered in [BPP97] is completely different, even though the hyperbolic plane has a tree-like structure.

3.4 Cylinder graphs

Other results on external DLA that worth mentioning have been proven in [BY08] on cylinder graphs $\mathsf{G} \times \mathbb{N}$.

For a graph G, the cylinder with base G, denoted by $\mathsf{G} \times \mathbb{N}$, is defined as: the vertex set of $\mathsf{G} \times \mathbb{N}$ is $V(\mathsf{G}) \times \mathbb{N}$, where $V(\mathsf{G})$ represents the vertex set of G. The edge set is defined by the following relations: for all $u, v \in \mathsf{G}$ and all $m, k \in \mathbb{N}$, $(u, m) \sim (v, k)$ if and only if $m = k$ and $u \sim v$, or $|m - k| = 1$ and $u = v$. The cylinder with base G is obtained by just placing infinitely many copies of G one over the other, and connecting each vertex in a copy to its corresponding vertices in the adjacent copies.

On $\mathsf{G} \times \mathbb{N}$, particles perform simple random walks (S_n) from infinity. Such random walks are recurrent on $\mathsf{G} \times \mathbb{N}$, and the harmonic measure from infinity can be in this case defined similar to the one on $\mathbb{Z}^2$, as in (2). That is, vertices are added to the existing cluster $\mathcal{E}_n$ according to the measure in (2). Denote by G_i the induced subgraph on the vertices of $\mathsf{G} \times \{i\}$, for all $i \in \mathbb{N}$, and call G_i the i-th level of $\mathsf{G} \times \mathbb{N}$. One of the results proven in [BY08] is that external DLA on $\mathsf{G} \times \mathbb{N}$ grows arms if the base graph G mixes fast. Recall that the mixing time $m(\mathsf{G})$ of the simple random walk on G is the time it takes for the random walk to come close in total-variation distance to the stationary distribution.

Theorem 3.13. *[BY08, Theorem 2.1] Let $2 \leq d \in \mathbb{N}$. There exists $n_0 = n_0(d)$, such that the following holds for all $n > n_0$: let G be a d-regular graph of size n, and mixing time $m(\mathsf{G}) \leq \dfrac{\log^2 n}{(\log \log n)^5}$. Let $(\mathcal{E}_t)$ be the external DLA process on $\mathsf{G} \times \mathbb{N}$ with $\mathcal{E}_0 = \mathsf{G}_0$, and for $m \in \mathbb{N}$, let T_m be the first time the DLA cluster reaches the layer m. Then, for all m, $\mathbb{E}[T_m] < \dfrac{4mn}{\log \log n}$.*

This phenomenon is often referred to as *the aggregate grows arms*, i.e. grows faster than order $|\mathsf{G}|$ particles per layer. As mentioned in [BY08], the result above is believed not to be optimal, and a stronger result is conjectured.

Conjecture 3.14. *[BY08, Conjecture 2.2] Let (G_n) be a family of d-regular graphs such that $\lim_{n \to \infty} |\mathsf{G}_n| = \infty$. There exists $0 < \gamma < 1$ and n_0 such that for all $n > n_0$ the following holds: set $\mathsf{G} = \mathsf{G}_n$ and let $(\mathcal{E}_t)$ be the external DLA process on $\mathsf{G} \times \mathbb{N}$ with $\mathcal{E}_0 = \mathsf{G}_0$, and T_m be the first time the external DLA cluster reaches level m. Then, for all m, $\mathbb{E}[T_m] \leq m|\mathsf{G}_n|^\gamma$.*

Concerning the density of the limit cluster at infinity $\mathcal{E}_\infty$, for cylinder graphs $\mathsf{G} \times \mathbb{N}$ with base G, in the same paper there are two results. For stating them, let us define the *fractional amount of particles in the finite cylinder* $\mathsf{G} \times \{1, \ldots, m\}$ as

$$D(m) = \frac{1}{mn} \sum_{i=1}^{m} |\mathcal{E}_\infty \cap \mathsf{G}_i|$$

and *the density at infinity* as $D = D_\infty = \lim_{m \to \infty} D(m)$. The next result relates the density at infinity to the average growth rate.

Theorem 3.15. *[BY08, Theorem 4.2] For the external DLA process on $\mathsf{G} \times \mathbb{N}$, where (G) is a d-regular graph of size n, we have*

$$D = \lim_{m \to \infty} \frac{1}{mn} \mathbb{E}[T_m]$$

In [BY08, Theorem 4.6] the previous result has been improved to $D \leq \frac{2}{3}$ for the case when the base graph G is a vertex transitive graph. Finally, for a family of base graphs with small mixing time, the following holds.
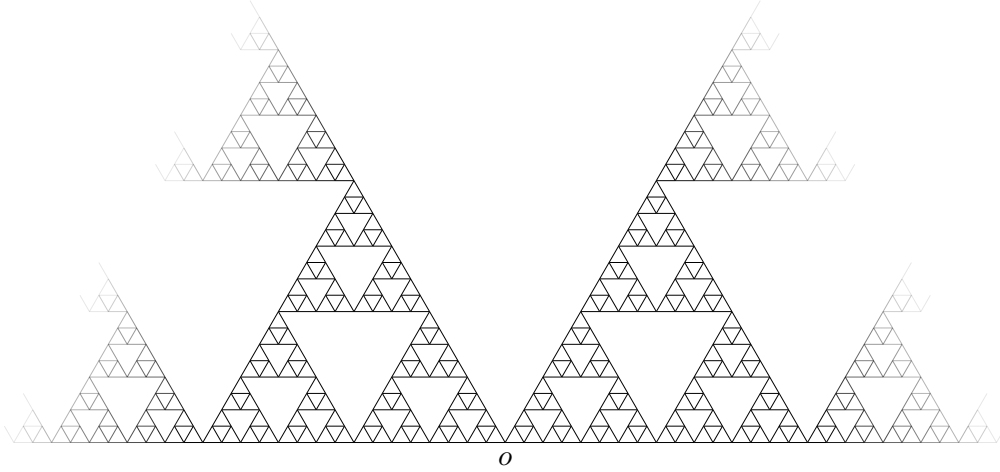
Figure 3: Doubly-infinite Sierpinski gasket graph SG.

Theorem 3.16. *[BY08, Theorem 4.8] Let (G_n) be a family of d-regular graphs $(d \geq 2)$ such that $\lim_{n\to\infty} |\mathsf{G}_n| = \infty$, and for all n, $m(\mathsf{G}_n) \leq \dfrac{\log^2 |\mathsf{G}_n|}{(\log\log |\mathsf{G}_n|)^5}$. Let $D(n)$ be the density at infinity of the external DLA process on $\mathsf{G}_n \times \mathbb{N}$. Then $\lim_{n\to\infty} D(n) = 0$.*

We refer to the last section of [BY08] for several open questions and problems concerning external DLA on cylinder graphs. Many of the bounds from the previous three results can be improved, with some careful technicalities and assumptions on the base graph G.

3.5 Fractal graphs

The appearance of fractal-like structures in DLA models (both internal and external) and their behavior on fractal graphs is the main theme of this paper, and we would like at this point to introduce two fractal graphs: Sierpinski gasket graph and Sierpinski carpet graph.

Sierpinski gasket graph SG is a pre-fractal associated with the Sierpinski gasket, defined as following. We consider in $\mathbb{R}^2$ the sets $V_0 = \{(0,0), (1,0), (1/2, \sqrt{3}/2)\}$ and

$$E_0 = \left\{ \big((0,0),(1,0)\big), \big((0,0),(1/2,\sqrt{3}/2)\big), \big((1,0),(1/2,\sqrt{3}/2)\big) \right\}.$$

Now recursively define $(V_1, E_1), (V_2, E_2), \ldots$ by

$$V_{n+1} = V_n \cup \left\{ \big(2^n, 0\big) + V_n \right\} \bigcup \left\{ \left(2^{n-1}, 2^{n-1}\sqrt{3}\right) + V_n \right\}$$

and

$$E_{n+1} = E_n \cup \left\{ (2^n, 0) + E_n \right\} \bigcup \left\{ \left(2^{n-1}, 2^{n-1}\sqrt{3}\right) + E_n \right\},$$

where $(x,y) + S := \{(x,y) + s : s \in S\}$. Let $V_\infty = \cup_{n=0}^\infty V_n$, $E_\infty = \cup_{n=0}^\infty E_n$, $V = V_\infty \cup \{-V_\infty\}$ and $E = E_\infty \cup \{-E_\infty\}$. Then the doubly infinite Sierpinski gasket graph SG is the graph with vertex set V and edge set E. See Figure 3 for a graphical representation of SG. Set the origin $o = (0,0)$. External DLA on SG seems to be an approachable problem, due to the fact that SG is a p.c.f. fractal, and the existence of cut points simplifies the understanding of the harmonic measure from infinity, which can be defined again as in (2), since the random walk on SG is recurrent.
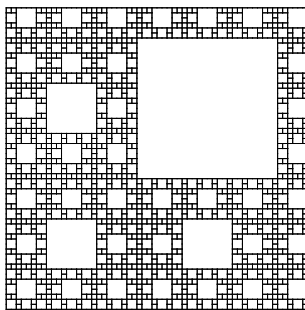
Figure 4: Sierpinski carpet graph SC_2

Sierpinski carpet graph SC_m, called also m-dimensional pre-Sierpinski carpet, is an infinite graph derived from the *Sierpinski carpet*. SC_2 is constructed from the unit square in $\mathbb{R}^2$ by dividing it into 9 equal squares and deleting the one in the center. The same procedure is then repeated recursively to the remaining 8 squares. We refer to [BB99] and the references therein for the rigorous construction of SC_m. See Figure 4 for a finite piece of Sierpinski carpet graph in dimension 2. For $m \geq 3$, simple random walk on SC_m is transient, and the harmonic measure from infinity $\mu_A(y)$, for a finite subset $A \subset \mathsf{SC}_m$ is defined by using the capacity of A and the equilibrium measure of A, similar to (3). More details on the construction can be found in [BY17], where upper bounds for the arms $r(\mathcal{E}_n)$ of external DLA on a large class of transient graphs, that include SC_m, $m \geq 3$, are proved. Their proofs are based on good control of heat-kernel estimates. The bounds for SC_m read as following.

Theorem 3.17. *[BY17, Theorem 5.5] Let SC_m be the m-dimensional Sierpinski carpet graph, and $(\mathcal{E}_m)_{m \geq 0}$ the external DLA process on SC_m started at $\mathcal{E}_0 = \{o\}$ (o is some fixed origin). Then almost surely,*

$$\limsup_{n \to \infty} n^{-\beta} r(\mathcal{E}_n) < \infty$$

where

$$\beta = \begin{cases} \frac{\log_2(13)-2}{3} = 0.5568 & , \quad \text{if } m = 3, \\ \frac{1}{2} & , \quad \text{if } m = 4. \end{cases}$$

When $m \geq 5$, we have almost surely,

$$\limsup_{n \to \infty} (\log n)^{-1} n^{-\frac{2}{d(m)-2}} r(\mathcal{E}_n) < \infty,$$

where $d(m) = \dfrac{\log(3^m - 1)}{\log(3^m - 1) - \log(3^{m-1} - 1)}$.

We would like to conclude the section on external DLA with a couple of problems/questions

Question 3.18. *Can one find an upper bound for the growth of arms in external DLA on SG and on SC_2, where the random walk is strongly recurrent? Can one extend the method of Kesten on $\mathbb{Z}^2$?*

Question 3.19. *Is the density at infinity of the cluster $\mathcal{E}_\infty$ on the Sierpinski gasket graph SG 0?*

Question 3.20. *Does the external DLA cluster on the Sierpinski gasket graph and on the Sierpinski carpet graph have infinitely many holes, with probability one, as in the case of $\mathbb{Z}^2$ from [EW99]?*

Other than SG and SC_m there is a variety of other fractal graphs one can look at, and investigate the behavior of external DLA, which can be easier than $\mathbb{Z}^d$.

Question 3.21. *Assuming that the Poisson boundary of the random walk on G is non trivial, is there a characterization of the Poisson boundary in terms of the number of ends of the external DLA cluster at infinity $\mathcal{E}_\infty$ on G?*

4 Internal DLA

Internal DLA can be defined on any infinite graph G; fix as above a vertex o of G and call it the origin. The internal DLA cluster is built up one site at a time, by letting the n-th particle perform a random walk until it exits the set of sites already occupied by the previous $n-1$ particles, the walk of the n-th particle being independent of the past. Similarly to external DLA, internal DLA is also a Markov chain on finite subsets of G.

Definition 4.1. *Let G be an infinite graph, and (S_n) a simple random walk on G starting at o. Internal DLA on G is a Markov chain $(\mathcal{I}_n)_{n \geq 0}$ on finite connected subsets of G, which evolves in the following way. Start with a single vertex $o \in$ G and set $\mathcal{I}_0 = \{o\}$. Given the process $\mathcal{I}_n$ at time n, let y_{n+1} be a random vertex in $\partial \mathcal{I}_n$ chosen according to the harmonic measure (from o) of $\partial \mathcal{I}_n$, as defined in (1). That is, y_{n+1} is the first exist location from $\mathcal{I}_n$ of the simple random walk (S_n) starting from o, independent of the past:*

$$\mathbb{P}[y_{n+1} = y | \mathcal{I}_n] = h_{\partial \mathcal{I}_n}(y), \quad for \ y \in \partial \mathcal{I}_n,$$

and we set $\mathcal{I}_{n+1} = \mathcal{I}_n \cup \{y_{n+1}\}$.

The set $\mathcal{I}_n$ is called the *internal DLA* cluster at time n, and it contains $n+1$ particles. As $n \to \infty$, we are interested in the asymptotic shape of internal DLA cluster $\mathcal{I}_n$, and the fluctuations of the cluster around the limiting shape. Due to the fact that the harmonic measure for "nice subsets" (for example balls) of G, when G is an Euclidean lattice, or a regular tree, is easier to understand than the harmonic measure from infinity as in the external DLA case, for the internal DLA model we have very precise estimates on many state spaces. Moreover, several variations of the classical internal DLA have been introduced.

4.1 Integer lattices $\mathbb{Z}^d$

The first result concerning the internal DLA goes back to [LBG92], where it is shown that the limit shape of internal DLA cluster is a ball, in the following sense. Let ω_d be the volume of the d-dimensional Euclidean ball of radius 1, and $\mathcal{B}_n$ be the d-dimensional "lattice ball" of radius n, that is, $\mathcal{B}_n = \{x \in \mathbb{Z}^d : |x| \leq n\}$, where $|x|$ denotes the Euclidean norm of x.

Theorem 4.2. *[LBG92, Theorem 1] At time $\lfloor \omega_d n^d \rfloor$, internal DLA cluster occupies a set of sites close to a d-dimensional ball of radius n. More precisely, for any $\epsilon > 0$, with probability 1*

$$\mathcal{B}_{n(1-\epsilon)} \subset \mathcal{I}_{\lfloor \omega_d n^d \rfloor} \subset \mathcal{B}_{n(1+\epsilon)}, \quad for \ n \ large.$$

In this first paper, a basic open question on fluctuations (deviation of $\mathcal{I}_n$ from the Euclidean ball) was asked: are the fluctuations of order $\sqrt{n}$, of order n^δ for some $\delta \in (0, \frac{1}{2})$, or even smaller?

LAWLER [Law95] proved that for $d \geq 2$, the fluctuations are subdiffusive and they are of order at most $n^{1/3}$. While it was conjectured that the fluctuations are at most logarithmic in the radius, this resisted a mathematical proof for about 20 years. Two independent groups JERISON, LEVINE, AND SHEFFIELD [JLS12, JLS13, JLS14a] and ASSELAH AND GAUDILLIÈRE [AG14, AG13a, AG13b], and by different methods have shown that indeed, for $d = 2$ there are $\log(n)$ fluctuations, and for $d \geq 3$, there are $\sqrt{\log(n)}$ fluctuations in the radius. A summary of their results reads as following.

Theorem 4.3. *If $d = 2$, there is an absolute constant c, such that with probability* 1,

$$\mathcal{B}_{n-c\log n} \subset \mathcal{I}_{\lfloor \pi n^2 \rfloor} \subset \mathcal{B}_{n+c\log n}, \text{ for all sufficiently large } n.$$

If $d \geq 3$, there is an absolute constant C, such that with probability 1,

$$\mathcal{B}_{n-C\sqrt{\log n}} \subset \mathcal{I}_{\lfloor \omega_d n^d \rfloor} \subset \mathcal{B}_{n+C\sqrt{\log n}}, \text{ for all sufficiently large } n.$$

A generalization of the classical internal DLA on $\mathbb{Z}^d$ was treated in [LP10], where instead of running all particles from the origin, the authors run the process from an arbitrary starting configuration of particles (initial density of particles) on finer and finer lattices, all particles still performing simple random walks. They then show that, as the lattice spacing tends to zero, the internal DLA has a deterministic scaling limit which can be described as the solution to a certain PDE free boundary problem in $\mathbb{R}^d$. We do not state here the rigorous result, which requires more notation and definition, but refer to the lengthy and complex paper [LP10]. In order to study this general model, a new model called *divisible sandpile* was introduced in [LP09], which uses a continuous amount of mass instead of discrete particles.

The *divisible sandpile model* can be briefly described as following: start with an initial mass μ at the origin o. A vertex is called *full* if it has mass at least 1. Any full site can topple by keeping mass 1 for itself and distributing the excess mass equally among its neighbors. At each time step, one chooses a full site and topple it. As time goes to infinity, provided each full site is eventually toppled, the mass approaches a limiting distribution in which each site has mass ≤ 1; this is proved in [LP09]. Individual topplings do not commute, but the divisible sandpile is *abelian* in the sense that any sequence of topplings produces the same limiting mass distribution; this is proved in [LP10, Lemma 3.1]. The set of sites with limit mass distribution equal to 1 is denoted by $\mathcal{S}_n$ and is called *the divisible sandpile cluster*. The asymptotic shape of the divisible sandpile cluster $\mathcal{S}_n$ is proven to be the same as the one of the internal DLA cluster on $\mathbb{Z}^d$ in [LP09], on regular trees in [Lev09], on comb lattices in [HS12] and on Sierpinski gasket graphs in [HSH18].

Random walks with drift on $\mathbb{Z}^d$. If one lets the particles which build up the internal DLA cluster $\mathcal{I}_n$ perform drifted random walk instead of simple random walk as in the classical model, one can ask about the shape of the limit cluster on any state space. On $\mathbb{Z}^d$, this was open for several years, and the cluster was believed to be represented by the level sets of the Green function for the drifted random walk. This fact has been disproved, and with the help of the divisible sandpile model, in [Luc14] it was proven that the internal DLA cluster is a true heat ball, because it gives rise to a mean-value property for caloric functions. He introduced there the *unfair divisible sandpile*, where the mass is not distributed equally to the neighbors, but according to the one-step probabilities of the drifted random walk; the limit shape for the unfair divisible sandpile on $\mathbb{Z}^d$ was also described there. The main result for the limit shape for drifted internal DLA can be found in [Luc14, Theorem 1.1], and for the limit shape of the unfair divisible sandpile cluster in [Luc14, Theorem 3.3].

Uniform starting points. To my knowledge, the most recent result for internal DLA on $\mathbb{Z}^d$, concerns the limit shape for the cluster when the particles do not start all in the same vertex o, but instead they choose the starting position uniformly at random in the existing cluster. Formally, one can define the internal DLA as in Definition 4.1, starting with $\mathcal{I}_0 = \{0\}$, and given the process at time n, let y_{n+1} be the first exit location from $\mathcal{I}_n$ of the simple random walk $S_n^{X_n}$ starting at X_n, where X_n is a point chosen uniformly on $\mathcal{I}_n$, independent of the past. Set $\mathcal{I}_{n+1} = \mathcal{I}_n \cup \{y_{n+1}\}$. It turns out, as shown in [BDCKL17], that this additional source of randomness arising from the choice of the initial position of the random walk, does not change the limit shape of the process, as the result below shows. Let $b_n := |\mathcal{B}_n|$.

Theorem 4.4. *[BDCKL17, Theorem 1.1] Let $d \geq 2$. There exists constants c_1, c_2, C_1 and C_2 depending only on the dimension d such that, almost surely, the internal DLA cluster $\mathcal{I}_n$ with uniform starting points satisfies*

$$\mathcal{B}_{n(1-C_1 n^{-c_1})} \subseteq \mathcal{I}_{b_n} \subseteq \mathcal{B}_{n(1+C_2 n^{-c_2})},$$

for n large enough.

Question 4.5. *What can we say about the fluctuations of the internal DLA cluster with uniform starting points around the limit shape? Are they bigger (smaller) that the fluctuations for internal DLA when all particles start their random walk in the same vertex o?*

Supercritical percolation cluster on $\mathbb{Z}^d$: In [She10], the state space for the internal DLA cluster was taken to be the supercritical bond percolation cluster on $\mathbb{Z}^d$, with the origin conditioned to be in the infinite cluster. It is shown in [She10, Theorem 1.1] that an inner bound for the internal DLA cluster is a ball in the graph metric. The picture for the outer bound was completed in [DCLYY13, Theorem 1.1], which show that also in this case the limit shape is a ball. The results in their paper hold in a more general setting: given the existence of a "good" inner bound for internal DLA, one can also prove a matching outer bound by using their methods. An interesting problem in the context of internal DLA model on a random graph is to understand the fluctuations.

4.2 Comb lattices $\mathcal{C}_2$

The 2-dimensional comb lattice $\mathcal{C}_2$ is the spanning tree of $\mathbb{Z}^2$, obtained by removing all horizontal edges, except the ones on the x-axis. While $\mathcal{C}_2$ is a simple graph, see Figure 5, it has some remarkable properties in what concerns the behavior of random walks: no form of the so-called Einstein relation for exponents associated with random walks hold on $\mathcal{C}_2$, see [Ber06]. PERES AND KRISHNAPUR [KP04] showed that on $\mathcal{C}_2$ two independent simple random walks meet only finitely often. The comb $\mathcal{C}_2$ is the first example where the limit shape of internal DLA is not a ball in the graph metric, or in any other metric. Indeed, the internal DLA cluster with n particles grows like $n^{2/3}$ in the y-direction, and like $n^{1/3}$ in the x-direction. See Figure 6 for a picture of the internal DLA cluster with 100, 500, and 1000 particles, respectively.
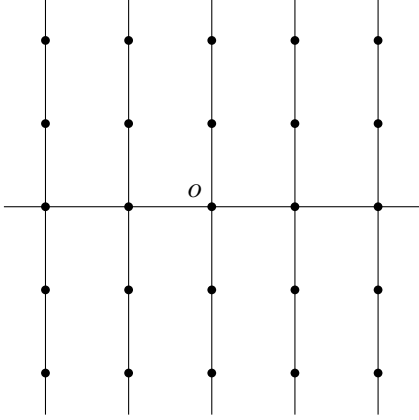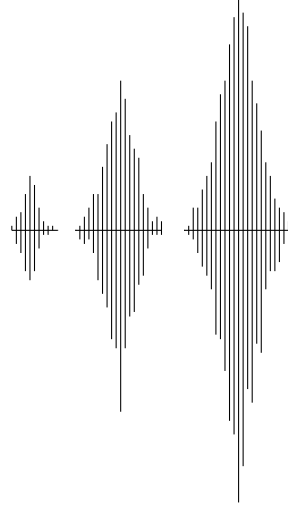
Figure 5: The comb $\mathcal{C}_2$.

Figure 6: Internal DLA cluster.

Let

$$\mathcal{D}_n = \left\{ (x,y) \in \mathcal{C}_2 : \ \frac{|x|}{k} + \left(\frac{|y|}{l} \right)^{1/2} \leq n^{1/3} \right\} \tag{5}$$

where

$$k = \left(\frac{3}{2} \right)^{2/3}, \quad l = \frac{1}{2} \left(\frac{3}{2} \right)^{1/3}.$$

The inner bound for the limit shape of internal DLA cluster on $\mathcal{C}_2$ was proven in [HS12, Theorem 4.2], while the outer bound together together with the fluctuations was proven in [AR16].

Theorem 4.6. *[AR16, Theorem 1.2] There is a positive constant a such that with probability 1, and n large enough*

$$\mathcal{D}_{n-a\sqrt{\log n}} \subset \mathcal{I}_n \subset \mathcal{D}_{n+a\sqrt{\log n}}.$$

Remark that this result does not mean that the fluctuations are sub-logarithmic, but rather gaussian. In [HS12, Theorem 3.5] we also prove that the limit shape for the divisible sandpile cluster on $\mathcal{C}_2$ is given by the set $\mathcal{D}_n$.

4.3 Trees $\mathbb{T}_d$

Internal DLA on discrete groups with exponential growths has been studied in [BB07]. The homogeneous tree $\mathbb{T}_d$ is a particular case (as a Cayley graph of a free group) of these state spaces, for which the authors have proven that the limit shape of internal DLA cluster is a ball in the graph metric, and they give lower bounds for the inner and outer error. The more general result is the following.

Theorem 4.7. *[BB07, Theorem 3.1] Let G be a finitely generated group of exponential growth, and consider the internal DLA model $(\mathcal{I}_n)$ on G, builded up with symmetric random walks with finitely supported increment, starting at the identity o of G. Then, for any constants $C_O > 2$ and $C_I > 3/K$,*

$$\mathbb{P}\left[\exists n_0 \ s.t. \forall n > n_0 : \quad \mathcal{B}_{n-C_I \ln n} \subset \mathcal{I}_{|\mathcal{B}_n|} \subset \mathcal{B}_{n+C_O\sqrt{n}} \right] = 1,$$

where K is a constant that ensures that the ball $\mathcal{B}_n$ contains the boundary $\partial \mathcal{B}_{n-1}$ and $\mathcal{B}_n$ is the ball of radius n around the identity, in the word metric of G.

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An extension of this result on non-amenable graphs, for a wide class of Markov chains was considered in [Hus08]. On discrete groups with polynomial growth, internal DLA has been considered in [Bla04].

4.4 Cylinder graphs

Like in Section 3.4, we consider here cylinder graphs $\mathsf{G} \times \mathbb{Z}$, and we let G to be the cycle graph $\mathbb{Z}_N$ on N vertices. Internal DLA on cylinder graphs $\mathbb{Z}_N \times \mathbb{Z}$ was investigated in [JLS14b], for the following initial setting. For $k \in \mathbb{Z}$, the set $\mathbb{Z}_N \times \{k\}$ is called the k-th level of the cylinder, and $R_k = \{(x, y) \in \mathbb{Z}_N \times \mathbb{N} : y \leq k\}$ the *rectangle of height k*. Let $\mathcal{I}_0 = R_0$, and given the cluster $\mathcal{I}_n$ at time n, let y_{n+1} be the first exit location from $\mathcal{I}_n$ of a random walk that starts uniformly at random on level zero of the cylinder, independent on the past, that is, the starting location is chosen with equal probability among the N sites $(x, 0)$, for $x \in \mathbb{Z}_N$. We then set $\mathcal{I}_{n+1} = \mathcal{I}_n \cup \{y_{n+1}\}$. It has been proven in [JLS14b, Theorem 2] that the limit shape of internal DLA clusters on $\mathbb{Z}_N \times \mathbb{Z}$ is logarithmically close to rectangles, result that we do not state in complete form here, but instead we state a more recent result due to LEVINE AND SILVESTRI [LS18, Theorem 1.1] which generalizes the previous one [JLS14b] (here the fluctuations are described in terms of the Gaussian Free Field exactly). Remark that in the cylinder setting, there are two parameters, the size N of the cycle base graph, and the time n.

Theorem 4.8. *[LS18, Theorem 1.1] Let $(\mathcal{I}_n)_{n \geq 0}$ be the internal DLA process on $\mathbb{Z}_N \times \mathbb{Z}$ starting from $\mathcal{I}_0 = R_0$. For any $\gamma > 0$, $m \in \mathbb{N}$ there exist a constant $C = C(\gamma, m)$ such that*

$$\mathbb{P}[R_{\frac{n}{N} - C \log N} \subseteq \mathcal{I}_n \subseteq R_{\frac{n}{N} + C \log N}] \geq 1 - N^{-\gamma}, \text{ for } N \text{ large enough.}$$

For other results concerning the fluctuations and the behavior of internal DLA clusters on $\mathbb{Z}_N \times \mathbb{Z}$, we refer to [LS18].

4.5 Fractal graphs

We would like to conclude the section about internal DLA with the behavior of the model on the Sierpinski gasket graphs SG. Recall the definition of the Sierpinski gasket graph SG and of the Sierpinski carpet graph SC_2, as given in Section 3.5. Due to the symmetry of SG, it is clear that the limit shape of the internal DLA cluster on SG is a ball in the graph metric, result proved in [CHSHT17].

Theorem 4.9. *[CHSHT17, Theorem 1.1] On SG, the internal DLA cluster of $|\mathcal{B}_n|$ particles occupies a set of sites close to a ball of radius n. That is, for all $\epsilon > 0$, we have*

$$\mathcal{B}_{n(1-\epsilon)} \subset \mathcal{I}_{|\mathcal{B}_n|} \subset \mathcal{B}_{n(1+\epsilon)}, \text{ for all } n \text{ sufficiently large}$$

with probability 1.

A limit shape for the divisible sandpile on SG was described in [HSH18]. Concerning the fluctuations for internal DLA, it is conjectured that there are sub-logarithmic.

Conjecture 4.10. *[Che18, Conjecture 4.1] There exists $C > 0$ such that*

$$\mathcal{B}_{n - C\sqrt{\log n}} \subset \mathcal{I}_{|\mathcal{B}_n|} \subset \mathcal{B}_{n - C\sqrt{\log n}}.$$

Many other questions concerning internal DLA on fractal graphs can be found in the final section of [Che18].

Question 4.11. *Is the limit shape for the internal DLA with uniform starting points on* SG*, again a ball in the graph metric? What about the fluctuations in this case?*

While SG is easier to work with because it is a finitely ramified fractal graph, fact that was important in understanding the divisible sandpile on SG, the absence of this property on SC_2 makes the analysis of the growth models much harder. See Figure 7 for internal DLA behavior on SC_2.
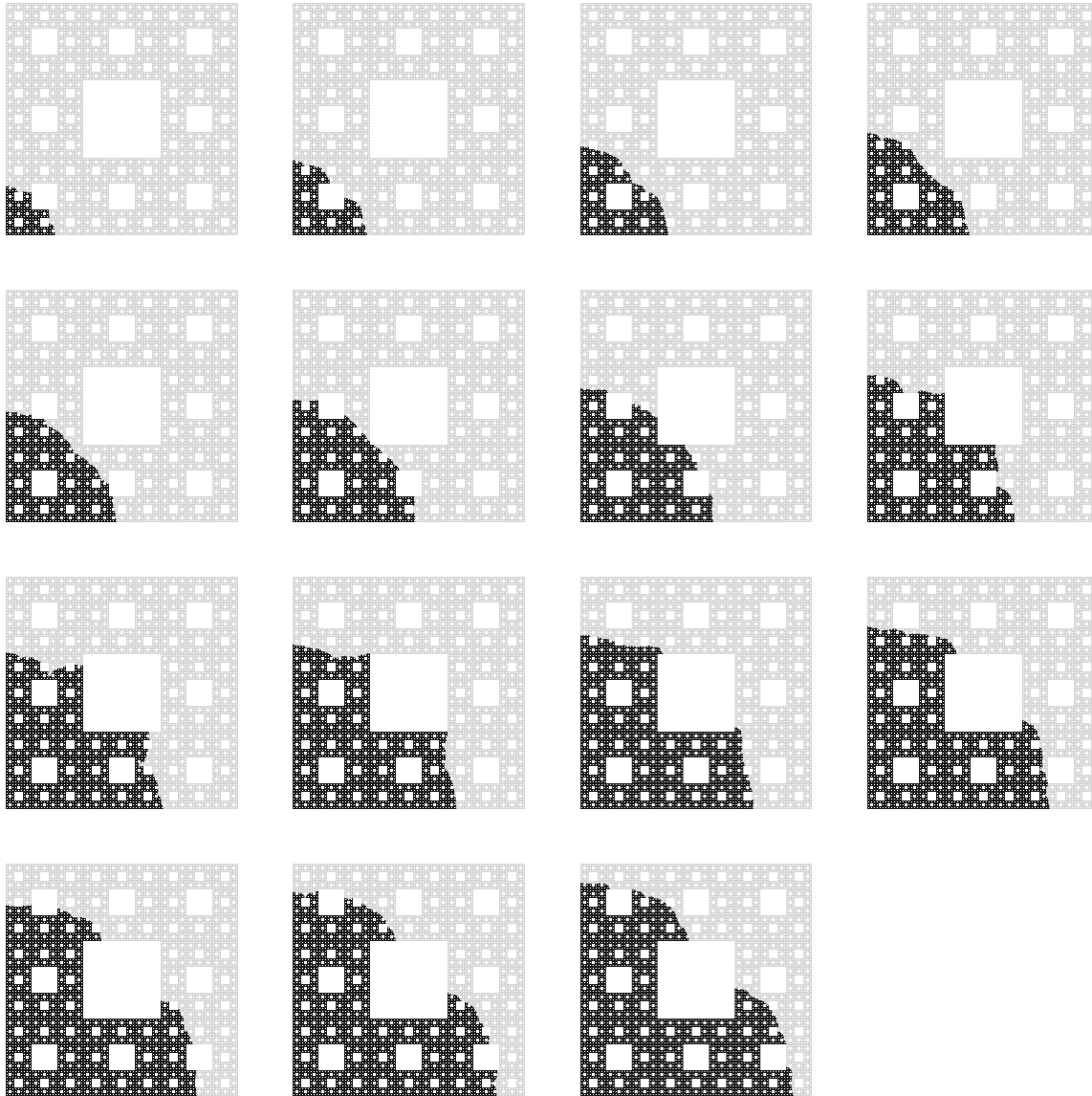

Figure 7: Internal DLA clusters on SC_2 for 10000 up to 150000 particles.
Simulations by W.Huss

Question 4.12. *Does the internal DLA cluster on the 2-dimensional Sierpinski carpet graph* SC_2 *have a (unique) scaling limit? What can one say about the boundary of the limit shape, which due to simulations appears to be of fractal nature?*

Question 4.13. *What is the limit shape of internal DLA on fractal graphs, other that* SG *(which is understood) and* SC_2 *(which seems hard to investigate)?*

Since in most cases, the limit shape for internal DLA is a ball (in the graph metric, or Euclidean metric, or word metric), a more general question is to ask about the state space for the process.

Question 4.14. *What properties should the state space G and the random walk on it have, in order for the internal DLA cluster on G to have a ball as limit shape?*

We would like to conclude this survey with the remark that fractals provide a class of state spaces with intriguing properties, both for the behavior of the external and internal DLA model, respectively. This behavior is definitely not understood on such graphs, and we hope to attract more people from the fractal community into the beauty of these topics.

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