DIOPHANTINE APPROXIMATIONS AND FRACTALS

PART I

1. Fractal sets and dimensions

1.1. **Definition of fractal dimensions.** Problem: How can we best define the dimension of a closed bounded set $X \subset \mathbb{R}^n$?

Ideally, we might want a definition which includes the following situations.

(i) If X is a manifold, then the value of the dimension is an integer which coincides with the usual notion of dimension;

(ii) If X is a more general set, then its dimension can be fractional;

(iii) If X is a countable union of points, then its dimension should be equal to zero.

Definition 1: The *topological dimension* of a topological space X is defined to be the minimum value of n, such that every open cover of X has an open refinement in which no point is included in more than $n+1$ elements. If no such minimal n exists, the space is said to be of infinite covering dimension.

The idea of topological dimension became a topic of considerable interest in the early 20th century. The core ideas were independently derived by Karl Menger, L. E. J. Brouwer, Pavel Urysohn and Henri Lebesgue.

Clearly, the topological dimension is always a non-negative integer. (For example, the topological dimension of the Cantor set C is zero). Unfortunately, this definition does not capture the requirement in (ii) above.

Definition 2: For $\epsilon > 0$, let $N(\epsilon)$ be the smallest number of ϵ-balls needed to cover X. The *box dimension* $\dim_B(X)$ of X is then defined by

$$\dim_B(X) = \lim_{\epsilon \to 0} \frac{\log N(\epsilon)}{\log(1/\epsilon)}.$$

Clearly, this definition is in line with the usual notion of dimensions for manifolds. However, the box dimension can be fractional (e.g., the box dimension of the Cantor set X is $\log 2/\log 3$ (Exercise!)). Unfortunately, this definition does not capture the requirement in (iii) above.

Lemma 1.1. *There exist countable sets such that the box dimension is strictly positive.*

Proof. Consider the countable infinite set

$$X = \left\{ \frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots \right\}.$$

For $\epsilon = 1/n^2$, consider an ϵ-cover of X. Since $1/k - 1/(k+1) = 1/(k+k^2) \sim 1/k^2$, one then immediately verifies that one requires roughly $2n$ intervals of size $\epsilon = 1/n^2$ to cover X. Therefore

$$\dim_B(X) = \lim_{1/n \to 0} \frac{\log N(1/n^2)}{-\log(1/n^2)} = \lim_{1/n \to 0} \frac{\log 2n}{\log(n^2)} = \frac{1}{2}.$$

□

Definition 3: The *Hausdorff dimension* $\dim_H(X)$ of X is defined as follows. For $\epsilon > 0$, consider open ϵ-covers

$$\mathcal{U}_\epsilon = \{U_i : X \subset \bigcup_{i\in\mathbb{N}} U_i, \operatorname{diam}(U_i) < \epsilon, U_i \text{ is open, for all } i \in \mathbb{N}\}$$

of X. For $s > 0$ we then define $\mathcal{H}^s_\epsilon(X) := \inf\{\sum_{i\in\mathbb{N}} (\operatorname{diam}(U_i))^s\}$, where the infimum is taken over all open ϵ-covers $\mathcal{U}_\epsilon$. Moreover, define $\mathcal{H}^s(X) := \lim_{\epsilon\to 0} \mathcal{H}^s_\epsilon(X)$. Finally, we then define

$$\dim_H(X) := \inf\{s : \mathcal{H}^s(X) < \infty\}(= \sup\{s : \mathcal{H}^s(X) = \infty\}).$$

Note that in here $\mathcal{H}^s(X)$ is also called the *s-dimensional Hausdorff measure* of X.

As for the previous two definitions this coincides with the usual notion of dimensions for manifolds. Furthermore, the Hausdorff dimension can be fractional (e.g., the Hausdorff dimension of the Cantor set X is again $\log 2/\log 3$ (Exercise!)). The advantage of this third definition is that this time the property (iii) also holds. This is the essence of the following proposition.

Proposition 1.2. *If X is countable then $\dim_H(X) = 0$.*

Proof. Let $X = \{x_n : n \in \mathbb{N}\}$. Given any $\epsilon, s > 0$, fix some sequence $(\epsilon_n)_{n\in\mathbb{N}}$ of positive real numbers such that $\sum_{n=1}^\infty \epsilon_n^s < \epsilon$. Then consider the open ϵ-cover $\mathcal{U}_\epsilon$ of X given by the balls $B(x_n, \epsilon_n/2)$ centred at x_n and of diameter ϵ_n. We then immediately obtain that $\mathcal{H}^s_\epsilon(X) < \epsilon$. Since this can be done for every $\epsilon > 0$, it follows that $\mathcal{H}^s(X) \le \lim_{\epsilon\to 0} \mathcal{H}^s_\epsilon(X) = 0$. The latter estimate holds for every $s > 0$, and therefore we obtain that $\dim_H(X) = 0$. □

EXAMPLES:

(1) $\frac{1}{3}$-*Cantor set*: Let C denote the middle third Cantor set. This is the closed set of points in the unit interval whose triadic expansion does not contain any occurrences of the the digit 1, that is

$$C = \left\{ \sum_{n=1}^\infty \frac{a_n}{3^n} : a_n \in \{0, 2\}, n \in \mathbb{N} \right\}.$$

Claim:

$$\dim_H(C) = \dim_B(C) = \frac{\log 2}{\log 3} (= 0.690...).$$

Proof. Let us here only give the calculation of the box dimension. The calculation of $\dim_H(C)$ will be given later as an application of the mass distribution principle. For $\dim_B(C)$ note that in the geometric construction of C we have that the n-th layer of the construction consists of 2^n well separated intervals of diameter 3^{-n}. In fact, two such intervals are separated by an interval of diameter at least 3^{-n}. Now let $\{\mathcal{U}_i\}_{i\in\mathbb{N}}$ be some covering of C by pairwise disjoint intervals of diameter $\epsilon > 0$ (sufficiently small). Without loss of generality we can assume that $U_i \cap C \neq \emptyset$. Then there exists $n \in \mathbb{N}$ such that $3^{-(n+1)} < \epsilon \le 3^{-n}$. Clearly, each of these intervals intersect at most 2 covering intervals of the n-th or of the $(n+1)$-th layer. Therefore, we have $N(\epsilon) \in [2^{n-1}, 2^{n+1}]$. Using this, it now follows that

$$\dim_B(C) = \lim_{\epsilon\to 0} \frac{\log N(\epsilon)}{\log(1/\epsilon)} = \lim_{n\to\infty} \frac{\log 2^n}{\log 3^n} = \frac{\log 2}{\log 3}.$$

□

(2) *von Koch curve*: The von Koch curve K is obtained by induction as follows. Start from the unit interval $K_0 := [0,1]$ and then assume that the curve K_{n-1} has been constructed such that it consists of 4^n straight segments each of length 3^{-n}. In order to obtain K_n from K_{n-1}, replace each middle third of each of the straight segments in K_{n-1} by the two sides of the equilateral triangle which is based at this middle third of the segment. The von Koch curve K is then the limit of this construction, for n tending to infinity. Employing a very similar argument as for the $\frac{1}{3}$-Cantor set above, one then finds the following. (Note that in the geometric construction of K we have that the n-th step of the construction consists of 4^n intervals of length 3^{-n}.)

$$\dim_H(K) = \dim_B(K) = \frac{\log 4}{\log 3}.$$

(3) *Sierpinski carpet*: The Sierpinski carpet S is given by

$$S := \left\{ \left(\sum_{n=1}^{\infty} \frac{a_n}{3^n}, \sum_{n=1}^{\infty} \frac{b_n}{3^n} \right) : (a_n, b_n) \in \{(i,j) : i,j \in \{0,1,2\}\} \setminus \{(1,1)\} \right\}.$$

Note that S is a connected set without interior. Again very similar to the argument for the $\frac{1}{3}$-Cantor set, one obtains the following. (Note that in the geometric construction of S we have that the n-th step of the construction consists of 8^n boxes of diameter 3^{-n}.)

$$\dim_H(S) = \dim_B(S) = \frac{\log 8}{\log 3}.$$

1.2. **Some properties of fractal dimensions.** A very simple, but nevertheless rather useful point of view is to think of Hausdorff (and/or box) dimension as being a way to distinguish between sets of zero Lebegue measure.

Lemma 1.3. *Let $X \subset \mathbb{R}^n$ and let λ_n denote the n-dimensional Lebesgue measure. If $\dim_H(X) < n$ then $\lambda_n(X) = 0$.*

Next note that in many examples one has $\dim_H(X) = \dim_B(X)$. But in general this is not true.

Proposition 1.4. *The Hausdorff dimension and the box counting dimension of a set X are related by*

$$dim_H(X) \le dim_B(X).$$

Proof. Let $\delta > 0$ be fixed. By definition of $\dim_B(X)$, we have that for each $\epsilon > 0$ sufficiently small we can cover X by $N(\epsilon) \le \epsilon^{-(\dim_B(X)+\delta)}$ balls of radius ϵ. Taking this as an open ϵ-cover $\mathcal{U}_\epsilon$, it follows that $\mathcal{H}_\epsilon^{\dim_B(X)+2\delta}(X) \le \epsilon^{-(\dim_B(X)+\delta)} \epsilon^{\dim_B(X)+2\delta} = \epsilon^\delta$. Therefore,

$$\mathcal{H}^{\dim_B(X)+2\delta}(X) \le \lim_{\epsilon \to 0} \mathcal{H}_\epsilon^{\dim_B(X)+2\delta}(X) = 0,$$

and hence $\dim_H(X) \le \dim_B(X) + 2\delta$. □

Note that for the set $X = \left\{\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\right\}$ we already saw that $\dim_B(X) = \frac{1}{2}$. On the other hand, by the above we have that $\dim_H(X) = 0$. This is a trivial example for that we can indeed have that Hausdorff dimension is strictly less than box dimension.

Lemma 1.5. *The box counting dimension is invariant under 'taking the closure'. That is,*

$$dim_B(X) = dim_B(\overline{X}).$$

Clearly, this indicates once more that the box dimension is not the ultimate tool for measuring the complexity of a fractal set in general. For instance, if X is the set of rational numbers in the unit interval then $X = \overline{X}$ and hence, $\dim_B(X) = 1$. A further useful property is that sets which are the same up to bi-Lipschitz maps have the same Hausdorff dimension. Recall that a surjective map $F : X \to Y$ is called *Lipschitz* if there exists a constant $C > 0$ such that

$$|F(x) - F(y) \leq C|x - y|, \text{ for all } x, y \in X.$$

Similarly, a bijection $G : X \to Y$ is called *bi-Lipschitz* if there exists a constant $K > 0$ such that

$$k^{-1}|x - y| \leq |G(x) - G(y) \leq K|x - y|, \text{ for all } x, y \in X.$$

Proposition 1.6.

(1) *If $F : X \to Y$ is a surjective Lipschitz map, then* $\dim_H(Y) \leq \dim_H(X)$.
(2) *If $G : X \to Y$ is a bijective bi-Lipschitz map, then* $\dim_H(X) = \dim_H(Y)$.

Proof. (1) Let $\mathcal{U}_\epsilon = \{U_i\}_{i \in \mathbb{N}}$ be an open ϵ-cover of X. Then $\mathcal{U}'_\epsilon := \{F(U_i)\}_{i \in \mathbb{N}}$ is an open $(\epsilon \cdot C)$-cover of Y. Therefore, it follows that $\mathcal{H}^s_{\epsilon C}(Y) \leq \mathcal{H}^s_\epsilon(X)$. Hence, by taking the limit for ϵ tending to zero, we obtain $\mathcal{H}^s(Y) \leq \mathcal{H}^s(X)$. By definition of Hausdorff dimension, it then follows that $\dim_H(X) \geq \dim_H(Y)$.
(2) Consider the map $G^{-1} : Y \to X$, which is a surjective Lipschitz map. Hence we can apply (1), which gives $\dim_H(X) \leq \dim_H(Y)$. Now, by applying (1) also to G, we also have that $\dim_H(Y) \leq \dim_H(X)$. □

Note that the proof of (1) can easily be generalised to show that if for some $\alpha \geq 0$ we have that $F : X \to Y$ is α-Hölder continuous (that is, $|F(x) - F(y) \leq C|x - y|^\alpha$, for all $x, y \in X$), then $\dim_H(X) \geq \alpha \cdot \dim_H(F(X))$.

The next result considers Hausdorff dimensions of sums and products of sets. For this recall that for $A, B \subset \mathbb{R}$ one defines

$$A \times B := \{(x, y) \in \mathbb{R}^2 : x \in A, y \in B\}$$

and

$$A + B := \{x + y : x \in A, y \in B\}.$$

Proposition 1.7. *For $A, B \subset \mathbb{R}$ we have*

(1) $\dim_H(A \times B) \geq \dim_H(A) + \dim_H(B)$ *and* (2) $\dim_H(A + B) \leq \dim_H(A) + \dim_H(B)$.

As an example consider the sum $C + C$ of the middle third Cantor set. By considering the representation of C in terms of triadic expansions one immediately sees that $C + C$ is an interval and hence has Hausdorff dimension equal to 1. On the other hand, the sum of the Hausdorf dimensions is equal to $\log 4 / \log 3$. In particular, this example shows that the inequality in (2) can not be expected to be strict in general. Note that the same holds for the inequality in (1) (see e.g. Example 7.8 in the book of Falconer).

1.3. **Techniques for calculating fractal dimensions.** One of the most useful technique for calculating Hausdorff dimensions is provided by the so called mass distribution principle. Historically, one of the preliminary versions of this principle is the following lemma due to Frostman.

Proposition 1.8 (Frostman's Lemma). *Let F be a bounded subset of $\mathbb{R}^n$. Assume that there exists a measure μ on F such that $0<\mu(F)<\infty$ and that there exists $s>0$ so that there exists constants $r_0, C>0$ such that*

$$\mu(B(x,r)) \le Cr^s, \quad \text{for all } x\in F, 0<r<r_0.$$

We then have that $\mu(F)\le C\mathcal{H}^s(F)$ and hence

$$\dim_H(F)\ge s.$$

Proof. Let $\{U_i\}_{i\in\mathbb{N}}$ be some arbitrary ϵ-cover of F. We then have

$$0<\mu(F)=\mu(\bigcup U_i)\le \sum \mu(U_i) \le C\sum \left(\frac{\operatorname{diam}(U_i)}{2}\right)^s.$$

By taking the infimum over all ϵ-coverings, we obtain $\mu(F)\le C\mathcal{H}^s_\epsilon(F)$. The result now follows by letting ϵ tend to zero. □

Proposition 1.9 (Mass distribution principle). *Let F be a bounded Borel set in $\mathbb{R}^n$, and assume that there exists a finite measure μ on F such that $\mu(F)=\mu(\mathbb{R}^n)$. Then the following hold, where $s>0$ denotes some constant.*

$$(1)\ \limsup_{r\to 0}\frac{\mu(B(x,r))}{r^s}\ll 1\ \forall x\in F \implies \mu(F)\ll \mathcal{H}^s(F) \implies \dim_H(F)\ge s;$$

$$(2)\ \limsup_{r\to 0}\frac{\mu(B(x,r))}{r^s}\gg 1\ \forall x\in F \implies \mu(F)\ll \mathcal{H}^s(F) \implies \dim_H(F)\le s.$$

Proof. (1) This follows by a slight improvement of the arguments in the proof of Frostman's Lemma.

(2) Fix $\epsilon>0$ and consider the family of balls

$$\mathcal{B}:=\{B(x,r): x\in F, 0<r<\epsilon, \mu(B(x,r))>Cr^s\}.$$

Then we have $F\subset\bigcup_{B\in\mathcal{B}}B$. (Note that if $\{B(x,r_x): x\in F, r_x>0\}$ is any collection of balls covering F (not necessarily countable), then we can always find a countable family $\{B(x_i,4r_{x_i}): x_i\in F\forall i\in\mathbb{N}\}$ of balls covering F such that the $B(x_i,r_{x_i})$ are pairwise disjoint (this is a standard covering lemma)). Hence, there exists a countable family $\{B(x_i,r_i): x_i\in F\forall i\in\mathbb{N}\}$ contained in $\mathcal{B}$ such that the $B(x_i,r_i)$ are pairwise disjoint and $F\subset\bigcup_{i\in\mathbb{N}}B(x_i,4r_{x_i})$. It then follows that

$$\mathcal{H}^s_{8\epsilon}(F)\le\sum_{i\in\mathbb{N}}(\operatorname{diam}(B(x_i,4r_{x_i})))^s \ll \sum_{i\in\mathbb{N}}(\operatorname{diam}(B(x_i,r_{x_i})))^s\ll 1.$$

By letting ϵ tend to zero, we obtain that $\mathcal{H}^s(F)$ is finite, and hence $\dim_H(F)\le s$. □

2. Iterated Function Systems

Here we introduce the basic constructions of an iterated function systems. These systems appear in a surprisingly large number of applications, including several that we have already described in the previous section. Moreover, sets X for which we have the most chance to compute fractal dimensions are those which are self-similar, that is if you magnify a piece of the set enough then the magnification looks roughly the same as the piece you are started of with. Often, if we have a local distance expanding map on a compact set we can view the natural associated invariant set as the limit set of an iterated function system of the inverse branches of this map. In the case of many linear maps, the dimension can be found implicitly in terms of an expression involving only the rates of contraction. In the non-linear case, the corresponding expression involves the so called pressure function.

2.1. **Iterated Function Systems: Definition and Basic Properties.**

Definition. Let $M \subset \mathbb{R}^n$ be an open set. A map $F : M \to M$ is a contraction if there exists $0 < r < 1$ (r is called the contraction rate) such that for all $x, y \in M$ we have

$$||F(x) - F(y)|| \leq r\,||x - y||.$$

(Here $||\cdot||$ denotes the usual Euclidean norm on $\mathbb{R}^n$.)

The following definition is fundamental to what follows.

Definition. An iterated function scheme on an open set $M \subset \mathbb{R}^n$ consists of a family of contractions $F_1, F_2, ..., F_k : M \to M$ (with contraction rates $r_1, ..., r_k$).

Proposition 2.1. *Let $F_1, F_2, ..., F_k : M \to M$ be a finite family of contractions. There exists a unique closed invariant set $\mathcal{L}$ such that $\mathcal{L} = \bigcup_{i=1}^{k} F_i(\mathcal{L})$.*

The set $\mathcal{L}$ is usually called the attractor or limit set of the system.

There is an alternative approach to constructing the limit set is as follows.

Definition. Let $M \subset \mathbb{R}^n$ be an open set, and let $F_1, F_2, ..., F_k : M \to M$ be an iterated function system with contraction rates $r_1, ..., r_k$. Fix any point $z \in M$ and let the limit set $\mathcal{L}_z$ be the set of all limit points of the iterates of z under the iterated function system. That is,

$$\mathcal{L}_z := \{\lim_{n\to\infty} F_{x_0} \circ F_{x_1} \circ ... \circ F_{x_n}(z) : (x_0, x_1, x_2, ...) \in \{1, ..., k\}^{\mathbb{N}}\}.$$

Clearly, each of these limits exists, since it arises from a nested sequence of compact sets. Moreover, since all of the maps are contractions it follows that each of these limits is a singleton.

Lemma 2.2. *The limit set $\mathcal{L}_z$ coincides with the attractor $\mathcal{L}$ defined above. In particular, $\mathcal{L}_z$ does not depend on the choice of z.*

Proof. Let F be the map of the set of compact subsets of M into the set of compact subsets of M, given by $F(A) := \bigcup_{i=1}^{n} F_i(A)$. One immediately verify that $F(\mathcal{L}_z) = \mathcal{L}_z$. Moreover, the contraction mapping theorem guarantees that $\mathcal{L}_z$ is the *unique* fixed point of F, and hence it must be equal to $\mathcal{L}$. □

This second way of interpreting $\mathcal{L}$ has the advantage that every point is coded by some infinite sequence. That is, we can define a metric d on the space of infinite sequences

$$\{1, ..., k\}^{\mathbb{N}}$$

by setting, for distinct elements $x = (x_1, x_2, ...)$ and $y = (y_1, y_2, ...)$,

$$d(x, y) := 2^{-n(x,y)}.$$

Here, $n(x, y) := \min\{n \geq 0 : x_i = y_i \text{ for } i = 1, 2, ..., n\}$. For $x = y$ we let $d(x, y) := 0$. This naturally leads to the coding map $\pi : \{1, ..., k\}^{\mathbb{N}} \to \mathcal{L}$, given by

$$\pi((x_1, x_2, ...)) := \lim_{n\to\infty} F_{x_1} \circ ... \circ F_{x_n}(z).$$

Also, the action of the map F on $\mathcal{L}$ is represented on this symbolical level by the left shift σ which is given by

$$F(\pi((x_1, x_2, ...)) = \pi(\sigma(x_1, x_2, ...)) = \pi((x_2, x_3, ...)).$$

Lemma 2.3. *The map π is Hölder continuous. That is, there exist $K, \beta > 0$ such that $||\pi(x) - \pi(y)|| \leq K\,||x - y||^{\beta}$, for all $x, y \in \{1, ..., k\}^{\mathbb{N}}$.*

Proof. Let $\pi(x), \pi(y) \in F_{x_1} \circ ... \circ F_{x_n}(\mathcal{L})$ be fixed. We then have

$$||\pi(x) - \pi(y)|| \leq \operatorname{diam}(F_{x_0} \circ ... \circ F_{x_n}(\mathcal{L})) \leq (\max r_i)^n \operatorname{diam}(\mathcal{L}) \leq \operatorname{diam}(\mathcal{L}) d(x,y)^\beta,$$

where $\beta := -\log_2(\max r_i)$. □

If the maps in an IFS are conformal contractions then it is called conformal IFS (or CIFS). In the one dimensional setting these systems are also called cookie cutters.

Examples: (1) Schottky groups; (2) Julia sets.

Definition. An IFS is said to satisfy the open set condition if there exists an open set $U \subset \mathbb{R}^n$ such that $F_i(U) \subset U$ and $F_i(U) \cap F_j(U) = \emptyset$, for all $i, j \in \{1, ..., k\} (i \neq j)$.

Proposition 2.4. *For the limit set $\mathcal{L}$ of a CIFS with open set condition, we have*

$$\dim_H(\mathcal{L}) = \dim_B(\mathcal{L}).$$

Proof. postponed. □

In particular, this applies to two of our favorite examples Julia sets and Schottky groups.

Corollary 2.5. *For limit sets of hyperbolic Julia sets and Schottky groups, Hausdorff dimension and box dimension are equal.*

2.2. **Self-similar IFS's.** Let us consider the very special class of contractions: similarities. A map $F : \mathbb{R}^n \to \mathbb{R}^n$ is called similarity if there exists $r > 0$ such that $||F(x) - F(y)|| = r||x - y||$, for all $x, y \in \mathbb{R}$. This condition is of course much stronger than asking that F is conformal.However, for this class of iterated function schemes it is easiest to find an expression for the dimension.

Definition. Let $\mathcal{L}$ be the limit set of an IFS for which the contractions are similarities. Then $\mathcal{L}$ is called a self-similar set.

Examples: (1) The middle third Cantor set: Here, the similarities are given by

$$F_1 : x \mapsto \frac{x}{3} \text{ and } F_2 : x \mapsto \frac{x+2}{3}.$$

(2) The Sierpinski Carpet: Here, the similarities are given by

$$F_{i,j} : (x,y) \mapsto \left(\frac{x+i}{3}, \frac{x+j}{3}\right), \text{for } i, j \in \{0,1,2\}, (i,j) \neq (1,1).$$

(3) von Koch Curve: Here, the similarities are given by

$$F_1 : (x,y) \mapsto \left(\frac{x}{3}, \frac{y}{3}\right), F_2 : (x,y) \mapsto \left(\frac{x+2}{6}, \frac{y}{2\sqrt{3}}\right),$$

$$F_3 : (x,y) \mapsto \left(\frac{x+3}{6}, \frac{1-y}{2\sqrt{3}}\right), F_4 : (x,y) \mapsto \left(\frac{x+2}{3}, \frac{y}{3}\right).$$

We now have the following crucial result. Note that our proof here is inspired by arguments given by Sullivan in order to show that the Hausdorff dimension of the limit set of a convex cocompact Kleinian group is equal to the Poincaré exponent of the Kleinian group.

Theorem 2.6 (Hutchinson, Moran). *The Hausdorff dimension δ of the limit set $\mathcal{L}$ of an IFS (with open set condition) whose contractions $F_1, ..., F_k$ are similarities (with contraction rates $r_1, ..., r_k$) is given by*

$$\sum_{i=1}^{k} r_i^{\delta} = 1.$$

Proof. For ease of exposition, let us only consider the special 1-dimensional case $M = I_0 := (0,1)$. With $F_i(M) =: I_i$, we then have that $\{I_i : i = 1, ..., k\}$ is a family of pairwise disjoint intervals contained in I_0. Let $Q_0 \subset \mathbb{R}^2$ be the *open* unit square based at I_0, and for each $(i_1, ..., i_m) \in \{1, ..., k\}^m$ let $Q_{i_1 \ldots i_m}$ denote the interior of the square based at the interval $I_{i_1 \ldots i_m} := F_{i_1} \circ ... \circ F_{i_m}(I_0)$ (clearly, $\mathrm{diam}(I_{i_1 \ldots i_m}) = r_{i_1} \cdot ... \cdot r_{i_m}$). Also, let $\widehat{I}_{i_1 \ldots i_m}$ denote the top side of the square $Q_{i_1 \ldots i_m}$, that is the side of $Q_{i_1 \ldots i_m}$ opposite to $I_{i_1 \ldots i_m}$. Moreover, for $x \in I_0$ we let s_x denote the horizontal straight line segment which starts at the top side of Q_0 and ends at $x \in I_0$. We can then visualise the limit set $\mathcal{L}$ as follows:

$$\mathcal{L} = \{x \in I_0 : s_x \cap \widehat{I}_{i_1 \ldots i_m} \neq \emptyset \text{ for infinitely many } i_1, ..., i_m\}.$$

Alternatively, we can think of $\mathcal{L}$ as being the set of accumulation points of the set of top sides $\{\widehat{I}_{i_1 \ldots i_m} : (i_1, ..., i_m) \in \{1, ..., k\}^m, m \in \mathbb{N}\}$. Then, the Poincaré series associated with the IFS is given for $s \in \mathbb{R}$ by

$$\Sigma_s := \sum_{m \in \mathbb{N}} \sum_{i_1 \ldots i_m} (r_{i_1} \cdot ... \cdot r_{i_m})^s ,$$

and we let δ denote the exponent of convergence of this series. Now, let us first show that δ has the property that $\sum_{i=1}^{k} r_i^{\delta} = 1$. Indeed, this follows since for $s > \delta$ we have

$$\sum_{m \in \mathbb{N}} \sum_{i_1 \ldots i_m} (r_{i_1} \cdot ... \cdot r_{i_m})^s = \sum_{m \in \mathbb{N}} \left(\sum_{i=1}^{k} r_i^s \right)^m = \frac{\sum_{i=1}^{k} r_i^s}{1 - \sum_{i=1}^{k} r_i^s},$$

and hence, Σ_s converges if and only if $s > \delta$, where δ is given by $\sum_{i=1}^{k} r_i^{\delta} = 1$. Note that the IFS is of (what one calls) δ-divergence type, meaning that for $s = \delta$ we have that Σ_s is a divergent series. It now remains to show that

$$\dim_H(\mathcal{L}) = \delta.$$

The upper bound: For this note that for each $\epsilon > 0$ there exists $n(\epsilon) \in \mathbb{N}$ such that $\bigcup_{m \geq n(\epsilon)} \bigcup_{i_1 \ldots i_m} I_{i_1 \ldots i_m}$ is an ϵ-covering of $\mathcal{L}$ (to see this, choose $n(\epsilon)$ such that $r_{i_1} \ldots r_{i_k} < \epsilon$, for all $k \geq n(\epsilon)$). Therefore, for each $\epsilon, \rho > 0$ we have that

$$\mathcal{H}_{\epsilon}^{\delta+\rho}(\mathcal{L}) \leq \sum_{m \geq n(\epsilon)} \sum_{i_1 \ldots i_m} (\mathrm{diam}\,(I_{i_1 \ldots i_m}))^{\delta+\rho} .$$

Since $\Sigma_{\delta+\rho}$ converges, we have that the latter sum tends to zero for ϵ tending to zero. This gives that for the $(\delta+\rho)$-dimensional Hausdorff measure $\mathcal{H}^{\delta+\rho}(\mathcal{L})$ we have for each $\rho > 0$ that

$$\mathcal{H}^{\delta+\rho}(\mathcal{L}) = 0,$$

and this clearly finishes the proof of the assertion.

The lower bound: For the lower bound we construct a probability measure on $\mathcal{L}$, which we call Cantor-Patterson measure, and then use the mass distribution principle to finish the proof.

Let $(\epsilon_n)_{n \in \mathbb{N}}$ be some fixed sequence of positive numbers such that $\lim_{n \to \infty} \epsilon_n = 0$.

For each $n \in \mathbb{N}$, let the probability measure μ_n be defined for arbitrary Borel sets $A \subset Q_0 \cup I_0$ by

$$\mu_n(A) := \frac{1}{\Sigma_{\delta+\epsilon_n}} \cdot \sum_{m\in\mathbb{N}} \sum_{\substack{i_1\dots i_m \\ \widehat{I}_{i_1\dots i_m}\subseteq A}} (r_{i_1}\cdot\ldots\cdot r_{i_m})^{\delta+\epsilon_n}.$$

Note that $\mu_n(I_0) = 0$, for each $n \in \mathbb{N}$. However, if n increases the region where we find the measure μ_n concentrated on gets pushed down further and further towards the bottom line of Q_0. More precisely, if for example $\widehat{I}_{i_1\dots i_n}$ is one of the top sides in the construction, the we have

$$\mu_k(\widehat{I}_{i_1\dots i_n}) = \frac{r_{i_1\dots i_n}^{\delta+\epsilon_k}}{\Sigma_{\delta+\epsilon_k}} \to 0, \text{ for } k \text{ tending to infinity.}$$

Clearly, $\lim_{n\to\infty}\mu_n(E) = 0$, for each $E \subset Q_0$ (recall that $Q_0 \subset \mathbb{R}^2$ was chosen to be the *open* unit square based at I_0). Then let μ be a weak limit measure of the the sequence $(\mu_n)_{n\in\mathbb{N}}$. (Recall that if ν is a weak limit of a sequence of probability measures $(\nu_k)_k$, then $\nu(O) \le \liminf_k \nu_k(O)$ for all O open, and $\limsup_k \nu_k(C) \le \nu(C)$ for all C compact). One immediately verifies that μ is supported on the limit set $\mathcal{L}$. Indeed, if this would not be the case, then there would exist an open connected set D in the complement of $\mathcal{L}$ such that $\mu(D) > 0$ and such that the open square $\widetilde{D}$ based at D has empty intersection with the set of top sides $\{\widehat{I}_{i_1\dots i_m} : (i_1,\dots,i_m) \in \{1,\dots,k\}^m, m \in \mathbb{N}\}$. Since obviously $\mu_k(\widetilde{D}) = 0$ for each $k \in \mathbb{N}$, this would then immediately lead to the contradiction

$$0 < \mu(D) \le \liminf_k \mu_k(D \cup \widetilde{D}) = \liminf_k \mu_k(\widetilde{D}) = 0.$$

Next observe that for each $Q_{i_1\dots i_n}$ we have

$$\begin{aligned}\sum_{m\in\mathbb{N}} \sum_{\substack{j_1\dots j_m \\ \widehat{I}_{j_1\dots j_m}\subseteq Q_{i_1\dots i_n}}} (r_{j_1}\cdot\ldots\cdot r_{j_m})^{\delta+\epsilon_k} &= \sum_{m\in\mathbb{N}}\sum_{l_1\dots l_m} (r_{i_1}\cdot\ldots\cdot r_{i_n} r_{l_1}\dots r_{l_m})^{\delta+\epsilon_k} \\ &= (r_{i_1}\cdot\ldots\cdot r_{i_n})^{\delta+\epsilon_k} \sum_{m\in\mathbb{N}}\sum_{l_1\dots l_m} (r_{l_1}\dots r_{l_m})^{\delta+\epsilon_k} \\ &= (r_{i_1}\cdot\ldots\cdot r_{i_n})^{\delta+\epsilon_k}\,\Sigma_{\delta+\epsilon_k}.\end{aligned}$$

By dividing the latter by $\Sigma_{\delta+\epsilon_k}$, it follows for each $k \in \mathbb{N}$ that

$$\mu_k(Q_{i_1\dots i_n}) = (r_{i_1}\cdot\ldots\cdot r_{i_n})^{\delta+\epsilon_k}.$$

Since $Q_{i_1\dots i_n} \cup I_{i_1\dots i_n}$ is open in $Q_0 \cup I_0$, similar as above it follows that

$$\begin{aligned}\mu(I_{i_1\dots i_n}) &= \mu(Q_{i_1\dots i_n} \cup I_{i_1\dots i_n}) \le \liminf_k \mu_k(Q_{i_1\dots i_n} \cup I_{i_1\dots i_n}) \\ &= \liminf_k \mu_k(Q_{i_1\dots i_n}) = \liminf_k (r_{i_1}\cdot\ldots\cdot r_{i_n})^{\delta+\epsilon_k} \\ &\le (r_{i_1}\cdot\ldots\cdot r_{i_n})^{\delta} = (\mathrm{diam}(I_{i_1}\cdot\ldots\cdot r_{i_n}))^{\delta}.\end{aligned}$$

From here it is now straight forward, by using the bounded geometry of the IFS, to deduce that for each $x \in \mathcal{L}$ and each sufficiently small radius $r > 0$ we have

$$\mu(B(x,r)) \ll r^{\delta}.$$

By applying the mass distribution principle, it follows that $\dim_H(\mathcal{L}) \ge \delta$.

□

The previous theorem allows us to determine the Hausdorff dimensions of the various examples which we already considered before.

Examples:

(1) $\frac{1}{3}$-*Cantor Set* C*:* Here we have

$$\left(\frac{1}{3}\right)^{\frac{\log 2}{\log 3}} + \left(\frac{1}{3}\right)^{\frac{\log 2}{\log 3}} = 1,$$

and hence $\dim_H(C) = \log 2/\log 3$.

(2) *Sierpinski Carpet* S*:* Here we have

$$8 \cdot \left(\frac{1}{3}\right)^{\frac{\log 8}{\log 3}} = 1,$$

and hence $\dim_H(S) = 3\log 2/\log 3$.

(3) *von Koch Curve* K*:* Here we have that each of the contractions $F_1, ..., F_4$ has the contraction rate $1/3$, and hence

$$4 \cdot \left(\frac{1}{3}\right)^{\frac{\log 4}{\log 3}} = 1.$$

This gives $\dim_H(K) = 2\log 2/\log 3$.

2.3. **The Bowen formula.** In order to associate an iterated function scheme to an expanding map $F: X \to X$, for X compact, we need to introduce the concept of a Markov Partition (we will always assume that F is a conformal expanding $C^{1+\epsilon}$ map, that F' is ϵ-Hölder continuous, F' is the same in all directions and $|F'(x)| > \alpha > 1$). The contractions in the associated iterated function system will then essentially be the inverse branches to the expanding maps. Let

Definition. A finite collection $\mathcal{M} = \{M_i\}_{i=1,\dots,k}$ of closed subsets of X is called Markov partition if the following hold.

(1) $X = \bigcup_{i=1}^{k} M_i$.
(2) Each of the M_i is the closure of the interior of M_i (relative to X).
(3) For each $i \in \{1, ..., k\}$ there exists $l_1(i), ..., l_m(i) \in \{1, ..., k\}$ such that $F(M_i) = \bigcup_{j=1}^{m} M_{l_j(i)}$.
(4) $F|_{M_i}$ is a local homeomorphism, for each $i \in \{1, ..., k\}$.

Note that in many examples we actually have that $F(M_i) = X$, for all $i \in \{1, ..., k\}$. We now state the following standard result (without proof).

Lemma 2.7. *For each conformal expanding* $C^{1+\epsilon}$ *map there exists a Markov Partition.*

This result is very useful, since it allows us to consider the family of the local inverses $F_i : F(M_i) \to M_i$ (extended to suitable open neigh- bourhoods) as an iterated function system.

Example: *Limit sets for Schottky groups* G*:* In this case, we have k pairs of disjoint disks $(D_i^{(1)}, D_I^{(-1)})$, whose boundaries are the isometric circles associated with the set of generators $\{g_1, ..., g_k\}$ (and there inverses). We then have that g_i maps the interior of $D_i^{(1)}$ onto the exterior of $D_i^{(-1)}$, and g_i^{-1} maps the interior of $D_i^{(-1)}$ onto the exterior of $D_i^{(1)}$. With $L(G)$ denoting the limit set of G, we then define the map $F : L(G) \to L(G)$ for each $i = 1, ..., k$ by:

$$\text{If } z \in D_i^{(-1)} \text{ for } j \in \{-1, 1\}, \text{ then } F(z) := g_i^j(z).$$

We then have that F is an expanding map.

In order to state Bowen's Formula, we require to introduce the pressure function P associated with in IFS. This function is given for a a continuous function $\varphi : \mathcal{L} \to \mathbb{R}$ by

$$P(\varphi) := \lim_{n\to\infty} \frac{1}{n} \log \left(\sum_{(x_1,\dots,x_n)\in\{1,\dots,d\}^n} e^{\sup_{x\in[x_1,\dots,x_n]}(\varphi(\pi(x))+\varphi(\pi(\sigma(x)))+\dots+\varphi(\pi(\sigma^{n-1}(x))))} \right),$$

where as usual $[x_1, ..., x_n] := \{(y_1, y_2, ...) : y_i = x_i, \text{ for all } i = 1, ..., n\}$ denotes an n-cylinder. It can be shown that the limit in this definition does always exist (at least for the cases which we consider here). In the situation of an expanding map F, one very often considers the family of special potential functions $\{-s\log|F'(x)| : s \in [0, n]\}$ and for these the pressure function can then be written as

$$P(-s\log|F'|) := \lim_{n\to\infty} \frac{1}{n} \log \left(\sum_{\substack{x\in\mathcal{L} \\ F^n(x)=x}} |(F^n)'(x)|^{-s} \right).$$

Theorem 2.8 (Bowen's Formula). *Let F be a conformal expanding $C^{1+\epsilon}$ map. We then have that in $s \in [0, n]$ there exists a unique solution of the pressure equation*

$$P(-s\log|F'|) = 0,$$

and this occurs precisely at $s = \delta = \dim_H(\mathcal{L})$.

Proof. In the linear situation where F is given such that the inverse branches of F are linear maps $F_1, ..., F_k$ with contraction rates $r_1, ..., r_k$, we have that

$$\begin{aligned} P(-s\log|F'|) &= \lim_{n\to\infty} \frac{1}{n} \log \left(\sum_{\substack{x\in\mathcal{L} \\ F^n(x)=x}} |(F^n)'(x)|^{-s} \right) \\ &= \lim_{n\to\infty} \frac{1}{n} \log \left(\sum_{(i_1,\dots,i_n)\in\{1,\dots,k\}^n} (|r_{i_1}| \cdot \dots \cdot |r_{i_n}|)^s \right) \\ &= \lim_{n\to\infty} \frac{1}{n} \log \left(\sum_{i\in\{1,\dots,k\}} |r_i|^s \right)^n . \end{aligned}$$

Since in this situation we have that $\sum_{i\in\{1,\dots,k\}} |r_i|^\delta = 1$, it follows that

$$P(-\delta\log|F'|) = \lim_{n\to\infty} \frac{1}{n} \log 1 = 0.$$

For the more general case in which F is a conformal expanding $C^{1+\epsilon}$ map, one uses the conformality and the ϵ-Hölder continuity of F' in order to show that for each $(i_1, ..., i_n) \in \{1, ..., k\}^n$ and for all $x, y \in \mathcal{L}$ we have

$$|F'_{i_1\dots,i_n}(x)| \asymp |F'_{i_1\dots,i_n}(y)| \asymp \operatorname{diam}(M_{i_1\dots,i_n}),$$

and for each $s > 0$ we have

$$\sum_{(i_1,\dots,i_n)} |F'_{i_1\dots,i_n}(x)|^s \asymp \sum_{(i_1,\dots,i_n)} (\operatorname{diam}(U_{i_1\dots,i_n}))^s.$$

Here, $F_{i_1\dots,i_n} := F_{i_1} \circ \dots \circ F_{i_n}$. Also $\{M_1, ..., M_m\}$ denotes some Markov partition and the $U_{i_1\dots,i_n}$ refer to neighborhoods of $F_{i_n} \circ \dots \circ F_{i_2}(M_{i_1})$. Using these estimates, one then uses the Ruelle-Perron-Frobenius Theorem to obtain Bowen's Formula. □

2.4. **Multifractal/thermodynamic formalism in a nutshell.** We only give a sketch of the ideas of the multifractal (or more general, the thermodynamic) formalism by demonstrating it along the example of the measure of maximal entropy. The proofs are omitted. Also, we restrict ourself to the simplest of all cases: an expanding linear map $F:[0,1]\to[0,1]$ such that F has two inverse branches F_1 and F_2 which are linear with contraction rates r_1 and r_2. In this situation we have a priori two 'canonical' measures associated with the system: the Cantor-Patterson measure μ_δ (which is equivalent to the δ-dimensional Hausdorff measure; here, δ denotes the Hausdorff dimension of the limit set $\mathcal{L}$) and the measure of maximal entropy μ_0 (sometimes also referred to as the harmonic measure). That is, for each cylinder set $[x_1...x_n]$ we have (for ease of exposition, we make no distinction between $\{1,2\}^{\mathbb{N}}$ and the limit set $\mathcal{L}$)

$$\mu_\delta([x_1...x_n]) \asymp (r_{x_1}\cdot ....\cdot r_{x_n})^\delta \text{ and } \mu_0([x_1...x_n]) \asymp 2^{-n}.$$

Let us assume that these two measures are not equivalent (that is, they measure on different sets). Then introduce the following notation. Let $\varphi((x_1,x_2,...)) := \log F'_{x_1}((x_2,x_3,...))(=\log r_{x_1})$, $\psi_0((x_1,x_2,...)) := -\log 2$, and let $S_n f(x) := f(x) + f(\sigma(x)) + ... + f(\sigma^{n-1}(x))$. With these notations the above can be written for $x\in[x_1...x_n]$ as follows

$$\mu_\delta([x_1...x_n]) \asymp e^{\delta\log\left(r_{x_1}\cdot ....\cdot r_{x_n}\right)} = e^{\delta S_n\varphi(x)},$$

and

$$\mu_0([x_1...x_n]) \asymp e^{-n\log 2} = e^{S_n\psi_0(x)}.$$

It turns out that these two measures do in fact represent two points in a continuous spectrum of probability measures $\{\mu_s : s\in I\}$, for some suitable interval $I\subset\mathbb{R}$. More precisely, for $s\in I$ we have (measures with this property are called Gibbs measures)

$$\mu_s([x_1...x_n]) \asymp e^{sS_n\varphi(x)+\alpha(s)S_n\psi_0(x)},$$

where the function α is given by the pressure equation

$$P(s\varphi+\alpha(s)\psi_0)=0.$$

Let us remark that in the literature the multifractal formalism proceeds sometimes slightly differently at this stage. Namely, one could alternatively consider a function β which is defined on some suitable interval J and which is given by the pressure equation $P(\beta(q)\varphi+q\psi_0)=0$. One then shows that there exists a family $\{\nu_q : q\in J\}$ of probability measures, such that for each ν_q we have $\nu_q([x_1...x_n]) \asymp e^{\beta(q)S_n\varphi(x)+qS_n\psi_0(x)}$. Clearly, this approach is completely analogous to the one which we have chosen here. Coming back to our approach, let us remark that the existence of the function α and the existence of the family $\{\mu_s : s\in I\}$ are the first two corner stones of the formalism(s). Moreover, in our special situation where we have the measure of maximal entropy on board, one immediately finds that the function α can be expressed in terms of the pressure function as follows

$$\alpha(s)=P(s\varphi)/\log 2.$$

Indeed, we have that

$$\begin{aligned} P\left(s\varphi+\frac{P(s\varphi)}{\log 2}\psi_0\right) &= \lim_{n\to\infty}\frac{1}{n}\log\left(\sum_{(x_1,\ldots,x_n)}(r_{x_1}\ldots r_{x_n})^s e^{\frac{P(s\varphi)}{\log 2}\psi_0 n}\right) \\ &= \lim_{n\to\infty}\frac{1}{n}\log\left(e^{\frac{P(s\varphi)}{\log 2}\psi_0 n}\left(\sum_{(x_1,\ldots,x_n)}(r_{x_1}\ldots r_{x_n})^s\right)\right) \\ &= \lim_{n\to\infty}\frac{1}{n}\log\left(e^{\frac{P(s\varphi)}{\log 2}\psi_0 n}\right)+\lim_{n\to\infty}\frac{1}{n}\log\left(\sum_{(x_1,\ldots,x_n)}(r_{x_1}\ldots r_{x_n})^s\right) \\ &= -P(s\varphi)+P(s\varphi)=0. \end{aligned}$$

Summarising this, we now have

$$P\left(s\varphi+\alpha(s)\psi_0\right)=0 \text{ and } \mu_s([x_1\ldots x_n])\asymp e^{sS_n\varphi(x)+\alpha(s)S_n\psi_0(x)},$$

for $s\in I$ and where the latter holds for each $x\in[x_1\ldots x_n]$.
The next step is to determine the sets on which the measures μ_s are concentrated on. It turns out that these sets are given by level sets of the form

$$\mathcal{L}_t := \left\{x : \lim_{n\to\infty}\frac{S_n\psi_0(x)}{S_n\varphi(x)}=t\right\}.$$

Note that

$$\mathcal{L}_t=\left\{(x_1,x_2,\ldots):\lim_{n\to\infty}\frac{\log\mu_0([x_1\ldots x_n])}{\log r_{x_1}\ldots r_{x_n}}=t\right\}=\left\{(x_1,x_2,\ldots):\lim_{r\to 0}\frac{\log\mu_0(B(x,r))}{\log r}=t\right\}.$$

More precisely, for each $s\in I$ there exists a smallest number $\tau(s)\in\mathbb{R}$ (usually expressed in terms of a Legendre transformation) such that

$$\lim_{n\to\infty}\frac{S_n\psi_0(x)}{S_n\varphi(x)}=\tau(s),$$

and such that the set of points which have this property has full μ_s-measure, meaning that $\mu_s(\mathcal{L}_{\tau(s)})=1$. This represents the third corner stone of the formalism(s). We can now determine the Hausdorff dimension of the set $\mathcal{L}_{\tau(s)}$ as follows. For this we look at the power law of the the measure μ_s, and observe that for $x\in\mathcal{L}_{\tau(s)}$ and for each $\epsilon>0$ we have for n sufficiently large

$$\begin{aligned} e^{sS_n\varphi(x)+\alpha(s)S_n\psi_0(x)} &= e^{S_n\varphi(x)\left(s+\alpha(s)\frac{S_n\psi_0(x)}{S_n\varphi(x)}\right)} \\ &= (r_{x_1}\cdot\ldots\cdot r_{x_n})^{s+\alpha(s)\tau(s)\pm\epsilon} \end{aligned}$$

Therefore, we now have for each $x\in\mathcal{L}_{\tau(s)},\epsilon>0$ and for $r>0$ sufficiently small

$$r^{s+\alpha(s)\tau(s)+\epsilon}\ll\mu_s(B(x,r))\ll r^{s+\alpha(s)\tau(s)-\epsilon},$$

and hence the mass distribution principle implies

$$\dim_H(\mathcal{L}_{\tau(s)})=s+\alpha(s)\tau(s).$$

Note: Let us remark that in the multifractal analysis for self-similar measures, instead of starting with the measure of maximal entropy, one starts with a self-similar measure $\nu_{\underline{p}}$, given by a probability vector $\underline{p}=(p_1,p_2)$ (where $p_1,p_2>0$ and $p_1+p_2=1$) such that

$$\nu_{\underline{p}}(E)=\sum_{i=1}^{2}p_i\nu_{\underline{p}}(F_i^{-1}(E)), \text{ for all } A\subset[0,1] \text{ Borel}.$$

Clearly, the above arguments can then be adapted in a straight forward way to this situation.