Transverse Laplacians for Substitution Tilings

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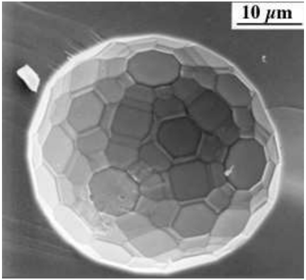
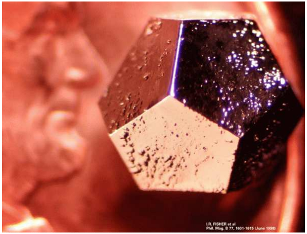
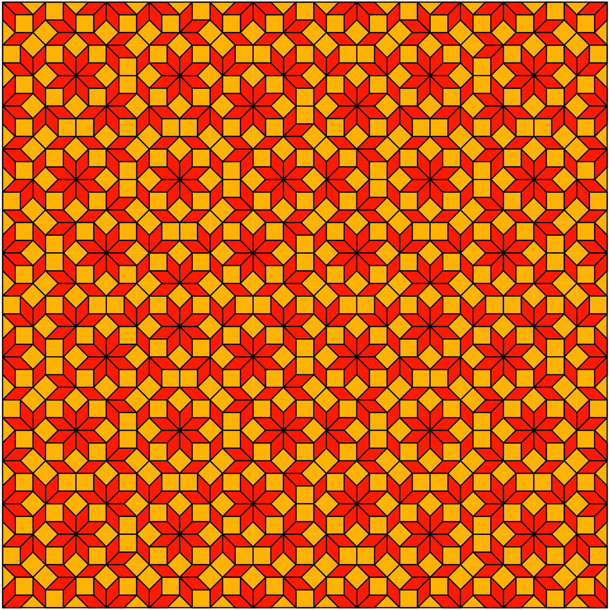
Aperiodic order and quasicrystals

Examples: $Al_{62.5}Cu_{25}Fe_{12.5}$, $Al_{70}Pd_{22}Mn_8$, $Al_{70}Pd_{22}Re_8$.





The Ammann–Benker “octogonal” tiling



Spectral triples for Cantor sets

The triadic Cantor set (Connes '94)

Metric Cantor sets (Christensen – Ivan '03)

Ultrametric Cantor sets (Pearson – Bellissard '08)

Self-similar cases and applications (Julien – Savinien '09 arXiv: 0908.1095 [math.OA])

Cantor set and Bratteli diagrams

Ultrametric Cantor set $(C, \rho) \leftrightarrow$ set of infinite paths in a Bratteli diagram B. (Michon '80)

Metric $\leftrightarrow$ weights $w : \Pi \to \mathbb{R}_+$
$w(\gamma) = \mathrm{diam}[\gamma], \text{ diameter of cylinder}$.

Self-similarity $\leftrightarrow$ stationary diagram.

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Choice function $\tau : \Pi \to \partial B \times \partial B$
$\gamma \mapsto$ a pair $(\tau_-\gamma, \tau_+\gamma)$ of infinite paths with largest common prefix γ,
$\rho(\tau_-\gamma, \tau_+\gamma) = \text{diam}[\gamma]$.

Stationary diagrams and transversals

Tiling T of $\mathbb{R}^d$: covering by (marked) tiles
Tiling space: closure of $T + \mathbb{R}^d$.

Transversal Ξ: subset of tilings with a marker at the origin, is a ultrametric Cantor set (combinatorial metric ρ_c).

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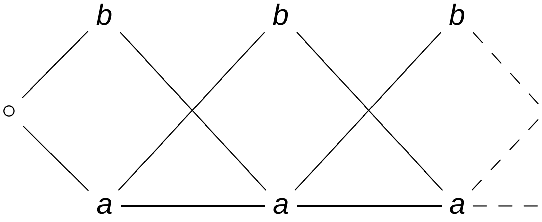
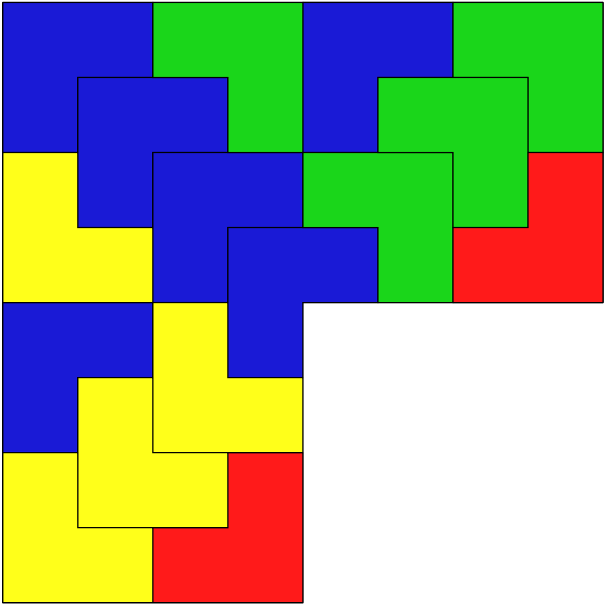
Substitution tiling, Abelianization matrix, and Bratteli diagram.

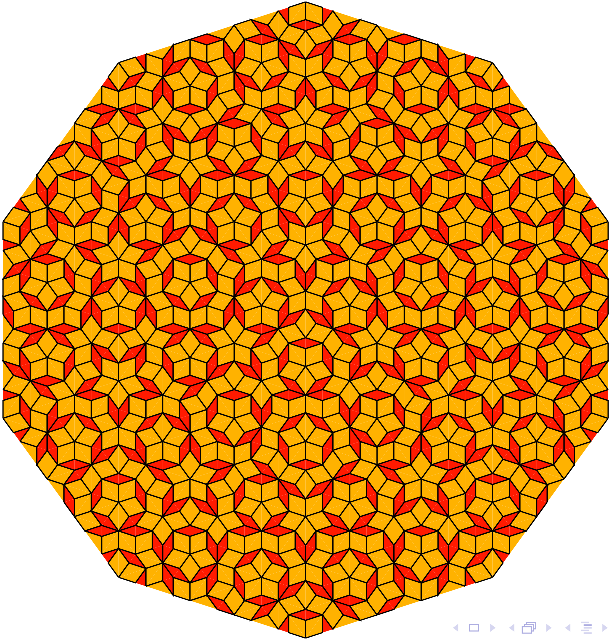
Example: The Fibonacci tiling, $\left\{ \begin{array}{lcl} a & \longrightarrow & ab \\ b & \longrightarrow & a \end{array} \right., \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix},$



The Ammann "chair" tiling



A Penrose tiling



The spectral triple of Pearson–Bellissard

Weighted Bratteli (B, w).

Algebra $\mathcal{A} = C_{\mathrm{lip}}(\partial B)$, Hilbert space $\mathcal{H} = l^2(\Pi) \otimes \mathbb{C}^2$

Dirac operator $D = \bigoplus_{\gamma \in \Pi} \frac{1}{\mathrm{diam}[\gamma]} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

Representation $\pi_\tau(f) = \bigoplus_{\gamma \in \Pi} \begin{bmatrix} f(\tau_- \gamma) & 0 \\ 0 & f(\tau_+ \gamma) \end{bmatrix}$

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"$\partial_\tau f$" $= [D, \pi_\tau(f)] = \bigoplus_{\gamma \in \Pi} \frac{f(\tau_+\gamma) - f(\tau_-\gamma)}{\mathrm{diam}[\gamma]} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

Connes' formula
$\rho_w(x, y) = \sup\{|f(x) - f(y)| \ : \ f \in \mathcal{A}, \sup_\tau \|[D, \pi_\tau(f)]\| \leq 1\}$.

Weights and invariant measure

Substitution tiling of $\mathbb{R}^d$, transversal (Ξ, ρ_c).

Choice of weights $w(\gamma) \sim \Lambda^{-n/d}$,
γ path of length n,
Λ Perron–Froebenius eigenvalue of the Abelianization matrix.

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Proposition There is a bi-Lipschitz homeomorphism
$\psi : (\partial B, \rho_w) \to (\Xi, \rho_c)$.

Theorem $\psi^*(\mu_{\mathrm{Dix}}) = \mu_\Xi^{\mathrm{erg}}$. (the measure of “frequencies”)

The ζ-function and tilings complexity

ζ-function
$\zeta(s) = \frac{1}{2}\mathrm{Tr}\left(|D|^{-s}\right) = \sum_{\gamma\in\Pi} \mathrm{diam}[\gamma]^s$.
Abscissa of convergence s_0.

Complexity of a tiling
$p(n) = \#\{\text{patches of diameter } \leq \mathrm{n}\}$,
exponent of complexity: $\nu = \lim p(n)/\log(n)$, .

Theorem $s_0 = \nu = d$.

The Laplace–Beltrami operators

Dirichlet form on $L^2(\partial B, d\mu_{\mathrm{Dix}})$

$$Q_s(f,g) = \frac{1}{2}\int_{\mathcal{T}} \mathrm{Tr}\left(|D|^{-s}[D,\pi_\tau(f)]^*[D,\pi_\tau(g)]\right) d\nu(\tau) = \langle f, \Delta_s g\rangle$$

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Theorem (Pearson – Bellissard ’08) Δ_s is non-positive, definite, self-adjoint, has pure point spectrum.

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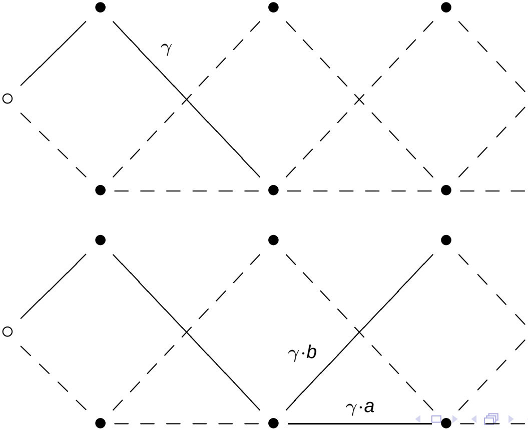
Theorem Spectrum of Δ_s: $E_\gamma = \left\langle \frac{1}{\mu_{\mathrm{Dix}}[\gamma\cdot a]}\chi_{[\gamma\cdot a]} - \frac{1}{\mu_{\mathrm{Dix}}[\gamma\cdot b]}\chi_{[\gamma\cdot b]}\right\rangle$



Cuntz–Krieger algebras and applications

Let A be the adjacency matrix of the edges in B, and $\mathcal{O}_A$ its associated Cuntz–Krieger algebra.

Proposition Given a path $\gamma = (\varepsilon, e_1, e_2, \cdots e_n)$, the eigenelements of Δ_s satisfy

$$\varphi_\gamma = U_\gamma(\varphi_\varepsilon), \quad \lambda_\gamma = u_\gamma(\lambda_\varepsilon),$$

where $U_\gamma = U_{e_1} U_{e_2} \cdots U_{e_n}$, $u_\gamma = u_{e_1} \circ u_{e_2} \circ \cdots u_{e_n}$, and the U_e and u_e form a faithful $*$-representation of $\mathcal{O}_A$ on $L^2(\partial B, d\mu_{\mathrm{Dix}})$ and $l^2(\mathrm{Sp}\,\Delta_s)$ respectively.

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Theorem Weyl asymptotics $\mathcal{N}_s(\lambda) \ \sim_{\lambda \to +\infty} \ \lambda^{d/(d-s+2)}$

Theorem Seeley equivalent $\mathrm{Tr}\,(e^{t\Delta_{s_0=d}}) \ \sim_{\lambda \downarrow 0} \ t^{-d/2}$