Dependence in probability, analysis and number theory: The mathematical work of Walter Philipp (1936 – 2006)

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Introduction

On July 19, 2006, Professor Walter Philipp passed away during a hike in the Austrian Alps as a result of a sudden heart attack. By the time of his death, Walter Philipp had been for almost 40 years on the faculty of the University of Illinois at Urbana-Champaign, for the last couple of years as professor emeritus. He is survived by his wife Ariane and his four children, Petra, Robert, Anthony and André. Walter Philipp is sorely missed by his family, but also by his many colleagues, coauthors and former students all over the world, to whom he was a loyal and caring friend for a long time, in some cases for several decades.

Walter Philipp was born on December 14, 1936 in Vienna, Austria, where he grew up and lived for most of the first 30 years of his life. He studied mathematics and physics at the University of Vienna, where he obtained his Ph.D. in 1960 and his habilitation in 1967, both in mathematics. From 1961 until 1967 he was scientific assistant at the University of Vienna. During this period, Walter Philipp spent two years as a postdoc in the US, at the University of Montana in Missoula and at the University of Illinois. In the fall of 1967 he joined the faculty of the University of Illinois at Urbana-Champaign, where he would stay for the rest of his life. Initially, Walter Philipp was on the faculty of the Mathematics Department, but in 1984 he joined the newly created Department of Statistics at the University of Illinois. From 1990 until 1995 he was chairman of this Department. While on sabbatical leave from the University of Illinois, Walter spent longer periods at the University of North Carolina at Chapel Hill, at MIT, at Tufts University, at the University of Göttingen and at Imperial College, London.

Walter Philipp received numerous recognitions for his work. Most outstanding of these was his election to membership in the Austrian Academy of Sciences.

As a student and postdoc at the University of Vienna, Walter Philipp worked under the guidance of Professor Edmund Hlawka, founder of the famous postwar

Austrian school of analysis and number theory. Other former students of Professor Hlawka include Johann Cigler, Harald Niederreiter, Wolfgang M. Schmidt, Fritz Schweiger and Robert Tichy. It was here that Walter Philipp got in touch with the classical topics from analysis and number theory that would guide a large part of his research for the rest of his life. Uniform distribution, discrepancy of sequences, number-theoretic transformations associated with various expansions of real numbers, additive number-theoretic functions, Diophantine approximation, lacunary series became recurrent themes in Walter Philipp's subsequent work. He studied these themes using techniques from probability theory, e.g. for mixing processes, martingales and empirical processes. He contributed greatly to the development of several branches of probability theory and solved much investigated, difficult problems in analysis and number theory with the help of the tools he developed.

Themes from analysis and number theory

A full understanding and appreciation of Walter Philipp's research requires the background of some topics from analysis and number theory. In what follows we shall briefly introduce the topics that were recurrent themes in Walter Philipp's work.

Uniform distribution mod 1. A sequence $(x_n)_{n\geq 1}$ of real numbers is called uniformly distributed mod 1 if for all $x \in [0,1]$

$$\lim_{N\to\infty} \frac{1}{N} \#\{1 \leq i \leq N : \{x_i\} \leq x\} = x,$$

where $\{x\}$ denotes the fractional part of x. More generally, a sequence $(x_n)_{n\geq 1}$ of vectors in $\mathbb{R}^d$ is called uniformly distributed mod 1 if

$$\lim_{N\to\infty} \frac{1}{N} \#\{1 \leq i \leq N : \{x_i\} \in A\} = \lambda(A)$$

for all rectangles $A \subset [0,1]^d$, where λ denotes Lebesgue measure and for $x \in \mathbb{R}^d$ the symbol $\{x\}$ is interpreted coordinatewise. The famous Weyl criterion (1916) states that a sequence $(x_n)_{n\geq 1}$ in $\mathbb{R}^d$ is uniformly distributed mod 1 iff

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} e^{2\pi i \langle k, x_n \rangle} = 0$$

for all $k \in \mathbb{Z}^d \setminus \{0\}$. An immediate consequence is the uniform distribution of the sequence $(n\alpha)_{n\geq 1} \subset \mathbb{R}$ for irrational α, but with suitable methods this leads to the uniform distribution of many other sequences in one and higher dimensions, for example, a large class of sequences of the type $\{n_k \alpha\}$ for increasing sequences $(n_k)_{k\geq 1}$ of positive integers.

Discrepancies. Given a sequence $(x_n)_{n\geq 1}$ of real numbers uniformly distributed mod 1, one can study the discrepancy

$$D_N := \sup_{0\leq a<b<1} \left| \frac{1}{N} \#\{1 \leq i \leq N : a \leq \{x_i\} < b\} - (b-a) \right|.$$

A Glivenko–Cantelli type argument shows that $D_N \to 0$, and one may then ask for the exact rate of convergence of D_N to 0. In the case of a d-dimensional sequence $(x_n)_{n\geq 1}$, one can define the discrepancy with respect to a class $\mathcal{C}$ of subsets of $[0,1]^d$ by

$$D_N(\mathcal{C}) := \sup_{A\in\mathcal{C}} \left| \frac{1}{N} \#\{1 \leq i \leq N : \{x_i\} \in A\} - \lambda(A) \right|.$$

In this case, the additional issue of the choice of suitable class $\mathcal{C}$ arises. Note that $D_N(\mathcal{C}) \to 0$ does not necessarily hold even if the sequence $(x_n)_{n\geq 1}$ is uniformly distributed.

θ-adic expansion of real numbers. Let $\theta > 1$, not necessarily an integer. Every real number $\omega \in [0,1)$ can be written as an infinite series

$$\omega = \sum_{n=1}^{\infty} \theta^{-n} x_n$$

where $0 \leq x_n < \theta$ are integers. Clearly $x_n = [T^n \omega]$ where the transformation $T : [0,1] \to [0,1]$ is defined by $T\omega := \{\theta\omega\}$ and $[x]$ denotes the integer part of x. Also, $T^n\omega = \sum_{k=1}^{\infty} \theta^{-k} x_{n+k}$. The basic asymptotic question here is the distribution of digits in the expansion, for example, one can ask if the limits of relative frequencies

$$F_k := \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} 1_{\{x_n=k\}}$$

exist. If θ is an integer, the transformation T is ergodic and has Lebesgue measure as an invariant measure, thus the ergodic theorem implies that the limits F_k exist and are equal to θ^{-1} for $k = 0,1,\ldots,\theta-1$ and almost every real number ω. With the usual terminology, almost every real number is normal with respect to base θ. This statement, proved first by Borel in 1909, is a typical result in the metric theory of numbers, stating that a certain property holds for almost every real number, without specifying the exceptional set. In fact, to determine whether a given number ω is normal is a very hard problem and only very few normal numbers are known explicitly. We do not know, for example, if $\sqrt{2}$, e or π are normal in any base. Note that the normality of ω in base $a \in \mathbb{N}$ is equivalent to the statement that the sequence $(a^n\omega)_{n\geq 1}$ is uniformly distributed mod 1.

If θ is not an integer, the transformation T is still ergodic, but Lebesgue measure is not an invariant measure. It is known from work of Alfréd Rényi (1957) that there exists a unique invariant measure μ which is equivalent to Lebesgue measure. In this case, the sequence $(\theta^n\omega)_{n\geq 1}$ is no longer uniformly distributed, but

$$\lim_{N\to\infty}\frac{1}{N}\#\{1\leq n\leq N : T^n\omega\leq x\}=\mu([0,x]),$$

for almost every ω.

Continued fraction expansion. Every real number $\omega\in(0,1]$ can be expressed as an infinite continued fraction

$$\omega=\cfrac{1}{x_1+\cfrac{1}{x_2+\cfrac{1}{x_3+\dots}}}=[x_1,x_2,\dots],$$

where $x_i\in\{1,2,\dots\}$. Closely related is the transformation $T:(0,1]\to(0,1]$, defined by

$$T\omega:=\{1/\omega\}.$$

The n-th digit in the continued fraction expansion is given by $x_n=[T^n\omega]$. As in the case of the θ-adic expansion, $T^n\omega$ can be written as a function of $x_{n+1},x_{n+2},\dots$ by

$$T^n\omega=[x_{n+1},x_{n+2},\dots].$$

T is an ergodic transformation with invariant measure given by the Gauss measure

$$\mu((a,b])=\frac{1}{\log 2}\int_a^b\frac{1}{1+x}dx.$$

As a consequence, the asymptotic distribution of the sequence $(T^n\omega)_{n\geq 1}$ is governed by the Gauss measure, i.e.

$$\lim_{N\to\infty}\frac{1}{N}\#\{1\leq n\leq N : T^n\omega\leq x\}=\frac{1}{\log 2}\int_0^x\frac{1}{1+t}dt.$$

From here, one obtains that the integer k occurs in the continued fraction expansion of a random number ω with relative frequency $\frac{1}{\log 2}\left(\log\frac{k}{k+1}-\log\frac{k+1}{k+2}\right)$, a fact already conjectured by Gauss.

Additive functions in number theory. A function $f:\mathbb{N}\longrightarrow\mathbb{R}$ is called additive if

$$f(mn)=f(m)+f(n),$$

whenever m and n are coprimes. A simple example of an additive function is $\omega(n)$, the number of prime divisors of n. Hardy and Ramanujan studied this function and showed that $\omega(n)$ is of the order $\log\log n$. More precisely, if P_N denotes the uniform distribution on the first N integers, then

$$P_N\left(n\leq N:\left|\frac{\omega(n)}{\log\log N}-1\right|\geq\varepsilon\right)\longrightarrow 0$$

for any $\varepsilon > 0$. Turán observed that the Hardy–Ramanujan theorem is a simple consequence of an easily verifiable inequality for the second moment of $\omega(n)$ and Chebychev's inequality and thus initiated the subject of probabilistic number theory. The Hardy–Ramanujan theorem was later strengthened by Erdős and Kac, who proved a central limit theorem

$$P_N\left(n \le N : \frac{\omega(n) - \log\log N}{\sqrt{\log\log N}} \le x\right) \longrightarrow \int_{-\infty}^{x} e^{-y^2/2} dy.$$

Diophantine approximation. Let f be a positive, continuous, nonincreasing function on $\mathbb{R}^+$. By a classical result of Khinchin, for almost all real α the inequality

$$|q\alpha - p| \le \frac{f(q)}{q} \tag{1}$$

has infinitely many or only finitely many solutions in integers p, q according as $\sum \frac{f(k)}{k}$ diverges or converges. Probabilistically, this is the Borel–Cantelli lemma for certain dependent events; the main difficulty is to deal with the dependence in the case $\sum \frac{f(k)}{k} = \infty$. Khinchin's result has been generalized and improved upon in many directions by Cassels, W. M. Schmidt, Erdős, LeVeque, Szüsz, Gallagher, Ennola, Billingsley and many others. The simplest proof depends on a connection of the problem with continued fraction theory. Call a fraction p/q a best approximation to α if it minimizes $|q'\alpha - p'|$ over fractions p'/q' with denominator q' not exceeding q. The successive best approximations to α are the convergents $p_n(\alpha)/q_n(\alpha)$, $n = 1, 2, \dots$ of its continued fraction expansion and thus the value of $|q\alpha - p|$ for the n-th in the series of best approximation is $d_n(\alpha) = |q_n(\alpha)\alpha - p_n(\alpha)|$. Thus the study of number of solutions of (1) is equivalent to the study of growth of the sequence $d_n(\alpha)$. Khinchin proved that

$$\frac{1}{n} \log d_n(\alpha) \to -\frac{\pi^2}{\log 2}$$

for almost every α.

Lacunary sequences. Probabilistic methods play an important role in harmonic analysis and there is a profound connection between probability theory and trigonometric series. From a purely probabilistic point of view, the trigonometric system $(\cos 2\pi nx, \sin 2\pi nx)_{n\ge1}$ is a sequence of orthogonal (i.e. uncorrelated) random variables over $[0,1]$, which, however, are strongly dependent. For example, the r.v.'s $\sin 2\pi nx$ have the same distribution, but their partial sums $\sum_{n\le N} \sin 2\pi nx$ remain bounded for any fixed x, a behavior very different from that of i.i.d. random variables. However, it has been known for a long time that for rapidly increasing $(n_k)_{k\ge1}$, the sequences $(\sin 2\pi n_k x)_{k\ge1}$ and $(\cos 2\pi n_k x)_{k\ge1}$ behave like sequences of independent random variables. For example, Salem and Zygmund (1947) proved that if $(n_k)_{k\geq 1}$ satisfies the Hadamard gap condition $n_{k+1}/n_k \geq q > 1 \ \ (k = 1, 2, \ldots)$, then $(\sin 2\pi n_k x)_{k\geq 1}$ obeys the central limit theorem, i.e.

$$\lim_{N\to\infty} \lambda\{x \in (0,1) : \sum_{k\leq N} \sin 2\pi n_k x < t\sqrt{N/2}\} = (2\pi)^{-1/2} \int_{-\infty}^{t} e^{-u^2/2} du. \qquad (2)$$

Erdős (1962) proved that the CLT (2) remains valid if the Hadamard gap condition is weakened to $n_{k+1}/n_k \geq 1 + c_k/\sqrt{k}$, $c_k \to \infty$ and this result is sharp. Similar results hold if $\sin n_k x$ is replaced by $f(n_k x)$, where f is a real measurable function satisfying

$$f(x+1) = f(x), \quad \int_0^1 f(x)dx = 0, \quad \int_0^1 f^2(x)dx < \infty.$$

For example, if $f \in \mathrm{Lip}\,(\alpha)$, $\alpha > 0$ and $n_{k+1}/n_k \to \infty$, then the CLT and LIL hold for $f(n_k x)$. (Takahashi (1961, 1963)). If we assume only $n_{k+1}/n_k \geq q > 1$, both the CLT and LIL can fail, a fact discovered by Erdős and Fortet. Gaposhkin (1970) showed that the validity of the CLT for $f(n_k x)$ is closely connected with the number of solutions (k,l) of the Diophantine equation

$$an_k + bn_l = c, \qquad l \leq k, l \leq N.$$

Walter Philipp's work

Given that Walter Philipp published close to 90 research papers, it is impossible to mention every single result he ever obtained. We will instead try to focus on the main lines of his research. Philipp's earliest work, originating from his Ph.D. thesis, concerns uniform distribution mod 1. Weyl (1916) had shown that the sequence $(a_n\,\omega)_{n\geq 1}$ is uniformly distributed modulo 1 for almost all $\omega \in [0,1]$, if $(a_n)_{n\geq 1} \subset \mathbb{R}$ is a sequence of positive numbers satisfying $a_{n+1} - a_n \geq \delta > 0 \ \ (n = 1, 2, \ldots)$ for some $\delta > 0$. Walter Philipp studied this question for d-dimensional sequences, i.e. for the sequence $(A_n\omega)_{n\geq 1}$ where $\omega \in \mathbb{R}^d$ and $(A_n)_{n\geq 1}$ is a sequence of d-dimensional matrices satisfying some growth condition. As a corollary, uniform distribution of the sequence $(A^n\omega)_{n\geq 1}$ for almost all $\omega \in \mathbb{R}^d$ can be obtained, provided the matrix A has all eigenvalues strictly larger than 1.

In 1967 Walter Philipp published the first in a series of papers on the asymptotic behavior of weakly dependent stochastic processes. This topic would dominate his research interests for the next 15 years and keep his close attention for the rest of his life. In the second half of the 1960s and during all of the 1970s Walter Philipp was internationally recognized as a leader in the development of new limit theory for weakly dependent processes and their applications to problems in analysis and number theory. In this period the focus of his research changed substantially: instead of an analyst and number theorist using tools from probability theory he became a probabilist applying his results to problems in analysis and number theory.

In all of the topics from analysis and number theory mentioned above, there are sequences of random variables in the background, most of them defined on the probability space $[0,1]$, equipped with some measure equivalent to Lebesgue measure. In

Weyl's almost sure equidistribution theory, we have the random variables $\omega \mapsto a_n \omega$. In each of the different expansions of real numbers $\omega \in (0,1]$, the n-th digit maps $\omega \to x_n = x_n(\omega)$ are random variables, and so are the n-th iterates $\omega \mapsto T^n \omega$. With the notable exception of the digits in the expansion to an integer base a, none of these random variables are independent. But the dependence is weak, in some sense yet to be defined. Walter Philipp soon realized that the theory of weakly dependent stochastic processes, then only recently created by publications of Rosenblatt (1956), Ibragimov (1962) and Billingsley (1968), provides the right framework for the problems he wanted to attack.

There is no such thing as a universal definition of weak dependence that would imply the validity of all limit theorems known for independent processes. There are many notions, each of them allowing, under additional technical assumptions, the proof of some of the classical limit theorems of probability theory. The stronger the notion, the more limit theorems can be established, but at the same time fewer examples satisfy the conditions. The earliest and most classical notions of weak dependence are α-mixing (also called strong mixing, but not to be confused with the same notion in ergodic theory) and ϕ-mixing (also called uniform mixing). Let $(X_n)_{n\geq 1}$ be a stochastic process, and define for integers k,l with $k \leq l$ the σ-fields $\mathcal{F}_k^l = \sigma(X_k, \ldots, X_l)$. We then define the mixing coefficients

$$\begin{aligned} \alpha(k) &:= \sup_{n\geq 1} \sup_{A\in\mathcal{F}_1^n,\, B\in\mathcal{F}_{n+k}^\infty} |P(A\cap B) - P(A)P(B)|, \\ \phi(k) &:= \sup_{n\geq 1} \sup_{A\in\mathcal{F}_1^n, B\in\mathcal{F}_{n+k}^\infty, P(A)>0} \frac{|P(A\cap B) - P(A)P(B)|}{P(A)}. \end{aligned}$$

The process $(X_n)_{n\geq 1}$ is called α-mixing if $\lim_{n\to\infty} \alpha(n) = 0$ and ϕ-mixing if $\lim_{n\to\infty} \phi(n) = 0$. Rosenblatt (1956) and Ibragimov (1962) established central limit theorems for α- and ϕ-mixing random variables, requiring a combination of moment conditions and conditions on the speed at which the mixing rates converge to zero. Later, several other related mixing concepts (ψ, ρ mixing, absolute regularity, etc.) were introduced and studied in detail. For stationary ϕ-mixing processes $(X_n)_{n\geq 1}$, Ibragimov conjectured that the central limit theorem holds if $EX_1^2 < \infty$ and $\mathrm{Var}\left(\sum_{k=1}^n X_k\right) \to \infty$, but until today this conjecture has not been verified. This conjecture inspired some of Walter Philipp's deepest results in the field of mixing random variables: his 1986 joint paper with Dehling and Denker, giving a necessary and sufficient condition for the CLT for ρ-mixing sequences without any rate or moment conditions and his 1998 joint paper with Berkes, giving a complete characterization of the law of the iterated logarithm and the domain of partial attraction of the Gaussian law for ϕ-mixing sequences, again without any moment or rate conditions.

In his first papers on limit theorems for weakly dependent processes, culminating his 1975 AMS memoir with William Stout, Philipp solved the central limit problem (characterizing the limit distributions of arrays with corresponding criteria for convergence to specific limits) in the case of bounded variances, proved Berry–Esseen bounds for the speed of convergence in the CLT and obtained laws of the iterated logarithm under various weak dependence conditions. In addition to the mixing conditions already mentioned, he studied the ψ-mixing coefficients defined by

$$\psi(k) := \sup_{n\geq 1} \sup_{A\in\mathcal{F}_1^n, B\in\mathcal{F}_{n+k}^\infty} \frac{|P(A\cap B)-P(A)P(B)|}{P(A)P(B)},$$

and ψ-mixing processes. The notion of ψ-mixing is stronger than any of the other mixing conditions and ψ-mixing processes satisfy most of the classical i.i.d. limit theorems. The digits of a random number $\omega\in(0,1]$ in the continued fraction expansion form a ψ-mixing process. A second weak dependence condition investigated by Walter Philipp is a correlation condition for mixed products, requiring that for all integers $0\leq i_1<\ldots<i_r$, $1\leq j\leq r$, $p_\nu\geq 0$

$$\left|E\left(X_{i_1}^{p_1}\cdot\ldots\cdot X_{i_r}^{p_r}\right)-E\left(X_{i_1}^{p_1}\cdot\ldots\cdot X_{i_j}^{p_j}\right)E\left(X_{i_{j+1}}^{p_{j+1}}\cdot\ldots\cdot X_{i_r}^{p_r}\right)\right| \\ \leq L(j)c(i_{j+1}-i_j)\sup_{1\leq i\leq i_r} E|X_i|^{\sum p_\nu}.$$

The standard method to prove limit theorems for mixing processes, employed in the pioneering works of Rosenblatt (1956) and Ibragimov (1962), was the Bernstein blocking technique, giving an approximation of the characteristic function of partial sums of mixing sequences by the characteristic function of sums of independent random variables via suitable correlation inequalities. This method leads to sharp results in the case of the CLT and LIL, but its applicability beyond them is rather limited: for example, upper-lower class refinements of the LIL require delicate tail estimates for the considered r.v.'s which are beyond the scope of the method. In his 1975 AMS Memoir with William Stout, Walter Philipp showed that sufficiently separated block sums of weakly dependent sequences are, after suitable centering, close to a martingale difference sequence, and thus using Skorohod embedding and Strassen's strong approximation technique, the partial sums of such sequences can be closely approximated by a Wiener process. This observation not only opens the way to prove a vast class of refined asymptotic results for mixing sequences, but the near martingale property can be easily verified for several other types of weak dependent processes for which the previous theory does not work, or leads to great difficulties: Markov processes, retarded mixing sequences, Gaussian processes, lacunary series, etc. Note that a different type of martingale approximation was used earlier by Gordin (1969) in case of stationary sequences; the two methods complement each other and lead to sharp asymptotic results in many important situations. In his 1979 joint paper with Berkes, Philipp made a further important step in the study of weak dependent behavior, showing that block sums of weakly dependent processes can be directly approximated by independent random variables, via the Strassen–Dudley existence theorem. This observation frees the investigations from moment conditions and works not only for real valued random variables, but for random variables taking values in abstract spaces. In the context of Banach space valued random variables, this method yields new results even for i.i.d. random variables, as a 1980 joint paper of Philipp and Kuelbs shows. This paper was the first in a long series of papers of Walter Philipp dealing with limit theorems of independent and weakly dependent B-valued random variables and Hilbert space valued martingales. The infinite dimensional setup also opens the way to study unform Glivenko–Cantelli type results and uniform limit theorems for random variables indexed by sets, a popular and much studied topic in the 1970's and 1980's. Walter Philipp's contribution in this field is very substantial; see e.g. his profound joint paper with Dudley (1983). In his last papers, Walter Philipp returned again to this topic, showing that metric entropy can be used to provide deep information on pseudorandom behavior and in the theory of uniform distribution. This completes a long circle in Philipp's mathematical work and at the same time opens a new direction in the study of weakly dependent behavior.

The importance of Walter Philipp's contributions to the asymptotic theory for weakly dependent processes can only be appreciated in the light of the many applications to problems in analysis and number theory. Some early applications are given in two papers entitled *Some metrical theorems in number theory* which are entirely devoted to such applications. In these papers Walter Philipp investigated the distribution of the sequence $(T^n\omega)_{n\geq 1}$ for the maps $T:(0,1]\to(0,1]$ associated with the θ-adic expansion and the continued fraction expansion. If $(I_n)_{n\geq 1}$ is a sequence of intervals, $I_n\subset(0,1]$, one can study the quantity

$$A(N,\omega):=\sum_{n=1}^{N}1_{\{T^n\omega\in I_n\}}.$$

If μ denotes the invariant measure associated with T, then the expected value of $A(N,\omega)$ becomes

$$\phi(N)=\sum_{n=1}^{N}\mu(I_n).$$

If all the intervals are identical, i.e. $I_n=I$, we obtain from the ergodic theorem that $A(N,\omega)=\phi(N)+o(N)$ a.s. Walter Philipp sharpened this result considerably by showing that for any $\varepsilon>0$,

$$A(N,\omega)=\phi(N)+O(\phi^{1/2}(N)\log^{3/2+\varepsilon}\phi(N)), \tag{3}$$

for almost all $\omega\in(0,1]$. While the accuracy of this approximation is limited by the second order method used in the proof, shortly thereafter Philipp went much further: he observed that the considered sequences $(T^n\omega)_{n\geq 1}$ are ϕ-mixing with exponential rate and thus using blocking techniques and applying some of his asymptotic theorems obtained earlier, one can prove a whole series of highly attracting limit theorems for the digits in various expansions and Diophantine approximation. These not only improve several earlier results in the literature, but actually provide the precise asymptotics in a number of important questions of metric number theory. Let us formulate a few such results here. Let $x=[a_1(x),a_2(x),\ldots]$ be the continued fraction expansion of $x\in(0,1)$ and let $\varphi(n)\to\infty$ be a sequence of integers with $\sum\frac{1}{\varphi(n)}=\infty$.

Denote by $A(N,x)$ the number of integers $n \le N$ with $a_n(x) \ge \varphi(n)$ and put

$$\phi(N) = \frac{1}{\log 2} \sum_{n \le N} \log\left(1 + \frac{1}{\varphi(n)}\right).$$

Then

$$\lambda\left\{x : \frac{A(N,x) - \phi(N)}{\sqrt{\phi(N)}} < z\right\} \to \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} \exp(-t^2/2)dt$$

and for almost all x

$$\limsup_{N\to\infty} \frac{|A(N,x) - \phi(N)|}{\sqrt{2\phi(N)\log\log\phi(N)}} = 1.$$

Also, letting $L_N(x) = \max_{1\le k\le N} a_n(x)$, Philipp proved

$$\liminf_{N\to\infty} \frac{\log\log N}{N} L_N(x) = \frac{1}{\log 2}$$

for almost all x, verifying an old conjecture of Erdős. Further, let f be a continuous, positive, nonincreasing function on $\mathbb{R}^+$ such that

$$\phi(n) = 2\sum_{k\le n} \frac{f(k)}{k} \to \infty$$

and let $N_{\alpha,f}(n)$ denote the number of solutions (p,q) of (1) in integers $q \le n$ and p. Then under mild additional regularity conditions on f, Walter Philipp proved

$$\lambda\left\{\alpha : \frac{N_{\alpha,f}(n) - \phi(n)}{\sqrt{\phi(n)}} < z\right\} \to \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} \exp(-t^2/2)dt$$

and for almost all α

$$\limsup_{n\to\infty} \frac{|N_{\alpha,f}(n) - \phi(n)|}{\sqrt{2\phi(n)\log\log\phi(n)}} = 1.$$

A further much studied connection between probability theory and number theory is the distribution of values of additive functions. Walter Philipp's contribution in his field can be found in his 1971 AMS Memoir written with the ambitious goal to unify probabilistic number theory and to deduce at least the most typical results as special cases of limit theorems for mixing random variables. Because of the very different type of weak dependence conditions in number theory, there is little hope that all applications of probability to number theory could be put in a general framework. But at least for applications in Diophantine approximation, continued fraction and related expansions, discrepancies and the distribution of additive functions, Walter Philipp succeeded in this program to a remarkable degree. He also studied weak convergence of additive function paths to Brownian motion, extending earlier results of Billingsley.

The basic motivation for the introduction of mixing conditions was to understand the asymptotic properties of weakly dependent structures in stochastics and driven by its intrinsic needs, the theory made a tremendous progress starting from the 60's and by now it is a closed, complete and beautiful theory, giving a nearly complete answer for the basic asymptotic questions connected with mixing structures. For a comprehensive treatise of the theory see the recent monograph of R. Bradley (2007).

While questions on weak dependence kept Walter Philipp's attention in his whole career, this did not prevent him from making fundamental contributions in other areas of probability theory, e.g. in the classical theory of independent random variables. In a short paper with M. Lacey in 1990 he proved that if $(X_n)_{n\geq 1}$ is a sequence of i.i.d. random variables with mean 0 and variance 1 then letting $S_n = \sum_{k=1}^n X_k$ we have

$$\lim_{N\to\infty} \frac{1}{\log N} \sum_{k=1}^{N} \frac{1}{k} I\left\{ \frac{S_k}{\sqrt{k}} \leq x \right\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2} dt$$

with probability 1 for all $x \in \mathbb{R}$. This remarkable 'pathwise' form of the central limit theorem was already stated (without proof and without specifying conditions) by Lévy in 1937 and was proved independently by Brosamler (1988), Fisher (1987) and Schatte (1988) under the assumption of higher moments. Due to these papers, almost sure central limit theory became extremely popular overnight and has not lost its attraction until today. The paper of Philipp and Lacey not only yields the final, optimal form of this theorem, but the method they used became the basic method in this field.

In a series of three papers, published in the mid 1980s jointly with Dehling and Denker, Walter Philipp investigated the asymptotic behavior of degenerate U-statistics. Given a symmetric function $h : \mathbb{R}^m \to \mathbb{R}$ and an i.i.d. process $(X_n)_{n\geq 1}$, the m-variate U-statistic with kernel h is defined as

$$U_n(h) = \sum_{1\leq i_1<\ldots<i_m\leq n} h(X_{i_1}, \ldots, X_{i_m}).$$

The kernel is called degenerate if $E(h(X, x_2, \ldots, x_m)) = 0$ for almost all $x_2, \ldots, x_m$. Dehling, Denker and Philipp proved a strong approximation of $U_n(h)$ by m-fold Wiener–Itô integrals

$$I_n(h) = \int \ldots \int h(x_1, \ldots, x_m)\, dW(x_1) \ldots dW(x_m),$$

where $W(t)$ is a mean-zero Gaussian process. The results of this research enabled Dehling in a subsequent paper to establish the functional law of the iterated logarithm for degenerate U-statistics and for multiple Wiener–Ito integrals. In the course of their work, Dehling, Denker and Philipp also investigated the empirical U-process, defined by

$$\sqrt{n} \frac{1}{\binom{n}{m}} \big(\#\{1 \leq i_1 < \ldots < i_m \leq n : h(X_{i_1}, \ldots, X_{i_m}) \leq t\} - P(h(X_1, \ldots, X_m) \leq t) \big)_{t\in\mathbb{R}},$$

and established an almost sure invariance principle for this process.

The investigations on the asymptotic behavior of degenerate U-statistics lead directly to questions concerning the asymptotic behavior of certain Hilbert space valued martingales. For the U-statistic applications, a bounded law of the iterated logarithm was sufficient. In later work, carried out jointly with Monrad, Walter Philipp established a Skorohod embedding of Hilbert space valued martingales.

Another favorite topic of Walter Philipp's research was lacunary series: he investigated such series already in his early papers in the 1960's and in his very last papers in 2006 he returned once more to this topic. By Weyl's (1916) theorem quoted above, given any sequence $(n_k)_{k\geq 1}$ of positive numbers with $n_{k+1} - n_k \geq \delta > 0$ $(k = 1,2,\ldots)$, the sequence $\{n_k\omega\}$ is uniformly distributed mod 1 for almost all ω. In contrast to the simplicity of this result, proving sharp bounds for the discrepancy of $\{n_k\omega\}$ is very difficult, and the only precise results known before 1970 were the results of Khinchin (1924) and Kesten's (1964) for the case $n_k = k$. Kesten's result states that the discrepancy $D_N(\omega)$ of the sequence $\{k\omega\}$ satisfies

$$\lim_{N\to\infty} \frac{\log N \log\log N}{N} D_N(\omega) = \frac{2}{\pi^2} \qquad \text{in probability.}$$

In 1975 Walter Philipp proved that if $(n_k)_{k\geq 1}$ is a sequence of integers satisfying the Hadamard gap condition $n_{k+1}/n_k \geq q > 1$ $(k = 1,2,\ldots)$, then the discrepancy $D_N(\omega)$ of the sequence $\{n_k\omega\}$ satisfies the law of the iterated logarithm, i.e.

$$\frac{1}{4\sqrt{2}} \leq \limsup_{N\to\infty} \sqrt{\frac{N}{\log\log N}} D_N(\omega) \leq C \tag{4}$$

for almost all ω, where $C = C(q) = 166 + 664(q^{1/2} - 1)^{-1}$. This remarkable result verified a long standing conjecture of Erdős and Gál, and showed that, as far as its discrepancies are concerned, $\{n_k\omega\}$ behaves like a sequence of independent random variables. For the partial sums of $\sin n_k x$ such phenomena have already been observed by Salem and Zygmund in 1947, but the discrepancy situation is much more delicate: as in a later paper Philipp (1994) showed, for a suitable sequence $(n_k)_{k\geq 1}$ the limsup in (4) is greater than $C \log\log\frac{1}{q}$ with an absolute constant C and thus for q close to 1 the limsup can be as large as we wish. Very recently, Fukuyama (2008) succeeded in computing the limsup for the sequences $n_k = \theta^k$, $\theta > 1$.

For sequences $(n_k)_{k\geq 1}$ growing slower than exponentially, the LIL (4) is generally false, and the behavior of the discrepancy of $\{n_k\omega\}$ becomes very complicated. R. C. Baker (1981) proved that for any n_k, $ND_N(\omega) = O((\log N)^{3/2+\varepsilon})$ for almost all ω and Philipp and Berkes (1994) showed that the constant $3/2$ here cannot be replaced by any number less than 1/2. Despite these fairly precise results on the extremal behavior of discrepancies, the exact order of magnitude of $D_N(\omega)$ for "concrete" n_k remains open. In one of his last papers, written jointly with Berkes and Tichy, Walter Philipp made a substantial step in clearing up this phenomenon as well: he showed that the asymptotic behavior of the discrepancy of $\{n_k\omega\}$ is intimately connected with the number of solutions of Diophantine equations of the type $a_1 n_{k_1} + \cdots + a_p n_{k_p} = b$. The discovery of this remarkable arithmetical connection is again a characteristic achievement of Walter Philipp, linking probabilistic phenomena with asymptotic results in analysis and number theory.

In conclusion we mention an interesting result of Philipp on the extremal behavior of exponential sums. From the Carleson convergence theorem for Fourier series in L^2 it follows that if f is a nondecreasing positive function on $\mathbb{R}^+$ satisfying

$$\sum_{k=1}^{\infty} \frac{1}{k f(k)^2} < \infty \tag{5}$$

then for any increasing sequence $(n_k)_{k \geq 1}$ of positive integers we have

$$\left| \sum_{k=1}^{N} e^{2\pi i n_k x} \right| = O(N^{1/2} f(N)) \qquad \text{a.e.} \tag{6}$$

In particular, we have for any $(n_k)_{k \geq 1}$

$$\left| \sum_{k=1}^{N} e^{2\pi i n_k x} \right| = O(N^{1/2} (\log N)^{1/2+\varepsilon}) \qquad \text{a.e.}$$

for any $\varepsilon > 0$. As early as 1930, Walfisz proved that for $n_k = k^2$ the left hand side of (6) exceeds $N^{1/2}(\log N)^{1/4}$ for infinitely many N, but this does not reach Carleson's upper bound. Walter Philipp proved that if f is a nondecreasing positive function on $\mathbb{R}^+$ satisfying mild regularity conditions such that the sum in (5) diverges, then the exponential sum in (6) exceeds $N^{1/2} f(N)$ a.e. for infinitely many N. This provides a complete solution of the problem of extremal speed of exponential sums and provides yet another example for the power of weak dependence techniques in problems of classical analysis.

Physics

In the last years of his life, Walter Philipp became interested in certain problems concerning the foundation of physics. Nearly 100 years after the birth of quantum mechanics, the problem of existence of "hidden parameters" in the theory (a question first investigated in depth by John von Neumann in 1932) is still not settled, due to unsatisfactory probabilistic models traditionally used to disprove the existence of such parameters. In a series of papers written jointly with Karl Hess, Walter Philipp provided refreshing new ideas in this field, inevitably causing great controversy in physics circles. It is a great loss to science that Walter Philipp's death in 2006 put an end to these investigations, leaving the solution of this important problem to the future.

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