The Visibility Conjecture

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Abstract

I will introduce the concept of 'visibility' in the context of fractal sets, with emphasis on subsets of $\mathbb{R}^2$. I will describe the so called 'Visibility Conjecture' and survey some recent work on the subject, sketch a couple of basic proofs and finish by briefly discussing my own research.

Definition: For an angle $\theta \in [0, 2\pi)$ we define the line $l_\theta \subset \mathbb{R}^2$ to be the half space given by

$$l_\theta = \{x = (r\cos\theta, r\sin\theta) : r \geqslant 0\}$$

and for a compact set $K \subset \mathbb{R}^2$ we define the visible part of K at θ to be

$$V_\theta K = \Big\{x \in K : (x + l_\theta) \cap K = \{x\}\Big\}.$$

Theorem: Let $K \subset \mathbb{R}^2$ be a Borel set.

i) If $\dim_H K \leqslant 1$ we have $\dim_H \mathrm{proj}_\theta K = \dim_H K$ for Lebesgue almost all $\theta \in [0, 2\pi)$.

ii) If $\dim_H K > 1$ we have $\dim_H \mathrm{proj}_\theta K = 1$ for Lebesgue almost all $\theta \in [0, 2\pi)$.

The Visibility Conjecture: For $K \subset \mathbb{R}^2$ such that $\dim_H K > 1$ we have

$$\dim_H V_\theta K = 1$$

for Lebesgue almost all $\theta \in [0, 2\pi)$.

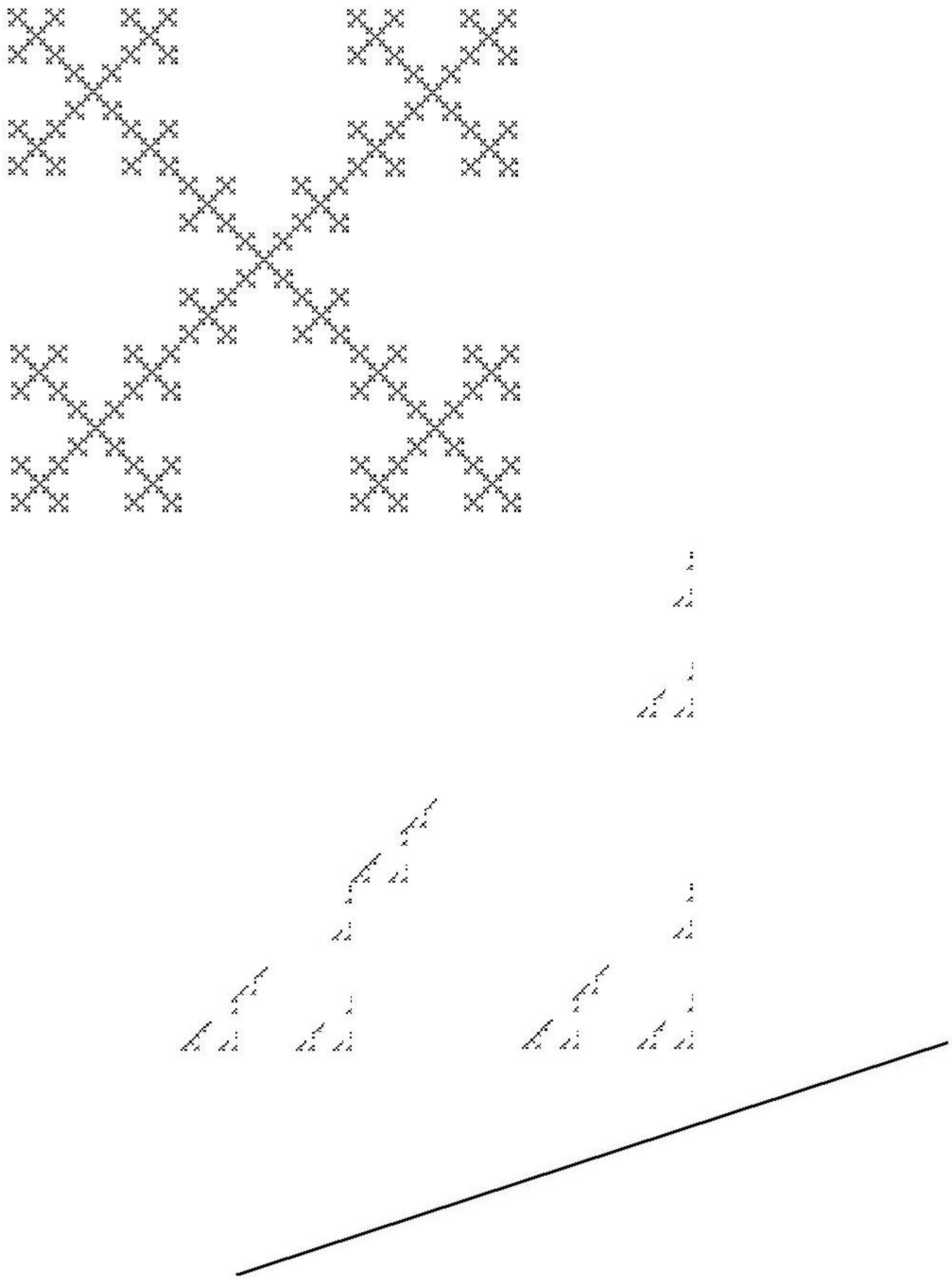

Figure 1: A self-similar Sierpinski carpet and the visibility set corresponding to $\theta = \arctan(\frac{1}{3}) + 3\pi/2$.

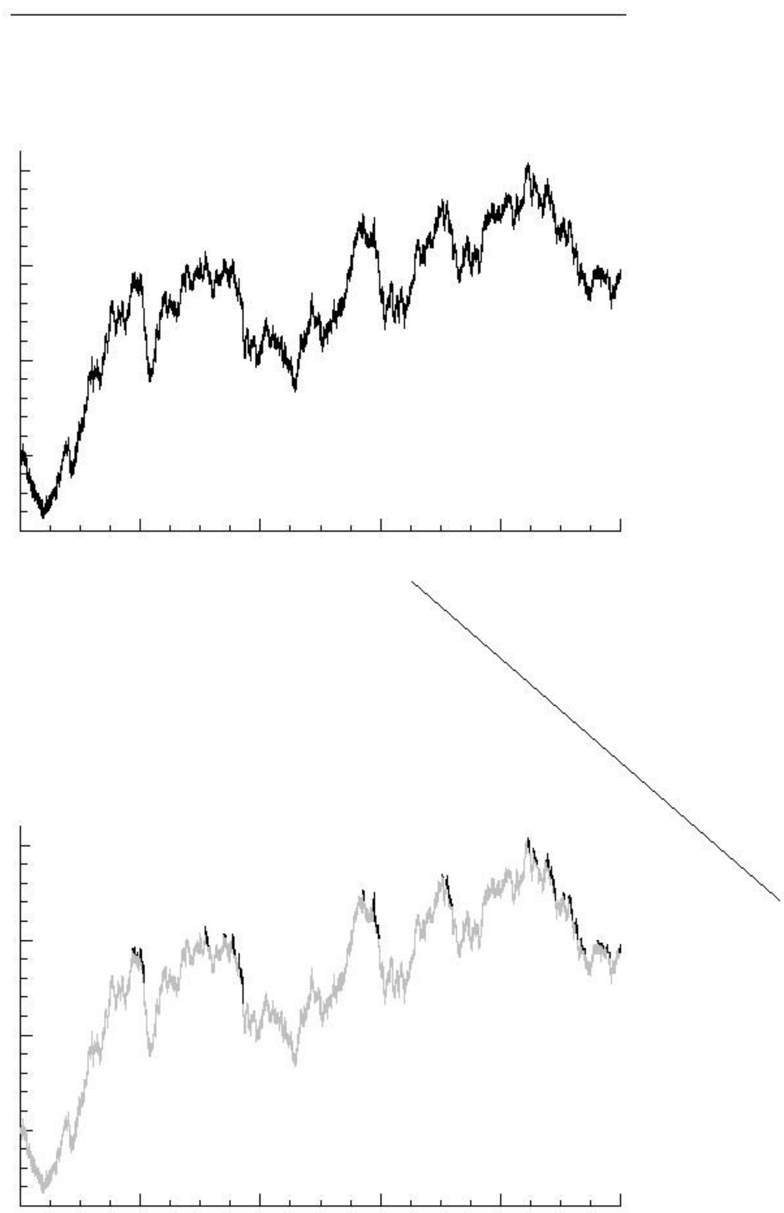

Figure 2: A fractal graph showing the visibility set corresponding to two different angles.

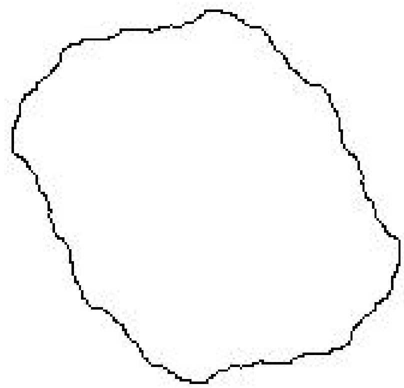

Figure 3: A Julia set corresponding to the mapping $f(z) = z^2 + i/4$.

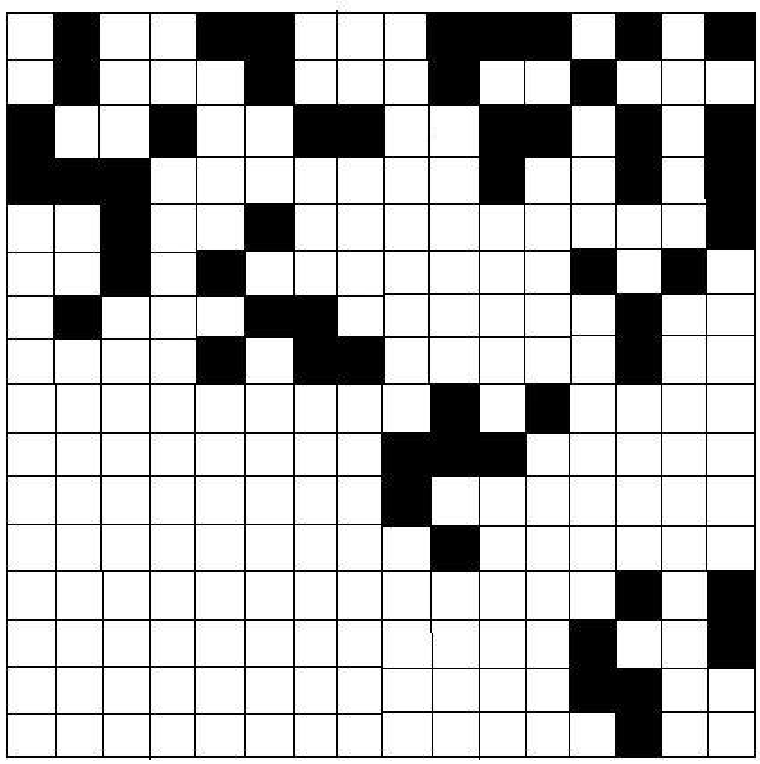

Figure 4: An example of fractal percolation.

Theorem: If $K \subset \mathbb{R}^2$ is a connected compact set then for (Lebesgue) almost all points $x \in \mathbb{R}^2$ we have

$$\dim_H K_x \leqslant \frac{1}{2} + \sqrt{\dim_H K - \tfrac{3}{4}} < \dim_H K$$

where K_x denotes the visible part of K from the point x (O'Neil [1]).

Definition: The upper box dimension of a non-empty bounded set $K \subset \mathbb{R}^2$ is defined as

$$\overline{\dim}_B F = \limsup_{\delta \to 0} \frac{\log N_\delta(F)}{-\log \delta}$$

where $N_\delta(F)$ is the smallest number of sets possible in a δ cover of F.

$$\dim_H F \leqslant \overline{\dim}_B F$$

References

[1] T. C. O'Neil. The hausdorff dimension of visible sets of planar continua. *Transactions of the AMS*, 359(11):5141–5170, 2007.