Diophantine Approximations and Fractals

Part II

1 Basics on regular continued fractions

For $a_0 \in \mathbb{N}_0$ (i.e. a_0 is a non-negative integer) and $a_1, a_2, a_3, \ldots \in \mathbb{N}$ (i.e. a_i is a positive integers for $i = 1, 2, 3, \ldots$) we consider an expression of the form

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 \ \ldots}}}.$$

Such an expression is called a *regular continued fraction* (which we shall abreviate by 'CF'), and the numbers a_i are called *elements* of the CF. The number of elements in a CF may be finite or infinite. For ease of notation we write

$$[a_0; a_1, a_2, \ldots] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ \ldots}}.$$

Since a finite CF results from a finite number of rational operations, it is clear that every finite CF represents a positive real number (in fact, we shall see that this statement also holds for the infinite CFs, and furthermore that also the converse is true, namely that every positive real number can be represented in an unique way by a CF). In particular, the number represented by a finite(!) CF must be a rational number (whereas, as we shall see, a real number is represented by an infinite CF if and only if it is an irrational number).

Examples:

A rational number:

$$\frac{17}{5} = \frac{15+2}{5} = 3 + \frac{2}{5} = 3 + \frac{1}{\frac{5}{2}} = 3 + \frac{1}{\frac{4+1}{2}} = 3 + \frac{1}{2 + \frac{1}{2}} = [3; 2, 2].$$

An irrational number:

$$\sqrt{2} = 1 + (\sqrt{2} - 1) = 1 + \frac{1}{\frac{1}{\sqrt{2}-1}} = 1 + \frac{1}{\sqrt{2}+1} = 1 + \frac{1}{2 + (\sqrt{2} - 1)}$$

$$= 1 + \frac{1}{2 + (\sqrt{2} - 1)} = \ldots = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \ \ldots}} = [1; 2, 2, 2, 2, \ldots].$$

Note that in this form for a finite CF we could have that the last element is equal to 1, which then gives two ways of representing a rational number by a CF. We resolve this by demanding that in a finite CF the final element is always greater than 1.
For instance, in the above example we then would have obtained

$$\frac{17}{5} = \ldots = 3 + \frac{1}{2 + \frac{1}{2}} = 3 + \frac{1}{2 + \frac{1}{1 + \frac{1}{1}}} = [3; 2, 1, 1],$$

but we simply do not allow this representation.

The CF-algorithm: *Given a positive real number* ξ *. Let* $[[\xi]]$ *denote the greatest integer less than or equal to* ξ *. Put* $a_0 = [[\xi]]$ *. If* $a_0 = \xi$ *then we are finished, that is the CF of* ξ *is equal to* $[a_0]$ *. If* $a_0 \neq \xi$ *, then there exists a real number* $r_1 > 1$ *such that*

$$\xi = a_0 + \frac{1}{r_1}.$$

Consider r_1 *, and let* $[[r_1]]$ *denote the greatest integer less than or equal to* r_1 *. Put* $a_1 = [[r_1]]$ *. Either we have that* $\xi = a_0 + \frac{1}{a_1}$ *and we are finished, that is the CF of* ξ *is equal to* $[a_0; a_1]$ *, or we have* $\xi \neq a_0 + \frac{1}{a_1}$ *. If we are in the latter case, then there exists a real number* $r_2 > 1$ *such that*

$$\xi = a_0 + \frac{1}{a_1 + \frac{1}{r_2}}.$$

Consider r_2 *, and let* $[[r_2]]$ *denote the greatest integer less than or equal to* r_2 *. Put* $a_2 = [[r_2]]$ *, and proceed as before.*

More generally, the mechanism of finding a_n *, given that* $a_0, \ldots, a_{n-1}$ *have been found, is as follows.*

Assume that $a_0, \ldots, a_{n-1}$ *have been found. If we have that*

$$\xi = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ \ldots \ \cfrac{1}{a_{n-1}}}},$$

then we are finished and ξ *is a rational number with CF equal to* $[a_0; a_1, \ldots, a_{n-1}]$ *. Whereas if*

$$\xi \neq a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ \ldots \ \cfrac{1}{a_{n-1}}}},$$

then there exists a real number $r_n > 1$ *such that*

$$\xi = a_0 + \cfrac{1}{a_1 + + \cfrac{1}{a_2 + \ \ldots \ \cfrac{1}{a_{n-1} + \frac{1}{r_n}}}}.$$

Consider r_n *, and let* $[[r_n]]$ *denote the greatest integer less than or equal to* r_n *. Put* $a_n = [[r_n]]$ *.*

Clearly, this process of finding a_n *can either stop at some stage (in which case* ξ *is a rational number), or it carries on forever (in which case* ξ *is an irrational number).*

Definition 1.1 *For* $[a_0; a_1, a_2, \ldots]$ *a given CF (which can be finite or infinite) we define*

- $\frac{p_n}{q_n} = [a_0; a_1, a_2, \ldots, a_n]$;
- $r_n = [a_n; a_{n+1}, a_{n+2}, \ldots]$.

The number $\frac{p_n}{q_n}$ *is called* n *-th order convergent and* r_n *is called* n *-th order remainder.*

Theorem 1.2 *For* $[a_0; a_1, a_2, \ldots]$ *a given CF (which can be finite or infinite) we have for* $n \in \mathbb{N}_0$

1. $p_{n+1} = a_{n+1} p_n + p_{n-1}$;
2. $q_{n+1} = a_{n+1} q_n + q_{n-1}$;
3. $q_n p_{n-1} - p_n q_{n-1} = (-1)^n$.

Where we have set $p_{-1} = q_0 = 1, q_{-1} = 0$ *and* $p_0 = a_0$.

Proof: 1. **and** 2. **:** (by induction)
For $n = 0$, we have that

$$\frac{p_1}{q_1} = [a_0; a_1] = a_0 + \frac{1}{a_1} = \frac{a_1 a_0 + 1}{a_1} = \frac{a_1 p_0 + p_{-1}}{a_1 q_0 + q_{-1}}.$$

Hence, the statement is true for $n = 0$, giving the start of our induction.
Now assume that for $n \geq 1$ the statement is true all $k < n$ (for all CFs).
Let us first consider

$$[a_1; a_2, \ldots, a_n] = \frac{A_{n-1}}{B_{n-1}}, [a_1; a_2, \ldots, a_{n-1}] = \frac{A_{n-2}}{B_{n-2}} \text{ and } [a_1; a_2, \ldots, a_{n-2}] = \frac{A_{n-3}}{B_{n-3}}.$$

Since the inductive assumption is applicable in this situation, we have that

$$A_{n-1} = a_n A_{n-2} + A_{n-3} \text{ and } B_{n-1} = a_n B_{n-2} + B_{n-3}.$$

Furthermore, we have

$$\frac{p_{n-1}}{q_{n-1}} = a_0 + \frac{1}{[a_1; a_2, \ldots, a_{n-1}]} = a_0 + \frac{B_{n-2}}{A_{n-2}} = \frac{a_0 A_{n-2} + B_{n-2}}{A_{n-2}},$$

which implies that

$$p_{n-1} = a_0 A_{n-2} + B_{n-2} \text{ and } q_{n-1} = A_{n-2}.$$

Similarly, we derive

$$p_{n-2} = a_0 A_{n-3} + B_{n-3} \text{ and } q_{n-2} = A_{n-3}.$$

Using these observations, we now obtain

$$\frac{p_n}{q_n} = a_0 + \frac{B_{n-1}}{A_{n-1}} = a_0 + \frac{a_n B_{n-2} + B_{n-3}}{a_n A_{n-2} + A_{n-3}} = \frac{a_0(a_n A_{n-2} + A_{n-3}) + (a_n A_{n-2} + A_{n-3})}{a_n A_{n-2} + A_{n-3}}$$

$$= \frac{a_n(a_0 A_{n-2} + B_{n-2}) + a_0 A_{n-3} + B_{n-3}}{a_n A_{n-2} + A_{n-3}} = \frac{a_n p_{n-1} + p_{n-2}}{a_n q_{n-1} + q_{n-2}},$$

which finishes the proof of 1. and 2. .
In order to prove 3. , multiply the formula in 1. by q_n and the formula in 2. by p_n , and then subtract the first from the second. This gives

$$q_{n+1} p_n - p_{n+1} q_n = -(q_n p_{n-1} - p_n q_{n-1}),$$

and by iterating this ($(n+1)$ -times), we get

$$q_{n+1} p_n - p_{n+1} q_n = \ldots = (-1)^{n+1}(q_0 p_{-1} - p_0 q_{-1}) = (-1)^{n+1},$$

which then proves 3. . □

Corollary 1.3 *For all* $n \in \mathbb{N}$ *we have*

$$\frac{p_{n-1}}{q_{n-1}} - \frac{p_n}{q_n} = \frac{(-1)^n}{q_n q_{n-1}}.$$

Theorem 1.4 *For* $[a_0; a_1, a_2, \ldots]$ *a given CF (which can be finite or infinite) we have for* $n \in \mathbb{N}_0$

$$q_{n+1}p_{n-1} - p_{n+1}q_{n-1} = (-1)^n a_{n+1}.$$

Proof: Multiply 1. in the previous theorem by q_{n-1}, and 2. by p_{n-1}. Subtracting the so obtained first equality from the second, we get

$$q_{n+1}p_{n-1} - p_{n+1}q_{n-1} = a_{n+1}(q_n p_{n-1} - p_n q_{n-1}) = (-1)^n a_{n+1}.$$

□

Corollary 1.5 *For all* $n \in \mathbb{N}$ *we have*

$$\frac{p_{n-1}}{q_{n-1}} - \frac{p_{n+1}}{q_{n+1}} = \frac{(-1)^n a_{n+1}}{q_{n+1} q_{n-1}}.$$

Lemma 1.6 *For the denominators* q_n *of the convergents of a CF we have for* $n \in \mathbb{N}$

$$q_n q_{n-1} \geq \frac{1}{\sqrt{2}} 2^{n-1}.$$

Proof: We have

$$q_n = a_n q_{n-1} + q_{n-2} \geq q_{n-1}.$$

Hence, it follows

$$\begin{aligned} q_n &= a_n q_{n-1} + q_{n-2} \geq q_{n-1} + q_{n-2} \geq 2q_{n-2} \\ &\geq \ldots \geq \begin{cases} 2^{\frac{n}{2}} q_0 & \text{for} \quad n \text{ even} \\ 2^{\frac{n-1}{2}} q_1 & \text{for} \quad n \text{ odd} \end{cases} \\ &\geq 2^{\frac{n-1}{2}}. \end{aligned}$$

Using this estimate, we derive

$$q_n q_{n-1} \geq 2^{\frac{n-1}{2}} 2^{\frac{n-2}{2}} \geq \frac{1}{\sqrt{2}} 2^{n-1}.$$

□

Proposition 1.7 *For an infinite CF* $[a_0; a_1, a_2, \ldots]$ *we have the following.*

1. *The sequence* $\left(\frac{p_{2n}}{q_{2n}}\right)$ *of convergents of even order is increasing.*
2. *The sequence* $\left(\frac{p_{2n+1}}{q_{2n+1}}\right)$ *of convergents of odd order is decreasing.*
3. *Every convergent of odd order is greater than any convergent of even order, and vice versa, that is every convergent of even order is less than any convergent of odd order.*
4. *The distances between two consecutive convergents tend to zero, i.e.*

$$\lim_{n\to\infty} \left(\frac{p_{n+1}}{q_{n+1}} - \frac{p_n}{q_n}\right) = 0.$$

With other 'words', we have that

$$\frac{p_0}{q_0} < \frac{p_2}{q_2} < \frac{p_4}{q_4} < \ldots < \frac{p_{2n}}{q_{2n}} \nearrow \xi \swarrow \frac{p_{2n+1}}{q_{2n+1}} < \ldots \frac{p_5}{q_5} < \frac{p_3}{q_3} < \frac{p_1}{q_1},$$

where ξ is the limit of the convergents, that is $\xi = \lim_{n\to\infty} p_n/q_n$.
Proof: By Corollary 5 we have

$$\frac{p_{2n}}{q_{2n}} = \frac{p_{2n+2}}{q_{2n+2}} + \frac{(-1)^{2n+1}a_{2n+2}}{q_{2n}q_{2n+2}} = \frac{p_{2n+2}}{q_{2n+2}} - \frac{a_{2n+2}}{q_{2n}q_{2n+2}} < \frac{p_{2n+2}}{q_{2n+2}},$$

and

$$\frac{p_{2n-1}}{q_{2n-1}} = \frac{p_{2n+1}}{q_{2n+1}} + \frac{(-1)^{2n}a_{2n+1}}{q_{2n-1}q_{2n+1}} = \frac{p_{2n+1}}{q_{2n+1}} + \frac{a_{2n+1}}{q_{2n-1}q_{2n+1}} > \frac{p_{2n+1}}{q_{2n+1}}.$$

This proves the first two assertions in the proposition. To see the third, we use Corollary 3 and what we have just proven, and derive

$$\frac{p_{2n}}{q_{2n}} = \frac{p_{2n+1}}{q_{2n+1}} - \frac{1}{q_{2n}q_{2n+1}} < \frac{p_{2n+1}}{q_{2n+1}} < \frac{p_{2n-1}}{q_{2n-1}} < \frac{p_{2n-3}}{q_{2n-3}} < \ldots < \frac{p_1}{q_1}.$$

Hence for each $n \in \mathbb{N}$, we have $\frac{p_{2n}}{q_{2n}}$ is strictly less than all convergents of odd order less than or equal to $2n+1$. We are now going to prove (by way of contradiction) that this also holds for all convergents of odd order greater than $2n+1$.
Therefore, we assume that the statement is false, i.e. we assume that there exists an odd number $2k+1$ such that $2k+1 > 2n+1$ and

$$\frac{p_{2n}}{q_{2n}} \geq \frac{p_{2k+1}}{q_{2k+1}}.$$

By the first part of the proposition, we have

$$\frac{p_{2n}}{q_{2n}} < \frac{p_{2n+2}}{q_{2n+2}} < \ldots < \frac{p_{2k}}{q_{2k}}.$$

Hence, by combining these two latter estimates, we get

$$\frac{p_{2k+1}}{q_{2k+1}} < \frac{p_{2k}}{q_{2k}}.$$

This clearly contradicts the fact which we have just seen to be true, namely the fact that

$$\frac{p_{2k}}{q_{2k}} < \frac{p_{2k+1}}{q_{2k+1}},$$

which finishes the proof of the third assertion in the proposition.
The forth assertion is an immediate consequence of Corollary 3 and Lemma 6.

□

We summarize our considerations in the following theorem.

Theorem 1.8 *To every positive real number ξ there corresponds a unique CF with value equal to ξ (where in the finite case we assume that the final element in the CF is greater than 1) . This CF is finite if ξ is rational, and infinite if ξ is irrational.*
In particular, if the CF is infinite, then ξ is equal to the limit of its convergents, that is

$$\xi = \lim_{n\to\infty} \frac{p_n}{q_n},$$

where we have more precisely

$$\frac{p_0}{q_0} < \frac{p_2}{q_2} < \frac{p_4}{q_4} < \ldots < \frac{p_{2n}}{q_{2n}} \nearrow \xi \swarrow \frac{p_{2n+1}}{q_{2n+1}} < \ldots \frac{p_5}{q_5} < \frac{p_3}{q_3} < \frac{p_1}{q_1}.$$

Theorem 1.9 *For* $\xi = [a_0; a_1, a_2, \ldots]$ *and* $n \in \mathbb{N}$ *, we have*

$$\xi = \frac{p_n r_{n+1} + p_{n-1}}{q_n r_{n+1} + q_{n-1}}.$$

Proof: (by induction) For $n = 0$ we have

$$\frac{p_0 r_1 + p_{-1}}{q_0 r_1 + q_{-1}} = \frac{a_0 r_1 + 1}{r_1} = a_0 + \frac{1}{r_1} = \xi.$$

Now assume that the statement is true for n . Then

$$\begin{aligned}\xi &= \frac{p_n r_{n+1} + p_{n-1}}{q_n r_{n+1} + q_{n-1}} = \frac{p_n(a_{n+1} + \frac{1}{r_{n+2}}) + p_{n-1}}{q_n(a_{n+1} + \frac{1}{r_{n+2}}) + q_{n-1}} \\ &= \frac{p_n a_{n+1} r_{n+2} + p_n + p_{n-1} r_{n+2}}{q_n a_{n+1} r_{n+2} + q_n + q_{n-1} r_{n+2}} = \frac{(p_n a_{n+1} + p_{n-1}) r_{n+2} + p_n}{(q_n a_{n+1} + q_{n-1}) r_{n+2} + q_n} = \frac{p_{n+1} r_{n+2} + p_n}{q_{n+1} r_{n+2} + q_n}.\end{aligned}$$

□

Corollary 1.10 *For* $\xi = [a_0; a_1, a_2, \ldots]$ *and* $n \in \mathbb{N}$ *, we have*

$$r_{n+1} = \frac{-q_{n-1}\xi + p_{n-1}}{q_n \xi - p_n}.$$

Proof: Solve the equation in Theorem 9 for r_{n+1} . □

Definition 1.11

- *An irrational number* α *is called quadratic irrational number if there exist integers* $A, B, C \in \mathbb{Z}$ *such that*

$$A\alpha^2 + B\alpha + C = 0.$$

- *A number* $\xi = [a_0; a_1, a_2, \ldots]$ *is said to be a periodic CF if from some stage onwards the CF expansion of* ξ *is periodic, that is there exist numbers* $k, l \in \mathbb{N}$ *such that for all* $m \geq k$ *we have that* $a_{m+l} = a_m$ *, i.e.*

$$\xi = [a_0; a_1, \ldots, a_{k-1}, a_k, a_{k+1} \ldots, a_{k+l-1}, a_k, a_{k+1}, \ldots, a_{k+l-1}, a_k, a_{k+1}, \ldots].$$

(*In this situation, one usually then writes*

$$\xi = [a_0; a_1, \ldots, a_{k-1}, \overline{a_k, a_{k+1} \ldots, a_{k+l-1}}].)$$

Theorem 1.12 *A number* $\xi = [a_0; a_1, a_2, \ldots]$ *is a periodic CF if and only if* ξ *is a quadratic irrational number.*

Proof: Since ξ has a periodic CF expansion, we have that there are numbers $k, l \in \mathbb{N}$ such that

$$r_{m+l} = r_m \text{ for all } m \geq k.$$

Therefore, using Theorem 9, we deduce

$$\xi = \frac{p_{m-1} r_m + p_{m-2}}{q_{m-1} r_m + q_{m-2}} = \frac{p_{m+l-1} r_{m+l} + p_{m+l-2}}{q_{m+l-1} r_{m+l} + q_{m+l-2}} = \frac{p_{m+l-1} r_m + p_{m+l-2}}{q_{m+l-1} r_m + q_{m+l-2}},$$

and hence

$$\frac{p_{m-1} r_m + p_{m-2}}{q_{m-1} r_m + q_{m-2}} = \frac{p_{m+l-1} r_m + p_{m+l-2}}{q_{m+l-1} r_m + q_{m+l-2}}.$$

Clearly, by multiplying both sides of this equation with the two denominators, this equality can be written in the form

$$Ar_m^2 + Br_m + C = 0,$$

with appropriate integers A, B and C. Hence, the number r_m is a quadratic irrational number. With this knowledge we now return to the formula which we already derived before

$$\xi = \frac{p_{m-1}r_m + p_{m-2}}{q_{m-1}r_m + q_{m-2}}.$$

which implies that (see Corollary 10)

$$r_m = \frac{-q_{m-2}\xi + p_{m-2}}{q_{m-1}\xi - p_{m-1}}.$$

Inserting this in the quadratic equation above, we obtain

$$A\left(\frac{-q_{m-2}\xi + p_{m-2}}{q_{m-1}\xi - p_{m-1}}\right)^2 + B\left(\frac{-q_{m-2}\xi + p_{m-2}}{q_{m-1}\xi - p_{m-1}}\right) + C = 0,$$

which implies that

$$A(-q_{m-2}\xi + p_{m-2})^2 + B(q_{m-1}\xi - p_{m-1})(-q_{m-2}\xi + p_{m-2}) + C(q_{m-1}\xi - p_{m-1})^2 = 0.$$

Clearly (by getting rid of the brackets), the latter equality can now be written in the form

$$D\xi^2 + E\xi + F = 0,$$

with appropriate integers D, E and F. It follows that ξ is a quadratic irrational number. The proof of the opposite direction is slightly more involved and will be omitted. □

Summary on Basics on Continued Fractions

Every irrational number ξ can be approximated by a sequence of rationals p_n/q_n which are 'good approximations' in the sense that there exists a constant $c > 0$ such that

$$\left|\xi - \frac{p_n}{q_n}\right| < \frac{c}{q_n^2} \quad \text{for all } n \in \mathbb{N}.$$

The rationals p_n/q_n are called convergents (or 'approximants'). For

$$\xi = [a_0; a_1, a_2, \ldots] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ \ldots}},$$

they are given by

$$p_n/q_n = [a_0; a_1, a_2, \ldots, a_n]$$

(we shall always assume that $a_0 \geq 0$ and $a_{i+1} \geq 1$ for all $i \in \mathbb{N}$). There are useful formulas which relate these quantities.
For $n \in \mathbb{N}$ we have (with $p_{-1} = q_0 = 1, q_{-1} = 0$ and $p_0 = a_0$)

- $p_{n+1} = a_{n+1}p_n + p_{n-1}$;
- $q_{n+1} = a_{n+1}q_n + q_{n-1}$;
- $q_n p_{n-1} - p_n q_{n-1} = (-1)^n$.

Definition 1.13 *For $\xi = [a_0; a_1, a_2, \ldots]$ and $n \in \mathbb{N}$, let r_n and s_n be defined as follows.*

$$r_n := [a_n; a_{n+1}, a_{n+2}, \ldots] \quad \textit{and} \quad s_n := \frac{q_{n-1}}{q_n}.$$

For these quantities the following holds (for $n \in \mathbb{N}$).

- $r_n = a_n + \frac{1}{r_{n+1}}$;
- Since $q_{n+1} = a_{n+1}q_n + q_{n-1}$, we have for the ratio $\frac{q_{n+1}}{q_n} = a_{n+1} + \frac{1}{q_n/q_{n-1}}$. Clearly, this process may be continued until $q_1/q_0 = a_1$ is reached. Therefore,

$$s_{n+1} = \frac{1}{[a_{n+1}; a_n, \ldots, a_1]}.$$

Theorem 1.14 *For $\xi = [a_0; a_1, a_2, \ldots]$ and $n \in \mathbb{N}$, we have*

$$\xi = \frac{p_n r_{n+1} + p_{n-1}}{q_n r_{n+1} + q_{n-1}}.$$

Proof: (by induction) For $n = 0$ we have

$$\frac{p_0 r_1 + p_{-1}}{q_0 r_1 + q_{-1}} = \frac{a_0 r_1 + 1}{r_1} = a_0 + \frac{1}{r_1} = \xi.$$

Now assume that the statement is true for n. Then

$$\begin{aligned}
\xi &= \frac{p_n r_{n+1} + p_{n-1}}{q_n r_{n+1} + q_{n-1}} = \frac{p_n(a_{n+1} + \frac{1}{r_{n+2}}) + p_{n-1}}{q_n(a_{n+1} + \frac{1}{r_{n+2}}) + q_{n-1}} \\
&= \frac{p_n a_{n+1} r_{n+2} + p_n + p_{n-1} r_{n+2}}{q_n a_{n+1} r_{n+2} + q_n + q_{n-1} r_{n+2}} = \frac{(p_n a_{n+1} + p_{n-1}) r_{n+2} + p_n}{(q_n a_{n+1} + q_{n-1}) r_{n+2} + q_n} = \frac{p_{n+1} r_{n+2} + p_n}{q_{n+1} r_{n+2} + q_n}.
\end{aligned}$$

□

Corollary 1.15

$$\left|\xi - \frac{p_n}{q_n}\right| = \frac{1}{q_n^2(r_{n+1}+s_n)} \quad \textit{for all } n \in \mathbb{N}.$$

Proof:

$$\begin{aligned}\left|\xi - \frac{p_n}{q_n}\right| &= \left|\frac{p_n r_{n+1} + p_{n-1}}{q_n r_{n+1} + q_{n-1}} - \frac{p_n}{q_n}\right| = \left|\frac{p_n q_n r_{n+1} + p_{n-1} q_n - p_n q_n r_{n+1} - p_n q_{n-1}}{(q_n r_{n+1} + q_{n-1}) q_n}\right| \\ &= \left|\frac{q_n p_{n-1} - p_n q_{n-1}}{q_n^2(r_{n+1}+s_n)}\right| = \frac{1}{q_n^2(r_{n+1}+s_n)}.\end{aligned}$$

□

2 Elementary Diophantine Approximations

2.1 Hurwitz's Theorem

Theorem 2.1 *For all irrationals* $\xi = [a_0; a_1, a_2, \ldots]$ *and for all* $n \in \mathbb{N}$ *, we have that*

$$\left|\xi - \frac{p_i}{q_i}\right| \leq \frac{1}{2q_i^2}$$

is fulfilled for at least one element $i \in \{n, n+1\}$ *.*

Proof: By way of contradiction, assume that the statement in the theorem is false. This means that

$$\left|\xi - \frac{p_i}{q_i}\right| > \frac{1}{2q_i^2}$$

holds simultaneously for $i = n$ and $i = n+1$. Since $\left|\xi - \frac{p_i}{q_i}\right| = \frac{1}{q_i^2(r_{i+1}+s_i)}$, this is equivalent to

$$r_{i+1} + s_i < 2 \ \text{ for } \ i = n, n+1.$$

(a) For $i = n$ we get $2 > r_{n+1} + s_n = a_{n+1} + \frac{1}{r_{n+2}} + s_n,$ and hence,

$$\frac{1}{r_{n+2}} < 2 - (a_{n+1} + s_n) = 2 - \frac{1}{s_{n+1}}.$$

(b) For $i = n+1$ we get

$$r_{n+2} < 2 - s_{n+1}.$$

Combining (a) and (b), we derive $1 < 4 - 2s_{n+1} - 2s_{n+1}^{-1} + 1$,
and hence $0 < 2 - s_{n+1} - s_{n+1}^{-1}$, implying

$$0 > (s_{n+1} - 1)^2,$$

and hence we derive a contradiction. □

Theorem 2.2 *For all irrationals* $\xi = [a_0; a_1, a_2, \ldots]$ *and for all* $n \in \mathbb{N}$ *, we have that*

$$\left|\xi - \frac{p_i}{q_i}\right| \leq \frac{1}{\sqrt{5}\, q_i^2}$$

is fulfilled for at least one element $i \in \{n, n+1, n+2\}$ *.*

Note, the number $\frac{1}{\sqrt{5}}$ is called the *Hurwitz number*.

Proof: As in the proof of the previous theorem (with 2 replaced by $\sqrt{5}$), assume by way of contradiction that for each $i \in \{n, n+1, n+2\}$ we have

$$r_{i+1} + s_i < \sqrt{5}.$$

Proceeding for $i = n$ and $i = n+1$ as in (a) and (b) in the previous proof, we derive

$$s_{n+1}^2 - \sqrt{5}s_{n+1} + 1 < 0. \tag{1}$$

Analogously, for $i = n+1$ and $i = n+2$, we get

$$s_{n+2}^2 - \sqrt{5}s_{n+2} + 1 < 0. \tag{2}$$

By the quadratic formula, (1) and (2) give (with $\gamma := \frac{\sqrt{5}+1}{2}$ and $\gamma^* := \frac{\sqrt{5}-1}{2}$)

$$\gamma^* < s_i < \gamma \text{ for } i = n+1, n+2. \tag{3}$$

Using this, we get

$$s_{n+2} = \frac{1}{a_{n+2} + s_{n+1}} \leq \frac{1}{1 + s_{n+1}} < \frac{1}{1 + \gamma^*} = \gamma^*,$$

which contradicts (3). □

Theorem 2.3 (Hurwitz's theorem) *For the golden mean* $\gamma := \frac{\sqrt{5}+1}{2} = [1; 1, 1, 1, \ldots]$ *we have that*

$$\left|\gamma - \frac{p_n}{q_n}\right| \leq \frac{C}{q_n^2}$$

is satisfied for at most finitely many reduced p_n/q_n *if and only if* $C < \frac{1}{\sqrt{5}}$.

Proof: First note that $r_n = \gamma$ for all $n \in \mathbb{N}$. Secondly, note that

$$s_n^{-1} = [a_n; a_{n-1}, \ldots, a_1] = \gamma + ([a_n; a_{n-1}, \ldots, a_1] - [a_n; a_{n-1}, \ldots]) = \gamma + \delta_n,$$

where for δ_n we have that $\lim_{n\to\infty} \delta_n = 0$. Hence, it follows

$$s_n = \frac{1}{\gamma + \delta_n} = \frac{1}{\gamma} + \frac{1}{\gamma + \delta_n} - \frac{1}{\gamma} = \frac{1}{\gamma} + \frac{-\delta_n}{\gamma^2 + \gamma\delta_n} = \frac{1}{\gamma} + \epsilon_n,$$

where for ϵ_n we have that $\lim_{n\to\infty} \epsilon_n = 0$. These two observations then give

$$r_{n+1} + s_n = \gamma + \frac{1}{\gamma} + \epsilon_n = \sqrt{5} + \epsilon_n \to \sqrt{5} \text{ (for } n \to \infty).$$

Now, if $C < \frac{1}{\sqrt{5}}$ is given, say $C = \frac{1}{\sqrt{5}+\rho}$ for some fixed $\rho > 0$, then

$$\left|\gamma - \frac{p_n}{q_n}\right| = \frac{1}{q_n^2(r_{n+1} + s_n)} = \frac{1}{q_n^2(\sqrt{5} + \epsilon_n)} \leq \frac{1}{q_n^2(\sqrt{5} + \rho)},$$

where the latter inequality can be fulfilled only for finitely many n (due to the fact that $\sqrt{5} + \rho < \sqrt{5} + \epsilon_n$ can be satisfied for at most finitely many n). □

Corollary 2.4 *For each irrational number* ξ, *the inequality*

$$\left|\xi - \frac{p_n}{q_n}\right| \leq \frac{K}{q_n^2}$$

is fulfilled for infinitely many reduced p_n/q_n *as long as* $K \geq \frac{1}{\sqrt{5}}$.

2.2 The Lagrange Spectrum

Definition 2.5 • *Let* c *denote some positive real number. An irrational* ξ *is called* c-approximable *if an only if*

$$\left|\xi - \frac{p_n}{q_n}\right| < \frac{c}{q_n^2}$$

is satisfied for infinitely many reduced p_n/q_n *.*

- *To each irrational number* ξ *we associate a non-negative real number* $\nu(\xi)$ *, defined by*

$$\nu(\xi) := \inf\{c > 0 : \xi \text{ is } c\text{-approximable}\}.$$

- *Two irrational numbers* ξ, η *are called* equivalent *(and we write* $\xi \sim \eta$ *) if and only if there exist* $k, l \in \mathbb{N}$ *such that* $r_k(\xi) = r_l(\eta)$ *(i.e. eventually the continued fraction expansions of* ξ *and* η *coincide).*

Lemma 2.6 *Let* ξ, η *be irrational. If* $\xi \sim \eta$ *, then* $\nu(\xi) = \nu(\eta)$ *.*

Proof: Let ξ, η be irrational such that $\xi \sim \eta$. Then there exist $k, l \in \mathbb{N}$ such that $r_{k+i}(\xi) = r_{l+i}(\eta)$ for all $i \in \mathbb{N}$. Without loss of generality, assume that $l \geq k$. Then ξ and η must be of the form

$$\xi = [a_0; a_1, \ldots, a_k, c_1, c_2, c_3, \ldots] \text{ and } \eta = [b_0; b_1, \ldots, b_k, b_{k+1}, \ldots, b_l, c_1, c_2, c_3, \ldots].$$

In order to prove the assertion of the lemma, it is sufficient to show that

$$\left|\frac{1}{r_{k+n}(\xi) + s_{k+n-1}(\xi)} - \frac{1}{r_{l+n}(\eta) + s_{l+n-1}(\eta)}\right| \to 0 \text{ for } n \to \infty.$$

For this it is sufficient to show that

$$|r_{k+n}(\xi) + s_{k+n-1}(\xi) - (r_{l+n}(\eta) + s_{l+n-1}(\eta))| \to 0 \text{ for } n \to \infty.$$

But this follows, since

$$|r_{k+n}(\xi) + s_{k+n-1}(\xi) - (r_{l+n}(\eta) + s_{l+n-1}(\eta))| = |s_{k+n-1}(\xi) - s_{l+n-1}(\eta)|$$

$$= \left|\frac{1}{[c_{n-1}; \ldots, c_1, a_k, \ldots, a_0]} - \frac{1}{[c_{n-1}; \ldots, c_1, b_l, \ldots, b_0]}\right| \to 0 \text{ (for } n \to \infty).$$

□

Definition 2.7 *An irrational* $\xi \sim \gamma$ *is called* noble number *(i.e. the continued fraction expansion of a noble number has from some stage onward exclusively* 1 *'s as its entries).*

Corollary 2.8 • *For each irrational number* ξ *we have that* $\nu(\xi) \leq \frac{1}{\sqrt{5}}$ *.*

- *A number* η *is a noble number if and only if* $\nu(\xi) = \frac{1}{\sqrt{5}}$ *.*

Theorem 2.9 *Let* N *be some fixed positive integer. If* $\xi = [a_0; a_1, a_2, \ldots]$ *is irrational such that for some* $n \in \mathbb{N}$ *we have that*

$$\left|\xi - \frac{p_i}{q_i}\right| > \frac{1}{q_i^2\sqrt{N^2+4}}$$

is fulfilled for all $i \in \{n, n+1, n+2\}$ *, then it follows that* $a_{n+2} < N$ *.*

Proof: We proceed as in the proof of the first two theorem of the section (with 2, resp. $\sqrt{5}$, now replaced by $\sqrt{N^2+4}$). In this way, considering $i=n$ and $i=n+1$, we derive

$$s_{n+1}^2 - \sqrt{N^2+4}\, s_{n+1} + 1 < 0.$$

And also, by considering $i=n+1$ and $i=n+2$, we derive along the same lines

$$s_{n+2}^2 - \sqrt{N^2+4}\, s_{n+2} + 1 < 0.$$

Then, using the quadratic formula, we obtain

$$\frac{\sqrt{N^2+4}-N}{2} < s_i, s_i^{-1} < \frac{\sqrt{N^2+4}+N}{2} \quad \text{for } i = n+1, n+2.$$

Using this, we then have

$$a_{n+2} = s_{n+1} + a_{n+2} - s_{n+1} = s_{n+2}^{-1} - s_{n+1} < \frac{\sqrt{N^2+4}+N}{2} - \frac{\sqrt{N^2+4}-N}{2} = N.$$

□

Corollary 2.10 *For each irrational number ξ and for every $N \in \mathbb{N}$, exactly one of the following two alternatives occurs.*

Either:

$$\left|\xi - \frac{p_n}{q_n}\right| \leq \frac{1}{q_n^2\sqrt{N^2+4}}$$

is fulfilled for infinitely many p_n/q_n (or with other words, $\nu(\xi) \leq 1/\sqrt{N^2+4}$),

Or: *There exists a number $n_0 > 0$ (depending on N and ξ) such that*

$$a_n < N \text{ for all } n \geq n_0$$

(or with other words, $\xi \in \mathcal{B}_N$ (see Definition 2.18).).

Corollary 2.11 *For each non-noble irrational number ξ we have that*

$$\left|\xi - \frac{p_n}{q_n}\right| \leq \frac{1}{2\sqrt{2}\, q_n^2}$$

is fulfilled for infinitely many reduced p_n/q_n. (Or with other words, for each non-noble number ξ we have $\nu(\xi) \leq 1/(2\sqrt{2})$.)

In fact, by means of similar ideas as in the proof of Hurwitz's theorem (theorem 2.3), one derives that

$$\nu(\xi) = \frac{1}{2\sqrt{2}} \text{ if and only if } \xi \sim \sqrt{2} \ (= [1; 2, 2, 2, \ldots]).$$

Proof: This follows immediately, since if $\xi = [a_0; a_1, \ldots]$ is non-noble then we have $a_n \geq 2$, for infinitely many n. Hence, by theorem 2.9, we have

$$\left|\xi - \frac{p}{q}\right| \leq \frac{1}{\sqrt{8}\, q_n^2}$$

for infinitely many reduced p/q, which implies that $\nu(\xi) \leq 1/(2\sqrt{2})$. □

Lemma 2.12 *Let $\xi = [a_0; a_1, a_2, \ldots]$ be an irrational number such that $\nu(\xi)$ is neither equal to $\frac{1}{\sqrt{5}}$ nor to $\frac{1}{2\sqrt{2}}$, but such that $\xi \sim [b_0; b_1, b_2, \ldots]$ with $b_i \leq 2$ for all $i \in \mathbb{N}$. It then follows that*

$$\nu(\xi) \leq \frac{6}{17}.$$

Proof: Without loss of generality we can assume that there are infinitely many 1's and 2's in $[b_0; b_1, b_2, \ldots]$ (since otherwise ξ would be equivalent to either $1/\sqrt{5}$ or $1/(2\sqrt{2})$). Hence there are infinitely many values n such that $a_n = 1$ and $a_{n+1} = 2$. For these n, we have

$$r_{n+1} + s_n = [a_{n+1}; a_{n+2}, \ldots] + \frac{1}{[a_n; \ldots, a_1]} \geq 2 + \frac{1}{2 + \frac{1}{1}} + \frac{1}{1 + \frac{1}{1}} = \frac{7}{3} + \frac{1}{2} = \frac{17}{6}.$$

It follows that $\nu(\xi) \leq \frac{6}{17}$. □

Lemma 2.13 *If $\xi = [a_0; a_1, \ldots]$ is irrational such that $a_n \geq 3$ for infinitely many n, then $\nu(\xi) \leq \frac{1}{\sqrt{13}}$.*

Proof: By Theorem 2.9 we have that if $a_n \geq 3$ for infinitely many n, then

$$\left| \xi - \frac{p_{n-2}}{q_{n-2}} \right| \leq \frac{1}{\sqrt{3^2 + 4}\, q_{n-2}^2} \quad \left(= \frac{1}{\sqrt{13}\, q_{n-2}^2} \right)$$

must hold for infinitely many n. Hence, $\nu(\xi) \leq \frac{1}{\sqrt{13}}$. □

Definition 2.14 *The set of numbers*

$$\mathcal{L} := \{\nu(\xi) : \xi \text{ is irrational } \}$$

is called the Lagrange spectrum*.*

Also note that since $\frac{1}{3} > \frac{1}{\sqrt{13}}$, we have by Lemma 2.13 that irrational numbers in the Lagrange spectrum in $\left(\frac{1}{3}, \frac{1}{\sqrt{5}}\right]$ must have the property that they are equivalent to irrational numbers whose continued fraction expansion contain exclusively 1's and 2's.
As an immediate consequence of Hurwitz's Theorem (Theorem 2.3), we obtain the following theorem.
Theorem 2.15

$$\mathcal{L} \subset \left[0, \frac{1}{\sqrt{5}}\right].$$

Proposition 2.16 *For an irrational number ξ we have that $\nu(\xi) \in \mathcal{L} \cap \left[\frac{1}{\sqrt{12}}, \frac{1}{\sqrt{5}}\right]$ if and only if $\xi \sim [a_0; a_1, a_2, \ldots]$, for $[a_0; a_1, a_2, \ldots]$ such that $a_n \leq 2$ for all $n \in \mathbb{N}$.*

One can say much more about the structure of the Lagrange spectrum $\left(\frac{1}{3}, \frac{1}{\sqrt{5}}\right]$. It has the following very interesting properties. The proof of this theorem is slightly more involved and will be omitted.

Theorem 2.17 *The Lagrange spectrum $\mathcal{L}$ in $\left(\frac{1}{3}, \frac{1}{\sqrt{5}}\right]$ consists of a countable set of numbers, and these numbers accumulate only at the value $\frac{1}{3}$.*

There are still plenty of fascinating open problems concerning the Lagrange spectrum. We now list a few known results about it. Some of these we have already obtained.

- Each number in the Lagrange spectrum in $\left(\frac{1}{3}, \frac{1}{\sqrt{5}}\right]$ is of the form $1/\sqrt{9-\frac{4}{m^2}}$, where m is a positive integer solution of the equation $m^2+k^2+l^2=3mkl$, for k and l some positive integers. It is known that there are infinitely many solutions m of this equation. The first numbers in the Lagrange spectrum are

$$\frac{1}{\sqrt{5}}\left(=\frac{1}{\sqrt{9-\frac{4}{1^2}}}\right), \frac{1}{2\sqrt{2}}\left(=\frac{1}{\sqrt{9-\frac{4}{2^2}}}\right), \frac{5}{\sqrt{221}}\left(=\frac{1}{\sqrt{9-\frac{4}{5^2}}}\right),$$

$$\frac{1}{\sqrt{9-\frac{4}{13^2}}}, \frac{1}{\sqrt{9-\frac{4}{29^2}}}, \frac{1}{\sqrt{9-\frac{4}{34^2}}}, \frac{1}{\sqrt{9-\frac{4}{89^2}}}, \frac{1}{\sqrt{9-\frac{4}{194^2}}}, \frac{1}{\sqrt{9-\frac{4}{433^2}}}, \ldots.$$

  Note that since $1/\sqrt{9-\frac{4}{m^2}}$ accumulates at $1/3$ (for m tending to infinity), it is clear that the Lagrange spectrum in $\left(\frac{1}{3}, \frac{1}{\sqrt{5}}\right]$ accumulates at $1/3$.

- We have that $\nu(x) \geq \frac{1}{\sqrt{12}}$ if and only if x is equivalent to a number whose continued fraction expansion contains exclusively 1's and 2's.

- In the interval $\left(\frac{1}{\sqrt{13}}, \frac{1}{\sqrt{12}}\right)$ the Lagrange spectrum is empty. That is

$$\mathcal{L} \cap \left(\frac{1}{\sqrt{13}}, \frac{1}{\sqrt{12}}\right) = \emptyset.$$

- Let f be the so called Freimann number which is given by

$$f := \frac{491993569}{2221564096 + 283748\sqrt{462}}.$$

  One then knows that in the interval $[0, f)$ the Lagrange spectrum is continuous. This means that for every $c \in [0, f)$ there exists an irrational number x such that $\nu(x) = c$.

Hausdorff dimensions of parts of the Lagrange spectrum

We already know that $\mathcal{L} \cap \left(\frac{1}{3}, \frac{1}{\sqrt{5}}\right]$ is countable. Moreover, in the interval $[0, f)$ the Lagrange spectrum is continuous. Hence, we have the following immediate result.

Corollary 2.18 *The following hold.*

1. $\dim_H \left(\mathcal{L} \cap \left(\frac{1}{3}, \frac{1}{\sqrt{5}}\right]\right) = 0.$

2. $\dim_H (\mathcal{L} \cap (0, f]) = 1.$

There are numerous interesting results concerning Hausdorff dimensions of parts of the Lagrange spectrum. For instance, we have the following.

Theorem 2.19 *We have that the following hold.*

1. $\dim_H \left(\mathcal{L} \cap \left[\frac{1}{\sqrt{10}}, \frac{1}{\sqrt{5}}\right]\right) \approx 0.8121\ldots$ *(note that* $\frac{1}{\sqrt{10}} \approx \frac{1}{3.16\ldots}$ *.)*

2. $\dim_H\left(\mathcal{L}\cap\left[\frac{8}{\sqrt{689}},\frac{1}{\sqrt{5}}\right]\right)\approx 0.9716\ldots$ *(note that* $\frac{8}{\sqrt{689}}\approx\frac{1}{3.28\ldots}$ *.)*

This should not be confused with the following result.

Theorem 2.20 *We have that the Hausdorff dimension of the set of irrational numbers whose continued fraction expansion uses only 1's and 2's is lies between* 0.44 *and* 0.66*. Therefore,*

$$\dim_H\left(\left\{x\in[0,1):\nu(x)\in\mathcal{L}\cap\left[\frac{1}{\sqrt{12}},\frac{1}{\sqrt{5}}\right]\right\}\right)\in(0.44,0.66).$$

2.3 Badly Approximable Numbers

Definition 2.21 *For* $N\in\mathbb{N}$ *define*

$$\mathcal{B}_N:=\{\xi=[a_0;a_1,a_2,\ldots]\ \textit{irrational}:\exists\, n_0>0\ \textit{such that}\ a_n<N\ \forall\, n\geq n_0\}.$$

The set of badly approximable *numbers* $\mathcal{B}$ *is then defined as*

$$\mathcal{B}:=\bigcup_{N>0}\mathcal{B}_N=\{\xi\ \textit{irrational}:\exists\, N>0\ \textit{such that}\ \xi\in\mathcal{B}_N\}.$$

With other words, $\xi\in\mathcal{B}_N$ if and only if $\xi\sim\eta$, for some $\eta=[b_0;b_1,\ldots]$ with $b_i<N$ for all $i\in\mathbb{N}$. Furthermore, $\xi\in\mathcal{B}$ if and only if there exists $M\in\mathbb{N}$ such that $\xi\in\mathcal{B}_M$.
The following corollary clarifies why the elements in $\mathcal{B}$ are called 'badly approximable'.

Lemma 2.22

- *If* ξ *is an irrational number such that* $\xi\notin\mathcal{B}_N$ *for some* $N\in\mathbb{N}$*, then*

  $$\left|\xi-\frac{p_n}{q_n}\right|\leq\frac{1}{q_n^2\sqrt{N^2+4}}$$

  is fulfilled for infinitely many reduced p_n/q_n *(i.e.* $\nu(\xi)\leq 1/\sqrt{N^2+4}$ *).*
- *For each* $\xi\in\mathcal{B}$ *there exists a constant* $C>0$ *such that for all* $n\in\mathbb{N}$ *we have*

  $$\left|\xi-\frac{p_n}{q_n}\right|>\frac{C}{q_n^2}.$$

Proof: The first part is an immediate consequence of theorem 2.9. For the second part, consider $\xi=[a_0;a_1,\ldots]\in\mathcal{B}$. Then there exist numbers M and m_0 such that $a_n<M$ for all $n\geq m_0$. Using this, we derive $r_{n+1}+s_n<M+1+1=M+2$, and hence

$$\left|\xi-\frac{p_n}{q_n}\right|>\frac{1}{(M+2)\,q_n^2}\quad\text{for all}\ n\geq m_0.$$

For $n<m_0$ we have that there exists a number $c_n>0$ such that

$$\left|\xi-\frac{p_n}{q_n}\right|>\frac{c_n}{q_n^2}.$$

If we define $C:=\min\{1/(M+2),c_0,c_1,\ldots,c_{m_0-1}\}$ (i.e. C is the smallest number in this finite set of numbers), then the result follows. □

3 Metrical Diophantine Approximations

In this section we restrict the investigations to the unit interval $\mathcal{I}:=[0,1)$.

3.1 The Borel-Cantelli Lemma

Definition 3.1 *A set* Σ *of subsets of* $\mathcal{I}$ *is called a* σ-algebra *of* $\mathcal{I}$ *if the following conditions are satisfied.*

- $\mathcal{I} \in \Sigma$;
- *If* $A \in \Sigma$, *then* $A^c \in \Sigma$ *(where* $A^c := \mathcal{I} \setminus A$ *denotes the* complement *of* A *in* $\mathcal{I}$*);*
- $\bigcup_{n\in\mathbb{N}} A_n \in \Sigma$ *for all sequences* (A_n) *with* $A_n \in \Sigma$ *(for all* $n \in \mathbb{N}$*).*

Definition 3.2 *The* Borel-σ-algebra Σ_0 *of* $\mathcal{I}$ *is the smallest* σ*-algebra of* $\mathcal{I}$ *which contains all possible intervals of* $\mathcal{I}$ *of the form* $[x, y)$ *(for* $0 \le x < y < 1$*).*
The elements of Σ_0 *are called* Borel sets*.*

Definition 3.3 *Each element in* Σ_0 *can be measured by the* Lebesgue measure λ *in* $\mathcal{I}$*. In particular, if* A *is an interval (i.e.* $A = [x, y)$ *for some* $0 \le x < y < 1$*), then* $\lambda(A)$ *is just the 'length' of that interval (i.e.* $\lambda(A) = \lambda([x, y)) = y - x$*).*

Properties:

- $\lambda(\mathcal{I}) = 1$;
- $\lambda(A) \ge 0$ for all $A \in \Sigma_0$;
- $\lambda\left(\bigcup_{n\in\mathbb{N}} A_n\right) = \sum_{n\in\mathbb{N}} \lambda(A_n)$ for every sequence (A_n) of pairwise disjoint elements $A_n \in \Sigma_0$ (i.e. $A_i \cap A_j = \emptyset \, \forall \, i \ne j$).
- For $A \in \Sigma_0$ we have:
  $\lambda(A) = 0$ if and only if for all $\epsilon > 0$ there exists a sequence (A_n) of elements $A_n \in \Sigma_0$ such that
  $$A \subset \bigcup_{n\in\mathbb{N}} A_n \text{ and } \sum_{n\in\mathbb{N}} \lambda(A_n) < \epsilon.$$

Note, every countable set in $\mathcal{I}$ is of zero λ-measure.
More general, in order to find out if a given Borel set is of zero λ-measure, the following theorem is often helpful.

Theorem 3.4 (Borel-Cantelli lemma)
If (A_n) *is a sequence of elements* $A_n \in \Sigma_0$ *such that* $\sum_{n\in\mathbb{N}} \lambda(A_n) < \infty$, *then we have*

$$\lambda(A_\infty) = 0,$$

where the lim sup*-set* A_∞ *is defined by*

$$A_\infty := \{\xi \in \mathcal{I} : \xi \in A_n \textit{ for infinitely many } n\}.$$

Proof: The convergence of $\sum_{n\in\mathbb{N}} \lambda(A_n)$ implies that for each $\epsilon > 0$ there exists an integer n_0 such that

$$\sum_{n \ge n_0} \lambda(A_n) < \epsilon.$$

Now note that by definition of A_∞, we have that

$$A_\infty \subset \bigcup_{n \ge n_0} A_n.$$

Hence, it follows that

$$\lambda(A_\infty) \le \lambda\left(\bigcup_{n\ge n_0} A_n\right) \le \sum_{n\ge n_0} \lambda(A_n) < \epsilon.$$

□

3.2 Metrical Diophantine Approximations

Definition 3.5 *Let* $a_1,\dots,a_n \in \mathbb{N}\setminus\{0\}$ *be given. The* n*-cylinder* $I(a_1,\dots,a_n)$ *(also called 'fundamental interval of order* n*') is defined by (here we use the common notation* $[x_1,x_2,\dots] := [0;x_1,x_2,\dots]$ *)*

$$I(a_1,\dots,a_n) := \{\xi = [x_1,x_2,x_3,\dots] \in \mathcal{I} \text{ irrational } : x_i = a_i \text{ for all } 1 \le i \le n\}.$$

Properties:

- For every $\xi \in I(a_1,\dots,a_n)$ we have

  $$\xi = \frac{p_n r_{n+1}(\xi) + p_{n-1}}{q_n r_{n+1}(\xi) + q_{n-1}},$$

  where $p_n, p_{n-1}, q_n, q_{n-1}$ are fixed (depending only on $a_1,\dots,a_n$).

- $$I(a_1,\dots,a_n) = \begin{cases} \left(\frac{p_n}{q_n}, \frac{p_n+p_{n-1}}{q_n+q_{n-1}}\right) & \text{for} \quad n \text{ even} \\ \left(\frac{p_n+p_{n-1}}{q_n+q_{n-1}}, \frac{p_n}{q_n}\right) & \text{for} \quad n \text{ odd.} \end{cases}$$

- $$\lambda(I(a_1,\dots,a_n)) = \frac{1}{q_n^2(1+s_n)}.$$

Proof: These properties are immediate consequences of the following.
By Theorem 1.2, we have

$$\xi = \frac{p_n r_{n+1}(\xi) + p_{n-1}}{q_n r_{n+1}(\xi) + q_{n-1}} = \frac{p_n + p_{n-1}/r_{n+1}(\xi)}{q_n + q_{n-1}/r_{n+1}(\xi)}.$$

Since $1 \le r_{n+1}(\xi)$ and since $r_{n+1}(\xi)$ can get arbitrary large if ξ varies, we see that

$$\lambda(I(a_1,\dots,a_n)) = \left|\frac{p_n+p_{n-1}}{q_n+q_{n-1}} - \frac{p_n}{q_n}\right| = \left|\frac{p_nq_n + p_{n-1}q_n - p_nq_n - q_{n-1}p_n}{q_n^2(1+s_n)}\right|$$
$$= \left|\frac{p_{n-1}q_n - q_{n-1}p_n}{q_n^2(1+s_n)}\right| = \frac{1}{q_n^2(1+s_n)}.$$

Furthermore, observe that

$$\frac{p_n}{q_n} < \frac{p_n+p_{n-1}}{q_n+q_{n-1}} \text{ if and only if } p_nq_{n-1} - q_np_{n-1} < 0.$$

But we know (since $q_np_{n-1} - p_nq_{n-1} = (-1)^n$) that the left hand side of the latter inequality is equal to (-1) if and only if n is even. □

For the next theorem, recall the definition of the set of badly approximable irrational numbers (Definition 2.21).

Theorem 3.6 *For* $\mathcal{B}' := \mathcal{B} \cap \mathcal{I}$ *we have*

$$\lambda(\mathcal{B}') = 0.$$

Proof: For $n, N \in \mathbb{N}$ we define the sets

$$\mathcal{A}_N := \{\xi = [a_1, a_2, \ldots] \in \mathcal{I} \text{ irrational} : a_i < N \;\forall\, i \in \mathbb{N}\}, \;\; \mathcal{A} := \bigcup_{N \in \mathbb{N}} \mathcal{A}_N,$$

$$\mathcal{A}_N^{(n)} := \{\xi = [a_1, a_2, \ldots] \in \mathcal{I} \text{ irrational} : a_i < N \;\forall\, i \in \{1, \ldots, n\}\}.$$

We want to show that $\lambda(\mathcal{A}) = 0$. For this, since $\mathcal{A}_N \subset \mathcal{A}_N^{(n)}$, it is sufficient to show that $\lim_{n\to\infty} \lambda(\mathcal{A}_N^{(n)}) = 0$, and this is what we are now going to prove.
Note that $\mathcal{A}_N^{(n+1)} \subset \mathcal{A}_N^{(n)}$, and that each $\mathcal{A}_N^{(n+1)}$ can be written as a union of disjoint fundamental intervals as follows

$$\mathcal{A}_N^{(n+1)} = \bigcup_{\substack{(a_1,\ldots,a_{n+1}):\\ a_i<N, i=1,\ldots,n+1}} I(a_1, \ldots, a_n, a_{n+1}) = \bigcup_{\substack{(a_1,\ldots,a_n)\\ a_i<N, i=1,\ldots,n}} \bigcup_{\substack{k:\\ k<N}} I(a_1, \ldots, a_n, k).$$

For fixed $(a_1, \ldots, a_n)$, we now calculate the Lebesgue measure of $\bigcup_{k:k<N} I(a_1, \ldots, a_n, k)$ as follows.

$$\lambda\left(\bigcup_{1\le k<N} I(a_1, \ldots, a_n, k)\right) = \left|\frac{p_n + p_{n-1}}{q_n + q_{n-1}} - \frac{p_n N + p_{n-1}}{q_n N + q_{n-1}}\right| = \ldots$$

$$= \frac{N-1}{q_n^2(1+s_n)(N+s_n)} < \frac{N-1}{q_n^2\, N(1+s_n)} = \left(1 - \frac{1}{N}\right) \lambda(I(a_1, \ldots, a_n)).$$

Using the latter estimate, we get

$$\lambda(\mathcal{A}_N^{(n+1)}) = \lambda\left(\bigcup_{\substack{(a_1,\ldots,a_n):\\ a_i<N, i=1,\ldots,n}} \bigcup_{\substack{k:\\ k<N}} I(a_1, \ldots, a_n, k)\right) = \sum_{\substack{(a_1,\ldots,a_n):\\ a_i<N, i=1,\ldots,n}} \lambda\left(\bigcup_{\substack{k:\\ k<N}} I(a_1, \ldots, a_n, k)\right)$$

$$\le \sum_{\substack{(a_1,\ldots,a_n):\\ a_i<N, i=1,\ldots,n}} \lambda(I(a_1, \ldots, a_n))\left(1 - \frac{1}{N}\right) = \left(1 - \frac{1}{N}\right) \lambda(\mathcal{A}_N^{(n)}).$$

Applying this estimate n times, we derive

$$\lambda(\mathcal{A}_N^{(n+1)}) \le \left(1 - \frac{1}{N}\right) \lambda(\mathcal{A}_N^{(n)}) \le \left(1 - \frac{1}{N}\right)^2 \lambda(\mathcal{A}_N^{(n-1)}) \le \ldots \le \left(1 - \frac{1}{N}\right)^n \lambda(\mathcal{A}_N^{(1)}),$$

which then implies

$$\lambda(\mathcal{A}_N^{(n+1)}) \to 0 \;\text{ for }\; n \to \infty.$$

From this we obtain that (since $\mathcal{A}_N \subset \mathcal{A}_N^{(n+1)}$)

$$\lambda(\mathcal{A}_N) = 0 \;\text{ for all }\; N \in \mathbb{N},$$

and hence, since

$$\lambda(\mathcal{A}) = \lambda\left(\bigcup_{N\in\mathbb{N}} \mathcal{A}_N\right) \le \sum_{N\in\mathbb{N}} \lambda(\mathcal{A}_N) = 0,$$

we obtain the desired result

$$\lambda(\mathcal{A}) = 0.$$

Finally, observe that $\xi \in \mathcal{B}'$ if and only if $\xi \in \mathcal{A}$, from which we derive

$$\lambda(\mathcal{B}') = 0.$$

□

By inspection of the proof of the previous theorem, we find that in there we in fact proved slightly more than we actually formulated in the theorem. Namely, we have seen that the following is true.

Corollary 3.7 *For* $\mathcal{B}'_N := \mathcal{B}_N \cap \mathcal{I}$ *we have*

$$\lambda(\mathcal{B}'_N) = 0 \;\; \textit{for all} \;\; N \in \mathbb{N}.$$

Also, combining the previous theorem and Corollary 2.22, we immediately obtain the following result.

Corollary 3.8

$$\lambda\left(\left\{\xi \in \mathcal{I} \textit{ irrational} : \exists C > 0 \textit{ such that } \left|\xi - \frac{p}{q}\right| > \frac{C}{q^2} \textit{ for all } \frac{p}{q}\right\}\right) = 0.$$

We have now seen that the set of badly approximable numbers does not contribute to sets of irrational numbers of positive Lebesgue measure. Hence, if we want to investigate sets of positive measure, then we have to look for irrationals which are more rapidly approximated by their approximants than it is the case for badly approximable irrationals. The contrapositive of the following theorem gives a first indication of how an irrational number has to look like in order to have a chance to contribute to positive Lebesgue measure. In particular, the theorem specifies how fast the $a_n(\xi)$ have to increase **at least** such that ξ has a chance to contribute to positive Lebesgue measure.

Theorem 3.9 *If* $\phi : \mathbb{N} \to \mathbb{R}^+$ *is a function such that* $\sum_{n=1}^{\infty} 1/\phi(n)$ *diverges, then*

$$\lambda(\mathcal{B}_\phi) = 0,$$

where $\mathcal{B}_\phi := \{\xi = [a_1, a_2, \ldots] \in \mathcal{I} \textit{ irrational} : a_n < \phi(n) \; \forall \, n \in \mathbb{N}\}$.

Note: A good choice for ϕ would be $\phi(n) = n \log(n)$ (recall that $\sum_{n=1}^{\infty} \frac{1}{n \log(n)}$ diverges).

Proof: The proof is basically the same as the proof of the previous theorem. As before, we obtain that

$$\lambda\left(\bigcup_{\substack{k: \\ k < \phi(n+1)}} I(a_1, \ldots, a_n, k)\right) < \left(1 - \frac{1}{\phi(n+1)}\right) \lambda(I(a_1, \ldots, a_n)).$$

Hence, with $\mathcal{B}_\phi^{(n)} := \{\xi = [a_1, a_2, \ldots] \in \mathcal{I} \text{ irrational} : a_i < \phi(i) \; \forall \, i \in \{1, \ldots, n\}\}$, we get

$$\lambda(\mathcal{B}_\phi^{(n+1)}) < \left(1 - \frac{1}{\phi(n+1)}\right) \lambda(\mathcal{B}_\phi^{(n)}) < \ldots < \prod_{k=1}^{n} \left(1 - \frac{1}{\phi(k+1)}\right) \lambda(\mathcal{B}_\phi^{(1)}).$$

Using the fact that $1-x<e^{-x}$ for each $0<x<1$, we can continue as follows.

$$\lambda(\mathcal{B}_\phi^{(n+1)}) < e^{-\sum_{k=1}^{n}\frac{1}{\phi(k+1)}}\ \lambda(\mathcal{B}_\phi^{(1)}),$$

which implies (since $\sum_{k=1}^{n} 1/\phi(k+1)$ gets arbitrary large, due to the divergence condition in the theorem)

$$\lambda(\mathcal{B}_\phi^{(n+1)}) \to 0 \ \text{ for } \ n\to\infty,$$

and hence (since $\mathcal{B}_\phi \subset \mathcal{B}_\phi^{(n+1)}$ for all n),

$$\lambda(\mathcal{B}_\phi) = 0.$$

□

Note that with the special choice of ϕ, that is $\phi(n) = n\log(n)$, an immediate consequence of the previous theorem is (for this essentially consider the complement of $\mathcal{B}_\phi$ in $\mathcal{I}$)

$$\lambda\left(\{\xi=[a_1,a_2,\ldots]\in\mathcal{I}\text{ irrational}: a_n \geq n\log(n) \text{ for infinitely many } n\in\mathbb{N}\}\right) = 1.$$

In contrast to the previous theorem, we now investigate how fast the $a_n(\xi)$ can increase **at most** such that ξ has a chance to contribute to positive Lebesgue measure.

Theorem 3.10 *If $\varphi:\mathbb{N}\to\mathbb{R}^+$ is a function such that $\sum_{n=1}^{\infty} 1/\varphi(n)$ converges, then*

$$\lambda(\mathcal{W}_\varphi) = 0,$$

where $\mathcal{W}_\varphi := \{\xi=[a_1,a_2,\ldots]\in\mathcal{I}\ \text{irrational}: a_n > \varphi(n) \text{ for infinitely many } n\}$.

Note: A good choice for ϕ would be $\phi(n) = n\,(\log(n))^{1+\epsilon}$, for any fixed $\epsilon>0$ (recall that $\sum_{n=1}^{\infty}\frac{1}{n(\log(n))^{1+\epsilon}}$ converges, for every $\epsilon>0$).

Proof: We have that

$$\lambda\left(\bigcup_{\substack{k:\\ k\geq\varphi(n+1)}} I(a_1,\ldots,a_n,k)\right) = \left|\frac{p_n\varphi(n+1)+p_{n-1}}{q_n\varphi(n+1)+q_{n-1}} - \frac{p_n}{q_n}\right| = \ldots$$

$$= \frac{1}{q_n^2(1+s_n)}\ \frac{1+s_n}{\varphi(n+1)+s_n} < \frac{2}{\varphi(n+1)}\lambda(I(a_1,\ldots,a_n)).$$

Hence, with $\mathcal{W}_\varphi^{(n)} := \{\xi=[a_1,a_2,\ldots]\in\mathcal{I}\text{ irrational}: a_n>\varphi(n)\}$, we get

$$\lambda(\mathcal{W}_\varphi^{(n+1)}) = \lambda\left(\bigcup_{(a_1,\ldots,a_n)}\ \bigcup_{\substack{k:\\ k\geq\varphi(n+1)}} I(a_1,\ldots,a_n,k)\right)$$

$$< \frac{2}{\varphi(n+1)}\sum_{(a_1,\ldots,a_n)}\lambda(I(a_1,\ldots,a_n)) \leq \frac{2}{\varphi(n+1)}.$$

Now, an application of the Borel-Cantelli lemma (Theorem 3.4) finishes the proof. □

Note that with the special choice of φ, that is $\varphi(n) = n\,(\log(n))^{1+\epsilon}$, an immediate consequence of the previous theorem is (for this essentially consider the complement of $\mathcal{W}_\varphi$ in $\mathcal{I}$) that for each $\epsilon>0$,

$$\lambda\left(\{\xi=[a_1,a_2,\ldots]\in\mathcal{I}\text{ irrational}:\exists n_0\text{ such that }a_n<n\,(\log(n))^{1+\epsilon}\ \ \forall n\geq n_0\}\right)=1.$$

Combining this with the remark after Theorem 3.9, we hence have that the continued fraction expansion of an irrational number $\xi=[a_1,a_2,\ldots]$ which contributes to a set of full Lebesgue measure has the property that for each $\epsilon>0$ we have

$$a_n>n\log(n)\text{ for infinitely many }n,\text{ whereas }a_n<n\,(\log(n))^{1+\epsilon}\text{ eventually.}$$

By taking *log*'s and dividing by $\log n$, we can therefore summarise this result as follows. For λ-almost all $[a_1,a_2,\ldots]\in\mathcal{I}$ we have

$$\limsup_{n\to\infty}\frac{\log a_n}{\log n}=1.$$

Finally, we mention the following important theorem (without proof). In this theorem we use the notion of a (α,β)-Khintchine function, by which we mean the following.

- A (α,β) *-Khintchine function* $\psi:\mathbb{R}^+\to\mathbb{R}^+$ is a non-increasing function which is not 'decreasing too rapidly', in the sense that there exist positive numbers $\alpha<1$ and $\beta\leq 1$ such that for all $x\in\mathbb{R}^+$ we have that $\psi(x)\geq\beta\psi(\alpha x)$.

Theorem 3.11 (Khintchine's theorem)
For ψ *a* (α,β) *-Khintchine function let*

$$\mathcal{K}_\psi:=\{\xi\in\mathcal{I}:\left|\xi-\frac{p_n}{q_n}\right|<\frac{\psi(q_n)}{q_n^2}\text{ is fulfilled for infinitely many }n\}.$$

Then the following holds.

(i) $\lambda(\mathcal{K}_\psi)=0$ *if and only if* $\sum_{n\in\mathbb{N}}\psi(\alpha^n)$ *converges.*

(ii) $\lambda(\mathcal{K}_\psi)=1$ *if and only if* $\sum_{n\in\mathbb{N}}\psi(\alpha^n)$ *diverges.*

Remark: In case (i), a good choice for the function ψ would be $\psi(x)=(\log(x))^{-(1+\epsilon)}$ (for any $\epsilon>0$). And in case (ii), a good choice for the function ψ would be $\psi(x)=(\log(x))^{-1}$. With these choices, we then obtain that for ξ from a set of full λ-measure we have that the two inequalities

$$\frac{1}{q_n^2(\log(q_n))^{1+\epsilon}}<\left|\xi-\frac{p_n}{q_n}\right|<\frac{1}{q_n^2\log(q_n)},$$

are fulfilled simultaneously for infinitely many p_n/q_n (more precisely, the left-hand inequality is fulfilled even for all p_n/q_n apart from finitely many exceptions).

3.3 Further fractal Diophantine approximations

MYRBERG DENSITY THEOREM:
Consider the set
$\mathcal{M}:=\{\xi=[x_1,x_2,x_3,\ldots]\in(0,1):$ the infinite sequence $x_1,x_2,x_3,\ldots$ contains EVERY FINITE BLOCK of positive integers INFINITELY MANY TIMES $\}$

Theorem. (Myrberg)
The set $\mathcal{M}$ is of full 1-dimensional Lebesgue measure.

NAKADA's THEOREM:
For each $N \in \mathbb{N}$ consider the set

$$\mathcal{N}_N := \left\{ [x_1, x_2, ...] \in (0,1) : \lim_{n\to\infty} \frac{\sharp\{m : x_m \geq N \text{ for } 1 \leq m \leq n\}}{n + \sum_{i=1}^{n} \log x_i} = c_0 \log\left(1 + \frac{1}{N}\right) \right\}$$

where $c_0^{-1} := \log 2 + \sum_{n=1}^{\infty} \frac{\log n}{\log(1+\frac{1}{n(n+2)})} \approx \log 5.2\ldots$

Theorem. (Nakada)
The set $\mathcal{N}_N$ is of full 1-dimensional Lebesgue measure, for all $N \in \mathbb{N}$.

JARNIK's THEOREM:
Consider the set of *Badly Approximable Irrational Numbers*

$$\mathcal{B} := \left\{ \xi \in (0,1) : \exists c(\xi) > 0 \text{ such that } \left|\xi - \frac{p}{q}\right| > \frac{c(\xi)}{q^2} \text{ for all } (p,q) = 1 \right\}$$

One easily verifies that

$$\mathcal{B} := \{\xi = [x_1, x_2, x_3, \ldots] \in (0,1) : \exists N(\xi) \in \mathbb{N} \text{ such that } x_i \leq N(\xi) \forall i \in \mathbb{N}\}$$

(By Myrberg's Theorem, the 1-dimensional Lebesgue measure of $\mathcal{B}$ vanishes).
For $\dim_H \mathcal{B}$, the Hausdorff dimension of $\mathcal{B}$, we have

Theorem. (Jarník)

$$\dim_H \mathcal{B} = 1.$$

THEOREM of JARNIK and BESICOVITCH:
For $\sigma \geq 0$, consider the set of σ-*Well-Approximable Irrational Numbers*

$$\mathcal{J}_\sigma := \left\{ \xi \in (0,1) : \left|\xi - \frac{p}{q}\right| < \frac{1}{q^{2(1+\sigma)}} \text{ for infinitely many } (p,q) = 1 \right\}$$

For $\dim_H \mathcal{J}_\sigma$, the Hausdorff dimension of $\mathcal{J}_\sigma$, we have

Theorem. (Jarník, Besicovitch)

$$\dim_H \mathcal{J}_\sigma = \frac{1}{1+\sigma}.$$

MULTIFRACTAL DIOPHANTINE APPROXIMATIONS:

$$\begin{aligned}
\mathcal{T}_0 &= \{[\tfrac{0}{1},\tfrac{1}{1})\} \\
\mathcal{T}_1 &= \{[\tfrac{0}{1},\tfrac{1}{2}),[\tfrac{1}{2},\tfrac{1}{1})\} \\
\mathcal{T}_2 &= \{[\tfrac{0}{1},\tfrac{1}{3}),[\tfrac{1}{3},\tfrac{1}{2}),[\tfrac{1}{2},\tfrac{2}{3}),[\tfrac{2}{3},\tfrac{1}{1})\} \\
\mathcal{T}_3 &= \{[\tfrac{0}{1},\tfrac{1}{4}),[\tfrac{1}{4},\tfrac{1}{3}),[\tfrac{1}{3},\tfrac{2}{5}),[\tfrac{2}{5},\tfrac{1}{2}),[\tfrac{1}{2},\tfrac{3}{5}),[\tfrac{3}{5},\tfrac{2}{3}),[\tfrac{2}{3},\tfrac{3}{4}),[\tfrac{3}{4},\tfrac{1}{1})\} \\
&\vdots \\
\mathcal{T}_n &= \{T_{n,1}\ldots\ldots\ldots T_{n,k}\ldots\ldots\ldots T_{n,2^n}\} \\
&\vdots
\end{aligned}$$

Stern-Brocot intervals

Note

- For each $\xi \in [0,1)$ and $n \in \mathbb{N}$,
  there exists a unique $T_n(\xi) \in \mathcal{T}_n$ such that $x \in T_n(\xi)$.

Consider STERN-BROCOT PRESSURE

$$\mathcal{P}(t) := \lim_{n\to\infty} \frac{1}{n} \log \sum_{k=1}^{2^n} (\operatorname{diam}(T_{n,k}))^t$$

and LEVEL SETS

$$\mathcal{L}(s) := \left\{ x \in [0,1) : \lim_{n\to\infty} \frac{|\log \operatorname{diam}(T_n(x))|}{n} = s \right\}$$

Theorem. (Kesseböhmer/Stratmann)

(i) The Stern-Brocot pressure $\mathcal{P}$ is differentiable throughout $\mathbb{R}$,
real-analytic on $(-\infty,1)$ and vanishes on $[1,\infty)$.
Furthermore, $\mathcal{P}$ is real-analytic on $(-\infty,1)$ and vanishes on $[1,\infty)$.

(ii) For each $s \in [0, 2\log\frac{\sqrt{5}+1}{2}]$ we have, with the convention $\dim_H(\mathcal{L}(0)) := 1$,

$$\dim_H(\mathcal{L}(s)) = \frac{\inf_{t\in\mathbb{R}}\{\mathcal{P}(t)+st\}}{s}.$$

Further MULTIFRACTAL DIOPHANTINE APPROXIMATIONS:

Classical Results of Levy and Khintchine:
With $\frac{p_n(x)}{q_n(\xi)} := [x_1, x_2, \ldots, x_n]$ referring to the n-th approximant of $\xi \in (0,1)$, we have on a set of FULL LEBESGUE MEASURE

(Levy)

$$\ell_1(\xi) := \lim_{n\to\infty} \frac{2\log q_n(\xi)}{\sum_{i=1}^n x_i} = 0$$

Furthermore, on a set of FULL LEBESGUE MEASURE

(Khintchine) (Levy)

$$\ell_2(\xi) := \lim_{n\to\infty} \frac{n}{\sum_{i=1}^n x_i} = 0 \text{ and } \ell_3(\xi) := \lim_{n\to\infty} \frac{2\log q_n(x)}{n} = \frac{\pi^2}{6\log 2}$$

In particular

$$\dim_H(\{\ell_1(\xi) = 0\}) = \dim_H(\{\ell_2(\xi) = 0 \text{ and } \ell_3(\xi) = \frac{\pi^2}{6\log 2}\}) = 1.$$

QUESTION:

$$\dim_H(\{\ell_1(\xi) = \ ?\}) = \dim_H(\{\ell_2(\xi) = \ ? \text{ and } \ell_3(\xi) = \ ?\}) = \ ?$$

ANSWER:
Consider LEVEL SETS

$$\mathcal{L}_i(s) := \{\xi \in [0,1) : \ell_i(\xi) = s\} \text{ for } i = 1,2,3$$

We then have

Theorem. (Kesseböhmer/Stratmann)
For each $a \in [0, 2\log\frac{1+\sqrt{5}}{2}]$
there exist numbers a^* and $a^\sharp$ related to a by $a = a^* \cdot a^\sharp$, such that

$$\dim_H(\mathcal{L}_1(a)) = \dim_H(\mathcal{L}_2(a^\sharp) \cap \mathcal{L}_3(a^*)) = \frac{\inf_{t\in\mathbb{R}}\{\mathcal{P}(t) + at\}}{a}.$$