RATIONAL INTEGERS AS SUMS OF UNITS – THE QUADRATIC CASE

CHRISTOPHER FREI, MARTIN WIDMER, AND VOLKER ZIEGLER

ABSTRACT. How many natural numbers below X can be written as a sum of k units of the ring of integers of a given number field? We give the asymptotics as X gets large for quadratic number fields. This solves a problem of Jarden and Narkiewicz from 2007 for quadratic number fields.

1. Introduction

Jarden and Narkiewicz proved that if L is a number field then there is no natural number k such that every element of the ring of integers $\mathcal{O}_L$ is a sum of at most k units of $\mathcal{O}_L$. More precisely they proved [5, Corollary 6] that the rational integers n that are sums of at most k units have density zero. Their proof is short and elegant, based on van der Warden's theorem and a classcial finiteness result concerning unit equations, but does not shed any light on the asymptotics of the counting function. They proposed the following problem [5, Problem C].

Problem 1 (Jarden and Narkiewicz, 2007)**.** *Let L be a number field. Obtain an asymptotical formula for the number of positive rational integers $n \leq X$ which are sums of at most k units of $\mathcal{O}_L$.*

So far this problem has not been addressed in the literature. In this article we solve Problem 1 for quadratic number fields.

For imaginary quadratic fields all units are roots of unity. Hence, no natural number $n > k$ is a sum of at most k units, whereas clearly all other n are. So let us fix a *real* quadratic number field $L = \mathbb{Q}(\sqrt{d})$ with $d \geq 2$ and squarefree. For $\mathbf{w} = (w_1, \ldots, w_r) \in L^r$, we write

$$S_{\mathbf{w}} := w_1 + \cdots + w_r. \tag{1}$$

Throughout this paper, we let $X \geq 2$ and $k \in \mathbb{N} = \{1, 2, 3, \ldots\}$. We are interested in the set[1]

$$\mathrm{N}_{L,k} := \{n \in \mathbb{Z} \ : \ n = S_{\mathbf{u}} \text{ for some } \mathbf{u} \in (\mathcal{O}_L^\times)^r \text{ with } 0 \leq r \leq k\}$$

and its counting function

$$\mathrm{N}_{L,k}(X) := \#\{n \in \mathrm{N}_{L,k} : |n| \leq X\}. \tag{2}$$

Non-zero integers n in $\mathrm{N}_{L,k}$ come in pairs $n, -n$. Hence, $\mathrm{N}_{L,k}(X) - 1$ is twice the number of positive rational integers $n \leq X$ which are sums of at most k units of $\mathcal{O}_L$.

Our main result is the following.

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[1] We interpret $(\mathcal{O}_L^\times)^0$ as containing only the empty tuple $\mathbf{u}$, and $S_{\mathbf{u}} = 0$.

Theorem 1. *Let $\eta > 1$ be the fundamental unit of the real quadratic field L, let $k \in \mathbb{N}$ and define $\rho := \lfloor k/2 \rfloor$. Then, for $X \geq 2$,*

$$\mathrm{N}_{L,k}(X) = c_k \left(\frac{2\log X}{\log \eta}\right)^{\rho} + O_{k,L}((\log X)^{\rho-1}),$$

where

$$c_k = \begin{cases} 1/\rho! & \text{if } k \text{ is even,} \\ 3/\rho! & \text{if } k \text{ is odd.} \end{cases}$$

Although only very few rational integers n are sums of at most k units in a fixed real quadratc field L, every rational integer n is the sum of two units in *some* real quadratic field, e.g., for $|n| > 2$ we can take the sum of conjugate units $\frac{n \pm \sqrt{n^2-4}}{2}$.

Restricting to sums of *exactly* k units we define

$$\tilde{\mathrm{N}}_{L,k} := \{n \in \mathbb{Z} \ : \ n = S_{\mathbf{u}} \text{ for some } \mathbf{u} \in (\mathcal{O}_L^{\times})^k\},$$

and its counting function

$$\tilde{\mathrm{N}}_{L,k}(X) := \#\{n \in \tilde{\mathrm{N}}_{L,k} : |n| \leq X\}. \tag{3}$$

That is, $\tilde{\mathrm{N}}_{L,k}(X)$ is the number of integers n with $|n| \leq X$ that can be written as the sum of exactly k units. The following result is an immediate consequence of Theorem 1.

Corollary 1. *Let $\eta > 1$ be the fundamental unit of the real quadratic field L, let $k \in \mathbb{N}$ and define $\rho := \lfloor k/2 \rfloor$. Then, for $X \geq 2$,*

$$\tilde{\mathrm{N}}_{L,k}(X) = \tilde{c}_k \left(\frac{2\log X}{\log \eta}\right)^{\rho} + O_{k,L}((\log X)^{\rho-1}),$$

where

$$\tilde{c}_k = \begin{cases} 1/\rho! & \text{if } k \text{ is even,} \\ 2/\rho! & \text{if } k \text{ is odd.} \end{cases}$$

Other aspects of the sets $\tilde{\mathrm{N}}_{L,k}$ and $\mathrm{N}_{L,k}$, at least for $k = 2$, have been studied before. Nagell asked for which number fields L the number 1 is contained in $\tilde{\mathrm{N}}_{L,2}$. He called such number fields L exceptional. Nagell's considerations culminated in [8], where he classified all exceptional number fields L of unit rank ≤ 1. More recently Freitas–Kraus–Siksek [3] have shown that, for any given prime $p \geq 5$, there are only finitely many cyclic degree p fields L that are exceptional.

For cyclotomic fields $L = \mathbb{Q}(\zeta_p)$, Newman [9, p. 89] observed that $1, 2$ and 3 are all contained in $\tilde{\mathrm{N}}_{L,2}$ for all primes $p > 3$, and he posed the problem to explicitly determine $\mathrm{N}_{L,2}$ for cyclotomic number fields L.

Recently, Tinkova et.al. [11] considered the problem to completely determine the sets $\tilde{\mathrm{N}}_{L,2}$ for cubic fields L. They resolved the problem for all cubic fields L which are either cyclic or imaginary. Moreover, they showed that for number fields L that do not contain a real quadratic field the sets $\mathrm{N}_{L,2}$ are finite.

Moreover, quantities of similar spirit as $\mathrm{N}_{L,k}(X)$ were studied in [4, 2].

There has been much activity recently and in the past regarding statistics for the number of fibres admitting rational or integral solutions in certain families of

Diophantine equations. For an overview and references, we refer to the introduction of [7].

Corollary 1 can be interpreted in this vein, as counting asymptotically the number of fibres admitting integral points in a certain natural family of schemes parameterised by integers.

More precisely, let $A := \mathcal{O}_L[t, x_0, \ldots, x_{k-1}]/(g)$, where

$$g = x_0 \cdots x_{k-1}(t - x_1 - \cdots - x_{k-1}) - 1$$

and $\mathcal{X}' := \operatorname{Spec}(A)$. The inclusion of $\mathcal{O}_L[t]$ in A induces a morphism $\mathcal{X}' \to \mathbb{A}^1_{\mathcal{O}_L}$ of schemes. The Weil restriction (see [6])

$$\mathcal{X} := \mathrm{R}_{\mathbb{A}^1_{\mathcal{O}_L}/\mathbb{A}^1_{\mathbb{Z}}}(\mathcal{X}')$$

comes with a morphism $\mathcal{X} \to \mathbb{A}^1_{\mathbb{Z}}$. For every $n \in \mathbb{Z}$, we consider the pull-back $\mathcal{X}_n := \mathcal{X} \times_{\mathbb{A}^1_{\mathbb{Z}}} \operatorname{Spec}(\mathbb{Z})$ along the integral point $\operatorname{Spec}(\mathbb{Z}) \to \mathbb{A}^1_{\mathbb{Z}}$ induced by $\mathbb{Z}[t] \to \mathbb{Z}$, $t \to n$, which we call the *fibre over* n. Using standard properties of the Weil restriction, we see that

$$\mathcal{X}_n \cong \mathrm{R}_{\operatorname{Spec}(\mathcal{O}_L)/\operatorname{Spec}(\mathbb{Z})}(\mathcal{X}' \times_{\mathbb{A}^1_{\mathcal{O}_L}} \operatorname{Spec}(\mathcal{O}_L)) = \mathrm{R}_{\operatorname{Spec}(\mathcal{O}_L)/\operatorname{Spec}(\mathbb{Z})}(\operatorname{Spec}(A_n)),$$

where $A_n = A \otimes_{\mathcal{O}_L[t]} \mathcal{O}_L = \mathcal{O}_L[x_0, \ldots, x_{k-1}]/(g_n)$ with

$$g_n = x_0 \cdots x_{k-1}(n - x_1 - \cdots - x_{k-1}) - 1.$$

Hence, for every ring B, the set of B-points of $\mathcal{X}_n$,

$$\mathcal{X}_n(B) = \operatorname{Spec}(A_n)(B \otimes_{\mathbb{Z}} \mathcal{O}_L),$$

is in one-to-one correspondence with the set of solutions of the unit equation

$$u_1 + \cdots + u_k = n \tag{4}$$

with $u_i \in (B \otimes_{\mathbb{Z}} \mathcal{O}_L)^\times$. Therefore, the function $\tilde{\mathrm{N}}_{L,k}(X)$ studied in Corollary 1 counts the set of integers $n \in \mathbb{Z} \cap [-X, X]$ for which the fibre $\mathcal{X}_n$ of $\mathcal{X}$ above n has integral points, i.e. $\mathcal{X}_n(\mathbb{Z}) \neq \emptyset$.

Regarding local solubility, when $k \geq 2$ it is clear that $\mathcal{X}_n(\mathbb{R}) \neq \emptyset$, and straightforward to see that $\mathcal{X}_n(\mathbb{Z}_p) \neq \emptyset$ whenever $p \neq 2$. Moreover, $\mathcal{X}_n(\mathbb{Z}_2) \neq \emptyset$ if and only if $n \equiv k \bmod 2$ or 2 is inert in L. Hence, in contrast to the global situation described in Corollary 1, a positive proportion of the fibres $\mathcal{X}_n$ have points over $\mathbb{R}$ and all $\mathbb{Z}_p$. We give a detailed proof of the local solubility in Section 6.

We end this introduction with a brief overview over the remaining sections. In Section 2 we show that if n is a sum of at most k units, then n is a sum of traces of units and some summands from a finite set depending only on k and L. Hence, we must count vectors $\mathbf{u}$ with $\ell \leq \rho$ components of units whose trace sums have no vanishing subsums and are bounded in modulus by X. This is achieved in Section 3. Different unit vectors can lead to the same integer n by permuting the coordinates of the vector, but also in more subtle ways. Counting these clashes is the purpose of Section 4. In Section 5 we are ready to prove Theorem 1. The final Section 6 is devoted to the proof of the claims about local solubility of (4).

2. Reduction to unit trace sums

For $\mathbf{u} \in L^r$, we say that the sum $S_{\mathbf{u}} = u_1 + \cdots + u_r$ has *no vanishing subsum*, if

$$\sum_{i\in I} u_i \neq 0 \quad \text{for all} \quad \emptyset \neq I \subseteq \{1,\dots,r\},$$

and *no vanishing proper subsum*, if

$$\sum_{i\in I} u_i \neq 0 \quad \text{for all} \quad \emptyset \neq I \subsetneq \{1,\dots,r\}.$$

Moreover, we let $\mathbf{u}'$ denote its conjugate. I.e., $\mathbf{u}' := (u_1', \dots, u_r')$, where $u_i' = \sigma(u_i)$, with $\sigma : L \to L$ the non-trivial $\mathbb{Q}$-automorphism.

At several places we will use the well-known fact that for any $T \in \mathbb{N}$ the unit equation

$$v_1 + \cdots + v_T = 1, \tag{5}$$

has only finitely many *non-degenerate* solutions $\mathbf{v} \in (\mathcal{O}_L^\times)^T$, i.e. solutions in which no subsum of the left-hand side vanishes (e.g. [10, Theorem 2A in Chapter V]). We denote the set of these solutions, depending only on L and T, by $\mathcal{S}_T$.

While for general number fields this is a deep fact (based on the subspace theorem), proved by Evertse [1] as well as van der Poorten and Schlickewei [12], we only need the case when L is real quadratic, where it is a consequence of the simple Lemma 2.

Proposition 1. *There is a chain of finite subsets $\mathcal{U}_0 \subseteq \mathcal{U}_1 \subseteq \cdots$ of $\mathcal{O}_L^\times$, such that the following holds true.*

If $r \in \mathbb{N}$, $n \in \mathbb{Z} \smallsetminus \{0\}$ and $\mathbf{u} \in (\mathcal{O}_L^\times)^r$ such that $n = S_{\mathbf{u}}$ with no vanishing subsum, then we also have $n = S_{(\mathbf{v},\mathbf{v}',\boldsymbol{\xi})}$, where

$$\mathbf{v} \in (\mathcal{O}_L^\times)^\ell \quad \textit{and} \quad \boldsymbol{\xi} \in \mathcal{U}_s^s, \qquad \textit{with} \qquad \ell, s \in \mathbb{N}_0 \quad \textit{satisfying} \quad 2\ell + s \leq r.$$

Remark 1.

(1) We interpret $\boldsymbol{\xi} \in \mathcal{U}_0^0$ as the empty tuple, so that $(\mathbf{v},\mathbf{v}',\boldsymbol{\xi}) = (\mathbf{v},\mathbf{v}')$.
(2) If $\boldsymbol{\xi} \in (\mathcal{O}_L^\times)^1$ and $S_{(\mathbf{v},\mathbf{v}',\boldsymbol{\xi})} \in \mathbb{Z}$, then $\boldsymbol{\xi} \in \{\pm 1\}$.

Corollary 2. *In the conclusion of Proposition 1, we may additionally require that the sum $S_{(\mathbf{v},\mathbf{v}',\boldsymbol{\xi})}$ has no vanishing subsum.*

Proof. If $n = S_{(\mathbf{v},\mathbf{v}',\boldsymbol{\xi})}$ has a vanishing subsum, then the remaining summands form a tuple $\tilde{\mathbf{u}} \in (\mathcal{O}_L^\times)^{\tilde{r}}$ with $\tilde{r} \leq r-2$ and $n = S_{\tilde{\mathbf{u}}}$. We apply Proposition 1 to $\tilde{r}$ and $\tilde{\mathbf{u}}$ in place of r and $\mathbf{u}$, which yields a representation $n = S_{(\mathbf{v}_1,\mathbf{v}_1',\boldsymbol{\xi}_1)}$. If the latter has no vanishing subsums, we are done. Otherwise, repeat the above, leading to an even shorter representation of the form $n = S_{(\mathbf{v}_2,\mathbf{v}_2',\boldsymbol{\xi}_2)}$. This process has to stop with a tuple $(\mathbf{v}_s,\mathbf{v}_s',\boldsymbol{\xi}_s)$ with at most r coordinates, such that $n = S_{(\mathbf{v}_s,\mathbf{v}_s',\boldsymbol{\xi}_s)}$ has no vanishing subsums. □

2.1. Proof of Proposition 1.

Lemma 1. *Let $u_1, u_2 \in \mathcal{O}_L^\times$ with $u_1 + u_2 \in \mathbb{Z}$. Then one of the following three situations holds:*

(1) $u_2 = u_1'$,
(2) $u_2 = -u_1$,
(3) $\{u_1,u_2\} \in \left\{\left\{\frac{3\epsilon_1+\sqrt{5}}{2}, \frac{\epsilon_2-\sqrt{5}}{2}\right\}, \left\{\frac{3\epsilon_1-\sqrt{5}}{2}, \frac{\epsilon_2+\sqrt{5}}{2}\right\} : \epsilon_1,\epsilon_2 \in \{\pm 1\}\right\}$.

Proof. As $u_1, u_2 \in \mathcal{O}_L$ with $u_1 + u_2 \in \mathbb{Z}$, we can write

$$u_1 = \frac{a_1 + b\sqrt{d}}{2}, \quad u_2 = \frac{a_2 - b\sqrt{d}}{2}$$

with $a_1, a_2, b \in \mathbb{Z}$. As u_1, u_2 are units, we have $u_1 u_1' = \pm 1$ and $u_2 u_2' = \pm 1$.

If $u_1 u_1' = u_2 u_2'$, then $a_1^2 - db^2 = a_2^2 - db^2$ and thus $a_2 = \pm a_1$, yielding situation *(1)* or *(2)*.

Now suppose that $u_1 u_1' = -u_2 u_2'$, and without loss of generality $u_1 u_1' = 1$. Then $a_1^2 - db^2 = 4$ and $a_2^2 - db^2 = -4$, which implies that $a_1^2 - a_2^2 = 8$, and thus $(a_1, a_2) = (\pm 3, \pm 1)$.

This gives $9 - db^2 = 4$, and thus $d = 5, b = \pm 1$. Hence, we are in situation *(3)*. □

We construct the sets $\mathcal{U}_0 \subseteq \mathcal{U}_1 \subseteq \cdots$ in Proposition 1 as follows. Take $\mathcal{U}_0 = \mathcal{U}_1 := \{\pm 1\}$ and $\mathcal{U}_2$ to consist of ± 1 and possibly the elements appearing in case *(3)* of Lemma 1.

Now let $t \geq 3$ and assume that we have already constructed the sets $\mathcal{U}_0 \subseteq \cdots \subseteq \mathcal{U}_{t-1}$. For every non-degenerate solution $(v_1, \ldots, v_{2t-1}) \in \mathcal{S}_{2t-1}$ of the unit equation (5) with $T = 2t-1$, write $\mathbf{v} := (v_1, \ldots, v_t)$. For each of the at most $\#\mathcal{S}_{2t-1}$ choices of $\mathbf{v}$, there are at most two values of $u \in \mathcal{O}_L^\times$ with $uS_{\mathbf{v}} \in \mathbb{Z} \smallsetminus \{0\}$, and we take $\mathcal{U}_t$ to be the union of $\mathcal{U}_{t-1}$ with all coordinates uv_i of all tuples $\mathbf{u} := u\mathbf{v}$ as above.

Having described the sets $\mathcal{U}_t$, we now prove Proposition 1 by induction on r. For $r = 1$, the conclusion holds trivially. For $r = 2$, it follows from Lemma 1.

Hence, let $r \geq 3$ and assume that the proposition's conclusion holds for all sums of less than r terms.

From $n = S_{\mathbf{u}}$, we see that also $n = S_{\mathbf{u}'}$ and thus

$$0 = n - n = u_1 + \cdots + u_r - u_1' - \cdots - u_r'.$$

Hence, there are subsets $I, J \subseteq \{1, \ldots, r\}$, such that

$$\sum_{i \in I} u_i - \sum_{j \in J} u_j' \tag{6}$$

is a *minimal* vanishing subsum, i.e. no proper subsum vanishes. As $n = S_{\mathbf{u}}$ has no vanishing subsums, we conclude that $I, J \neq \emptyset$. Moreover, we may assume without loss of generality that $|I| \geq |J|$, as conjugating and multiplying by -1 allow us to exchange the roles of I and J. We observe that then

$$n = \sum_{i=1}^{r} u_i = \sum_{i=1}^{r} u_i - \left(\sum_{i \in I} u_i - \sum_{j \in J} u_j' \right) = \sum_{i \in I^c} u_i + \sum_{j \in J} u_j', \tag{7}$$

where $I^c = \{1, \ldots, r\} \smallsetminus I$. We now distinguish between four different cases.

Case 1: $|I| > |J|$. As the sum on the right-hand side of (7) has $|I^c| + |J| = r - |I| + |J| < r$ terms, we find a minimal subsum with $q < r$ terms which equals n. As the subsum is minimal, it has no vanishing subsums. Hence, the induction hypothesis yields a representation $n = S_{(\mathbf{v}, \mathbf{v}', \boldsymbol{\xi})}$ with $\mathbf{v} \in (\mathcal{O}_L^\times)^\ell$ and $\boldsymbol{\xi} \in \mathcal{U}_s^s$, such that $2\ell + s \leq q < r$. This is enough for the proposition's conclusion to hold.

Case 2: $I^c \cap J \neq \emptyset$. Let $j_0 \in I^c \cap J$ and $m := u_{j_0} + u'_{j_0} \in \mathbb{Z}$. Then by (7) we get the representation

$$n - m = \sum_{i \in (I \cup \{j_0\})^c} u_i + \sum_{J \smallsetminus \{j_0\}} u'_j.$$

There is a minimal subsum of the right-hand side that equals $n - m$, with $q \leq r - (|I| + 1) + |J| - 1 \leq r - 2$ terms. Again, the induction hypothesis yields a representation $n - m = S_{(\mathbf{v}_1, \mathbf{v}'_1, \boldsymbol{\xi})}$ with $\mathbf{v}_1 \in (\mathcal{O}_L^\times)^{\ell_1}$ and $\boldsymbol{\xi}_1 \in \mathcal{U}_s^s$, such that $2\ell_1 + s \leq q \leq r - 2$.

Then we may take $\ell := \ell_1 + 1$ and $\mathbf{v} := (\mathbf{v}_1, u_{j_0}) \in (\mathcal{O}_L^\times)^\ell$, giving $2\ell + s \leq r$ and $n = S_{(\mathbf{v}, \mathbf{v}', \boldsymbol{\xi})}$ as desired.

Case 3: $I = J = \{1, \ldots, r\}$. In this case, the sum $u_1 + \cdots + u_r - u'_1 - \cdots - u'_r = 0$ has no vanishing proper subsums, and hence so does the sum

$$v_1 + \cdots + v_{2r-1} = 1, \tag{8}$$

where

$$v_i := \frac{u_i}{u'_r} \; (1 \leq i \leq r), \quad v_i := \frac{-u'_{i-r}}{u'_r} \; (r + 1 \leq i \leq 2r - 1).$$

Hence, $(v_1, \ldots, v_{2r-1}) \in \mathcal{S}_{2r-1}$. Writing $\mathbf{v} := (v_1, \ldots, v_r)$, then $\mathbf{u} = u'_r \mathbf{v}$, which implies that $u'_r S_{\mathbf{v}} = S_{\mathbf{u}} = n \in \mathbb{Z} \smallsetminus \{0\}$. By construction of $\mathcal{U}_r$, this implies that all coordinates of $\mathbf{u} = u'_r \mathbf{v}$ are in $\mathcal{U}_r$. Hence, the proposition's conclusion is satisfied with $\ell = 0$ and $\boldsymbol{\xi} = \mathbf{u} \in \mathcal{U}_r^r$.

Case 4: $I = J \subsetneq \{1, \ldots, r\}$. In this case, we see from (6) that

$$\sum_{i \in I} u_i = \left(\sum_{u \in I} u_i \right)'$$

and thus $m := \sum_{i \in I} u_i \in \mathbb{Q} \cap \mathcal{O}_L = \mathbb{Z}$. As $S_{\mathbf{u}}$ has no vanishing subsums by hypothesis, we see that $m \notin \{0, n\}$, and also the above representation of m has no vanishing subsums. As $1 \leq q := |I| < r$, the induction hypothesis yields a representation $m = S_{(\mathbf{v}_1, \mathbf{v}'_1, \boldsymbol{\xi}_1)}$ with $\mathbf{v}_1 \in (\mathcal{O}_L^\times)^{\ell_1}$ and $\boldsymbol{\xi}_1 \in \mathcal{U}_{s_1}^{s_1}$, such that $2\ell_1 + s_1 \leq q$.

Moreover, we may write

$$n - m = \sum_{i=1}^{r} u_i - \sum_{i \in I} u_i = \sum_{i \in I^c} u_i,$$

again a representation without vanishing subsums, as $S_{\mathbf{u}}$ has no vanishing subsums. As $1 \leq r - q = |I^c| < r$, the induction hypothesis yields a representation $n - m = S_{(\mathbf{v}_2, \mathbf{v}'_2, \boldsymbol{\xi}_2)}$ with $\mathbf{v}_2 \in (\mathcal{O}_L^\times)^{\ell_2}$ and $\boldsymbol{\xi}_2 \in \mathcal{U}_{s_2}^{s_2}$, such that $2\ell_2 + s_2 \leq r - q$.

Then we may take $\ell = \ell_1 + \ell_2$, $s := s_1 + s_2$, $\mathbf{v} = (\mathbf{v}_1, \mathbf{v}_2) \in (\mathcal{O}_L^\times)^\ell$ and $\boldsymbol{\xi} := (\boldsymbol{\xi}_1, \boldsymbol{\xi}_2) \in \mathcal{U}_s^s$ to obtain $2\ell + s \leq r$ and $n = S_{(\mathbf{v}, \mathbf{v}', \boldsymbol{\xi})}$. □

3. Counting unit trace sums

Throughout this section we fix $\ell \in \mathbb{N}$. Let $\mathbf{c} = (c_1, \ldots, c_\ell) \in (L^\times)^\ell$. This section is devoted to study the following counting function

$$T^{\mathbf{c}}_{L,\ell}(X) := \# \left\{ (u_1, \ldots, u_\ell) \in \mathcal{O}_L^\times \ : \begin{array}{l} \left|\mathrm{Tr}_{L/\mathbb{Q}}(c_1 u_1) + \cdots + \mathrm{Tr}_{L/\mathbb{Q}}(c_\ell u_\ell)\right| \leq X; \\ c_1 u_1 + \cdots + c_\ell u_\ell + c_1' u_1' + \cdots + c_\ell' u_\ell' \\ \text{has no vanishing subsum;} \\ |u_i| \geq 1 \text{ for } 1 \leq i \leq \ell. \end{array} \right\}$$

If we drop the third condition $|u_i| \geq 1$ then we can replace any of the coordinates u_i by their conjugates u_i' so that for each $\mathbf{u}$ counted in $T^{\mathbf{c}}_{L,\ell}(X)$ we have at most 2^ℓ vectors. Hence, dropping the condition $|u_i| \geq 1$ gives a set of cardinality at most $2^\ell T^{\mathbf{c}}_{L,\ell}(X)$.

The main result of this section provides an asymptotic formula for $T^{\mathbf{c}}_{L,\ell}(X)$ as X gets large.

Proposition 2. *For $X \geq 2$ we have*

$$T^{\mathbf{c}}_{L,\ell}(X) = \left(\frac{2\log X}{\log \eta}\right)^\ell + O_{L,\ell,\mathbf{c}}\left((\log X)^{\ell-1}\right).$$

3.1. **Proof of Proposition 2.** We prove the upper and lower bound separately. To prove the required upper bound we need the following lemma. A different version was proved by the last author in [13, Proposition 3.2].

Lemma 2. *Let $q \geq 1$ be an integer, $\alpha \in \mathbb{C}$ with $|\alpha| > 1$, $\mathbf{c} = (c_1, \ldots, c_q) \in (\mathbb{C}^\times)^q$ and $n_1 \geq \cdots \geq n_q$ integers. Then there exists $C = C(\alpha, \mathbf{c}) > 0$ depending only on α and $\mathbf{c}$, such that*

$$|c_1\alpha^{n_1} + \cdots + c_q\alpha^{n_q}| > C\alpha^{n_1},$$

provided that $c_1\alpha^{n_1} + \cdots + c_q\alpha^{n_q}$ has no vanishing subsum.

Proof. This is trivial for $q = 1$. For $q \geq 2$ we need to show that

$$f_{\mathbf{c},\alpha}(\mathbf{m}) := \left|c_1 + c_2\alpha^{-m_2} + \cdots + c_k\alpha^{-m_q}\right| \geq C(\alpha, \mathbf{c}) > 0 \tag{9}$$

for every integer vector $\mathbf{m} = (m_2, \ldots, m_q)$ with $0 \leq m_2 \leq \cdots \leq m_q$, provided no subsum of $c_1 + c_2\alpha^{-m_2} + \cdots + c_q\alpha^{-m_q}$ vanishes. First suppose $q = 2$. Set $M_0 = (\log|2c_2/c_1|)/\log|\alpha|$, so that

$$f_{\mathbf{c},\alpha}(\mathbf{m}) = |c_1 + c_2\alpha^{-m_2}| \geq \left|\frac{c_1}{2}\right|$$

whenever $m_2 \geq M_0$. On the other hand, by the non-vanishing subsum hypothesis, we have

$$\min_{0 \leq m_2 \leq M_0} f_{\mathbf{c},\alpha}(\mathbf{m}) =: C_0 > 0.$$

This proves the claim for $q = 2$ with $C(\alpha, \mathbf{c}) = \min\{|c_1/2|, C_0\}$.

Now let $q \geq 2$ be given and suppose (9) holds, assuming the non-vanishing hypothesis. Let $c_{q+1} \in \mathbb{C}^\times$. Set

$$M_1 = \frac{\log\left|\frac{2c_{q+1}}{C(\alpha,\mathbf{c})}\right|}{\log|\alpha|},$$

and consider the integer q-tuple $(\mathbf{m}, m_{q+1})$ with $0 \le m_2 \le \cdots \le m_q \le m_{q+1}$. First suppose that $m_{q+1} \ge M_1$ then

$$\begin{aligned} f_{(\mathbf{c},c_{q+1}),\alpha}((\mathbf{m}, m_{q+1})) &= |c_1 + c_2\alpha^{-m_2} + \cdots + c_q\alpha^{-m_q} + c_{q+1}\alpha^{-m_{q+1}}| \\ &\ge f_{\mathbf{c},\alpha}(\mathbf{m}) - |c_{q+1}\alpha^{-M_1}| \\ &\ge \frac{C(\alpha, \mathbf{c})}{2}. \end{aligned}$$

Next suppose that $m_{q+1} \le M_1$. Using the non-vanishing subsum hypothesis, we note that

$$\min_{0 \le m_2 \le \cdots \le m_{q+1} \le M_1} |c_1 + c_2\alpha^{-m_2} + \cdots + c_{q+1}\alpha^{-m_{q+1}}| =: C_1 > 0.$$

Note that C_1 depends only on $\alpha, C(\alpha, \mathbf{c})$ and c_{q+1}. Hence, we conclude

$$f_{(\mathbf{c},c_{q+1}),\alpha}((\mathbf{m}, m_{q+1})) \ge \min\left\{\frac{C(\alpha, \mathbf{c})}{2}, C_1\right\}$$

for all integer k-tuple $(\mathbf{m}, m_{q+1})$ with $0 \le m_2 \le \cdots \le m_q \le m_{q+1}$. This completes the proof of the lemma. □

Before we apply Lemma 2 to derive the required upper bound for $T^{\mathbf{c}}_{L,\ell}(X)$ let us point out that Lemma 2 also implies the finiteness of the non-degenerate solutions $\mathbf{v} \in (\mathcal{O}_L^\times)^T$ to (5). For this it suffices to note that the conjugate $\mathbf{v}'$ is also a solution of (5), so that we can assume $n_1 \ge |n_T|$.

Lemma 3. *For $X \ge 2$ we have*

$$T^{\mathbf{c}}_{L,\ell}(X) \le \left(\frac{2\log X}{\log \eta}\right)^\ell + O_{L,\ell,\mathbf{c}}\left((\log X)^{\ell-1}\right).$$

Proof. Let us consider an ℓ-tuple $(u_1, \dots, u_\ell) \in T^{\mathbf{c}}_{L,\ell}(X)$ so that

$$\left|\mathrm{Tr}_{L/\mathbb{Q}}(c_1u_1) + \cdots + \mathrm{Tr}_{L/\mathbb{Q}}(c_\ell u_\ell)\right| \le X,$$

and no subsum of

$$c_1u_1 + \cdots + c_\ell u_\ell + c_1'u_1' + \cdots + c_\ell' u_\ell'$$

vanishes. Recall that $|u_i| \ge 1$, so that each u_i has the form $u_i = \pm\eta^{n_i}$ with $n_i \in \mathbb{N}_0$. Applying Lemma 2 with $\alpha = \eta$ and $q = 2\ell$ and taking logs gives

$$\begin{aligned} \max_{1\le i\le \ell}\{|n_i|\}\log\eta &\le \log\left|\mathrm{Tr}_{L/\mathbb{Q}}(c_1u_1) + \cdots + \mathrm{Tr}_{L/\mathbb{Q}}(c_\ell u_\ell)\right| + O_{\mathbf{c},L}(1) \\ &\le \log X + O_{\mathbf{c},L}(1). \end{aligned}$$

This immediately yields the upper bound

$$T^{\mathbf{c}}_{L,\ell}(X) \le \left(\frac{2\log X}{\log \eta}\right)^\ell + O_{\mathbf{c},L}\left((\log X)^{\ell-1}\right). \tag{10}$$

□

Next we prove the required lower bound for $T^{\mathbf{c}}_{L,\ell}(X)$.

Lemma 4. *For $X \ge 2$ we have*

$$T^{\mathbf{c}}_{L,\ell}(X) \ge \left(\frac{2\log X}{\log \eta}\right)^\ell + O_{L,\ell,\mathbf{c}}\left((\log X)^{\ell-1}\right).$$

Proof. Set

$$C_1 = C_1(\mathbf{c}) = \frac{\max_i\{1, |c_i|, |c_i'|\}}{\min_j\{1, |c_j|, |c_j'|\}}, \text{ and } C_2 = \frac{\log(2\ell C_1)}{\log \eta}.$$

It suffices to prove the bound for $X > 2C_1\ell$. Next let us count the $\mathbf{n} \in \mathbb{N}^\ell$ satisfying

1) $\ell C_1(\eta^{n_i} + 1) < X$ $(1 \le i \le \ell)$
2) $n_i \ge C_2$ $(1 \le i \le \ell)$
3) $|n_i - n_j| \ge C_2$ $(1 \le i < j \le \ell)$

The number of those $\mathbf{n}$ with 1) is

$$\left\lfloor \frac{\log(\frac{X}{C_1\ell} - 1)}{\log \eta} \right\rfloor^\ell = \left(\frac{\log X}{\log \eta}\right)^\ell + O_{L,\ell,\mathbf{c}}\left((\log X)^{\ell-1}\right).$$

And of those $\mathbf{n}$ only $O_{L,\ell,\mathbf{c}}\left((\log X)^{\ell-1}\right)$ fail 2) and only $O_{L,\ell,\mathbf{c}}\left((\log X)^{\ell-1}\right)$ fail 3). Hence, we have

$$\left(\frac{\log X}{\log \eta}\right)^\ell + O_{L,\ell,\mathbf{c}}\left((\log X)^{\ell-1}\right)$$

$\mathbf{n} \in \mathbb{N}^\ell$ that satisfy 1), 2) and 3) simultaneously. Each of these $\mathbf{n}$ produces exactly 2^ℓ unit vectors $\mathbf{u} \in {\mathcal{O}_L^\times}^\ell$ with modulus of the coordinates ≥ 1 via $u_i = \pm\eta^{n_i}$ $(1 \le i \le \ell)$. Note that these

$$\left(\frac{2\log X}{\log \eta}\right)^\ell + O_{L,\ell,\mathbf{c}}\left((\log X)^{\ell-1}\right)$$

unit vectors $\mathbf{u}$ are pairwise distinct. We claim that all these unit vectors $\mathbf{u}$ are counted in $T^{\mathbf{c}}_{L,\ell}(X)$. First note that $|u_i'| = \eta^{-n_i} < 1$, and thus it follows from 1) that

$$\left|\sum_{i=1}^{\ell} \mathrm{Tr}_{L/\mathbb{Q}}(c_i u_i)\right| < X.$$

Next suppose that

$$\sum_{i=1}^{\ell} \mathrm{Tr}_{L/\mathbb{Q}}(c_i u_i) = c_1u_1 + c_1'u_1' + \cdots + c_\ell u_\ell + c_\ell' u_\ell'$$

has a vanishing subsum, say

$$v_1 + \cdots + v_s = 0 \tag{11}$$

with $2 \le s \le 2\ell$. After permuting the coordinates of $\mathbf{c}$ we can assume that $v_i = d_i\eta^{m_i}$ where $d_i \in \{\pm c_i, \pm c_i'\}$, and $m_1 < m_2 < \cdots < m_s$ are integers with $m_s - m_{s-1} \ge C_2$. The latter is a consequence of 2) and 3) for the positive integers n_i. Dividing the zero-sum (11) by d_s yields

$$\begin{aligned} \eta^{m_s} &= |(d_1/d_s)\eta^{m_1} + \cdots + (d_{s-1}/d_s)\eta^{m_{s-1}}| \\ &\le (s-1)C_1\eta^{m_{s-1}} \\ &\le (2\ell - 1)C_1\eta^{m_s - C_2} \\ &= \frac{(2\ell-1)C_1}{2\ell C_1}\eta^{m_s} \\ &< \eta^{m_s}. \end{aligned}$$

This contradiction shows that no subsum vanishes, and therefore completes the proof of the lemma. ☐

Combining Lemma 3 and Lemma 4 proves Proposition 2

4. Upper bounds for non-unique tuples

Definition 1. *We define an equivalence relation on* $\Omega := \bigcup_{m\in\mathbb{N}} L^m$ *as follows: for* $\mathbf{u} \in L^m$ *and* $\mathbf{w} \in L^n$*, we have* $\mathbf{u} \sim \mathbf{w}$ *if and only if* $n = m$ *and* $\mathbf{w}$ *arises from* $\mathbf{u}$ *by a permutation of the coordinates.*

Let $\mathcal{U}_t \subseteq \mathcal{O}_L^\times$ be the finite subset from Proposition 1 and define

$$\mathcal{F}_t := \mathcal{U}_0^0 \cup \mathcal{U}_1^1 \cup \cdots \cup \mathcal{U}_t^t.$$

Recall that we defined $\rho := \lfloor k/2 \rfloor$. For $0 \leq \ell \leq \rho$, we consider the sets

$$\mathcal{T}_{k,\ell} := \left\{ \mathbf{u} = (\mathbf{v}, \mathbf{v}', \boldsymbol{\xi}) \ : \ \begin{array}{c} \mathbf{v} \in (\mathcal{O}_L^\times)^\ell, \ \boldsymbol{\xi} \in \mathcal{F}_{k-2\ell}; \\ S_{\mathbf{u}} \in \mathbb{Z} \text{ with no vanishing subsums;} \\ |v_i| \geq 1 \text{ for } 1 \leq i \leq \ell. \end{array} \right\} \tag{12}$$

and $\mathcal{T}_{k,\ell}(X) := \{\mathbf{u} \in \mathcal{T}_{k,\ell} \ : \ |S_{\mathbf{u}}| \leq X\}$. Next, we define the subset of $\mathcal{T}_{k,\rho}(X)$ of tuples $\mathbf{u}$ that do not represent $S_{\mathbf{u}}$ essentially uniquely,

$$\mathcal{E}_k(X) := \{\mathbf{u} \in \mathcal{T}_{k,\rho}(X) \ : \ \exists \widetilde{\mathbf{u}} \in \mathcal{T}_{k,\rho}(X) \text{ such that } \widetilde{\mathbf{u}} \not\sim \mathbf{u} \text{ and } S_{\widetilde{\mathbf{u}}} = S_{\mathbf{u}}\}.$$

The main result of this section is an upper bound for the size of $\mathcal{E}_k(X)$.

Proposition 3. *We have*

$$\#\mathcal{E}_k(X) \ll_{k,L} (\log X)^{\rho-1}.$$

4.1. **Proof of Proposition 3.** Let $\mathbf{u} \in (\mathcal{O}_L^\times)^r$, $\widetilde{\mathbf{u}} \in (\mathcal{O}_L^\times)^s$ with $1 \leq r, s \leq k$, $\mathbf{u} \not\sim \widetilde{\mathbf{u}}$ and $S_{\mathbf{u}} = S_{\widetilde{\mathbf{u}}} = n \in \mathbb{Z}$, such that both representations of n have no vanishing subsums and $1 \leq |n| \leq X$.

Then there are $I \subseteq \{1, \ldots, r\}$ and $J \subseteq \{1, \ldots, s\}$, such that

$$0 = S_{\mathbf{u}} - S_{\widetilde{\mathbf{u}}} = \sum_{i\in I} u_i - \sum_{j\in J} \widetilde{u}_j \tag{13}$$

is a *minimal vanishing subsum*, i.e. no subsum on the right-hand side vanishes. As both $S_{\mathbf{u}}$ and $S_{\widetilde{\mathbf{u}}}$ have no vanishing subsums, it follows that $I, J \neq \emptyset$.

If $|I| = |J| = 1$, then we consider the complements $I^c = \{1, \ldots, r\} \smallsetminus I$ and $J^c = \{1, \ldots, s\} \smallsetminus J$, and take a minimal vanishing subsum of

$$0 = \sum_{i\in I^c} u_i - \sum_{j\in J^c} \widetilde{u}_j.$$

Continuing this way, we either find a minimal vanishing subsum of the form (13) with $|I| + |J| \geq 3$, or $\mathbf{u} \sim \widetilde{\mathbf{u}}$. As the latter was excluded from the start, we may thus assume by symmetry that our minimal vanishing subsum (13) satisfies $|I| \geq 2$ and $|J| \geq 1$.

For any $j_0 \in J$, we thus have

$$\sum_{i\in I} \frac{u_i}{\widetilde{u}_{j_0}} - \sum_{j\in J\smallsetminus\{j_0\}} \frac{\widetilde{u}_j}{\widetilde{u}_{j_0}} = 1,$$

with no vanishing subsums. Write $u := \widetilde{u}_{j_0}$ for simplicity, then

$$\mathbf{w} := \frac{1}{u}((u_i)_{i\in I}, (-\widetilde{u}_j)_{j\in J\smallsetminus\{j_0\}}) \in (\mathcal{O}_L^\times)^T, \quad T = |I| + |J| - 1,$$

is a nondegenerate solution of the unit equation (5), whence $\mathbf{w} \in \mathcal{S}_T$. Hence, for one of at most $\#\mathcal{S}_T$ values of $\mathbf{c} = (c_i)_{i\in I}$, we have $u_i = c_i u$ for all $i \in I$.

We conclude that

$$\mathbf{u} \sim ((c_i u)_{i\in I}, (u_i)_{i\in\{1,\dots,r\}\smallsetminus I}).$$

As $|I| \geq 2$, we have decreased the number of free variables in $\mathcal{O}_L^\times$ by at least one, at the cost of introducing the coefficients $\mathbf{c}$.

For any $\mathbf{u} \in \mathcal{E}_k(X)$, we moreover know that $\mathbf{u} \in \mathcal{T}_{k,\rho}(X)$, and thus

$$((c_i u)_{i\in I}, (u_i)_{i\in\{1,\dots,r\}\smallsetminus I}) \sim \mathbf{u} = (\mathbf{v}, \mathbf{v}', \boldsymbol{\xi}) \text{ with } \mathbf{v} \in (\mathcal{O}_L^\times)^\rho \text{ and } \boldsymbol{\xi} \in \mathcal{F}_{k-\rho}.$$

Hence, if k is even, then $r = k$ and $\boldsymbol{\xi}$ is the empty tuple. If k is odd, then either $r = k-1$ and $\boldsymbol{\xi}$ is the empty tuple, or $r = k$ and $\boldsymbol{\xi} \in \{\pm 1\}$.

As $|I| \geq 2$, there are $1 \leq i_1 < i_2 \leq r$ with $\{i_1, i_2\} \subseteq I$, and thus $u_{i_j} = c_{i_j} u$ for $j = 1, 2$.

Fixing i_1, i_2 and the c_{i_j}, we now distinguish a few different cases, showing in each case that the number of tuples $\mathbf{u}$ satisfying the above conditions is $\ll_{k,L} (\log X)^{\rho-1}$.

Case 1: $i_2 = 2\rho + 1$. In this case, k is odd and $i_2 = r = k$. Therefore, $\mathbf{u} = (\mathbf{v}, \mathbf{v}', \pm 1)$ and one coordinate of $\mathbf{v}$ (and $\mathbf{v}'$) is also fixed, say with value $\pm a$, where a depends only on c_{i_1} and c_{i_2}. Using Proposition 2, we get at most

$$\ll T_{L,\rho-1}^{(1,\dots,1)}(X + 1 + |a + a'|) \ll_{k,L} (\log X)^{\rho-1}$$

possibilities for the value of $\mathbf{v}$, and thus of $\mathbf{u}$.

Case 2: $i_2 = i_1 + \rho \leq 2\rho$. In this case, $c_{i_2} u = u_{i_2} = u'_{i_1} = c'_{i_1} u'$, and thus

$$u^2 = \pm\frac{u}{u'} = \pm\frac{c'_{i_1}}{c_{i_2}}.$$

This leaves only finitely many values of u, and thus of $v_{i_1} = u_{i_1} = c_{i_1} u$. Fixing v_{i_1} and using again Proposition 2, we get $\ll_{k,L} (\log X)^{\rho-1}$ choices for $\mathbf{v}$, and thus for $\mathbf{u} = (\mathbf{v}, \mathbf{v}', \boldsymbol{\xi})$.

Case 3: $i_2 \leq 2\rho$ and $i_2 \neq i_1 + \rho$. We may assume that $i_1 < i_2 \leq \rho$, possibly replacing u_{i_j} by u'_{i_j}. In any case, the coordinates v_{i_1} and v_{i_2} of $\mathbf{v}$ are both determined by u, and the sum of the traces of these coordinates, i.e. $(v_{i_1} + v'_{i_1}) + (v_{i_2} + v'_{i_2})$, is the trace of $(c_{i_1} + c_{i_2})u$. Note that $c_{i_1} + c_{i_2} \neq 0$, as the c_i are coordinates of some $\mathbf{w} \in \mathcal{S}_T$. Using Proposition 2, we get that the number of $\mathbf{v}$ (and thus also the number of $\mathbf{u}$) is bounded by

$$\ll T_{L,\rho-1}^{(c_{i_1}+c_{i_2},1,\dots,1)}(X+1) \ll_{k,L} (\log X)^{\rho-1}.$$

□

5. Proof of Theorem 1

Recall Definition 1 of the equivalence relation $\sim$ on $\Omega = \bigcup_{m\in\mathbb{N}} L^m$. If $M \subseteq \Omega$, we write

$$M/\sim := \{[m] \ : \ m \in M\}$$

for the set of equivalence classes that have a representative in M. As the set $(A \smallsetminus B)/\sim$ clearly contains $(A/\sim) \smallsetminus (B/\sim)$, we see that

$$\mathrm{N}_{L,k}(X) \geq \#\left((\mathcal{T}_{k,\rho}(X) \smallsetminus \mathcal{E}_k(X))/\sim\right) \geq \#\left(\mathcal{T}_{k,\rho}(X)/\sim\right) - \#\left(\mathcal{E}_k(X)/\sim\right).$$

On the other hand, Proposition 1 and Corollary 2 show that

$$\mathrm{N}_{L,k}(X) \leq \sum_{\ell=0}^{\rho} \#\left(\mathcal{T}_{k,\ell}(X)/\sim\right).$$

Proposition 2 implies that

$$\mathcal{T}_{k,\ell}(X) \ll_{L,k,\ell} (\log X)^\ell \quad \text{for all} \quad 0 \leq \ell \leq \rho,$$

by fixing $\boldsymbol{\xi} \in \mathcal{F}_{k-2\ell}$ and counting all $\mathbf{v}$ with $|S_{(\mathbf{v},\mathbf{v}')}| \leq X + |S_{\boldsymbol{\xi}}| \ll_{k,L} X$. Together with Proposition 3, this shows that

$$\mathrm{N}_{L,k}(X) = \#(\mathcal{T}_{k,\rho}(X)/\sim) + O_{k,L}\left((\log X)^{\rho-1}\right).$$

Hence, it remains to evaluate $\#(\mathcal{T}_{k,\rho}(X)/\sim)$ asymptotically. Elements $\mathbf{u} \in \mathcal{T}_{k,\rho}(X)$ have one of the following shapes, all with $\mathbf{v} \in (\mathcal{O}_L^\times)^\rho$ and $|v_i| \geq 1$ for $1 \leq i \leq \rho$:

(S_1) $\mathbf{u} = (\mathbf{v}, \mathbf{v}')$,
(S_2) $\mathbf{u} = (\mathbf{v}, \mathbf{v}', 1)$,
(S_3) $\mathbf{u} = (\mathbf{v}, \mathbf{v}', -1)$.

If k is even, then only shape (S_1) is possible. If k is odd, then all three shapes can appear. Using that

$$|S_{(\mathbf{v},\mathbf{v}',\boldsymbol{\xi})}| - |S_{\boldsymbol{\xi}}| \leq |S_{(\mathbf{v},\mathbf{v}')}| \leq |S_{(\mathbf{v},\mathbf{v}',\boldsymbol{\xi})}| + |S_{\boldsymbol{\xi}}|,$$

we see that in each of the three cases we have at least $T_{L,\rho}^{(1,\ldots,1)}(X-1)$ and at most $T_{L,\rho}^{(1,\ldots,1)}(X+1)$ elements $\mathbf{u} \in \mathcal{T}_{k,\rho}(X)$. Hence, by Proposition 2, for $i \in \{1,2,3\}$ we have

$$\#\{\mathbf{u} \in \mathcal{T}_{k,\rho}(X) \ : \ \mathbf{u} \text{ of shape } (S_i)\} = \left(\frac{2\log X}{\log \eta}\right)^\rho + O_{k,L}((\log X)^{\rho-1}).$$

An easy application of Proposition 2 shows that the contribution to the above count of those $\mathbf{u}$ with two or more identical coordinates is $\ll_{k,L} (\log X)^{\rho-1}$. Hence, we can assume the coordinates of $\mathbf{v}$ are pairwise distinct and of modulus > 1. This means that for each $\mathbf{u}$ in $\mathcal{T}_{k,\rho}(X)$ there are exactly $\rho!$ many equivalent elements in $\mathcal{T}_{k,\rho}(X)$ (arising from permuting the first ρ coordinates). We conclude that

$$\#\left(\{\mathbf{u} \in \mathcal{T}_{k,\rho}(X) \ : \ \mathbf{u} \text{ of shape } (S_i)\}/\sim\right) = \frac{1}{\rho!}\left(\frac{2\log X}{\log \eta}\right)^\rho + O_{k,L}((\log X)^{\rho-1}).$$

This proves Theorem 1 for even k, as then only shape (S_1) is possible.

If k is odd, it only remains to note that all elements $\mathbf{u} \in \mathcal{T}_{k,\rho}$ that belong to the same equivalence class must share the same shape. Indeed, the shape of $\mathbf{u}$ is specified by the number of coordinates of $\mathbf{u}$ and the parity of the number of coordinates of $\mathbf{u}$ equal to 1, which are clearly constant in equivalence classes. Hence,

$$\begin{aligned}\#\left(\mathcal{T}_{k,\rho}(X)/\sim\right) &= \sum_{i=1}^{3} \#\left(\{\mathbf{u} \in \mathcal{T}_{k,\rho}(X) \ : \ \mathbf{u} \text{ of shape } (S_i)\}/\sim\right) \\ &= \frac{3}{\rho!}\left(\frac{2\log X}{\log \eta}\right)^{\rho} + O_{k,L}((\log X)^{\rho-1}). \quad \square\end{aligned}$$

6. Local solubility

Let p be a prime. We need to study the solubility of (4) with $u_i \in (\mathbb{Z}_p \otimes_{\mathbb{Z}} \mathcal{O}_L)^{\times}$. We start by investigating solutions with $u_i \in \mathbb{Z}_p^{\times}$.

If either $k, n \in \mathbb{Z}_p^{\times}$ or $k, n \notin \mathbb{Z}_p^{\times}$, then at least one of n/k and $(n-1)/(k-1)$ is in $\mathbb{Z}_p^{\times}$, and thus

$$\frac{n}{k} + \cdots + \frac{n}{k} = n \quad \text{or} \quad 1 + \frac{n-1}{k-1} + \cdots + \frac{n-1}{k-1} = n$$

is a solution in units of $\mathbb{Z}_p$.

If p is odd and $p \mid k$, then $p \nmid n - e$ for some $e \in \{1, 2\}$ and we get the solution

$$e + \frac{n-e}{k-1} + \cdots + \frac{n-e}{k-1} = n,$$

in units of $\mathbb{Z}_p$.

If p is odd and $p \nmid k$, $p \mid n$, then also $p \nmid k - e$ for some $e \in \{1, 2\}$ and we get the solution

$$1 + \frac{n-1}{k-1} + \cdots + \frac{n-1}{k-1} = n \quad \text{or} \quad 1 + 1 + \frac{n-2}{k-2} + \cdots + \frac{n-2}{k-2} = n$$

in units of $\mathbb{Z}_p$.

In conclusion, there are solutions in units of $\mathbb{Z}_p$ whenever p is odd or $p = 2$ and $n \equiv k \bmod 2$. Via $u \mapsto u \otimes 1$, these also give solutions in units of $\mathbb{Z}_p \otimes_{\mathbb{Z}} \mathcal{O}_L$.

If $p = 2$ is inert in $\mathcal{O}_L$, then $\mathbb{Z}_2 \otimes_{\mathbb{Z}} \mathcal{O}_L = \mathcal{O}_{\mathfrak{P}}$, the localisation of $\mathcal{O}_L$ at the unique prime ideal $\mathfrak{P}$ over 2. One easily sees that every element of $\mathbb{F}_4 = \mathcal{O}_L/\mathfrak{P}$ can be written as a sum of two units, and hence (4) has solutions over $\mathbb{F}_4$, in units $u_i \in \mathbb{F}_4^{\times}$. By Hensel's lemma, these solutions lift to solutions over $\mathcal{O}_{\mathfrak{P}}$, still in units $u_i \in \mathcal{O}_{\mathfrak{P}}^{\times}$. Hence, we have shown that $\mathcal{X}_n(\mathbb{Z}_p) \neq \emptyset$ whenever $p \neq 2$, when $p = 2$ is inert in L, or when $p = 2$ and $k \equiv n \bmod 2$.

When $k \not\equiv n \bmod 2$ and 2 is split or ramified in L, let $\mathfrak{P}$ be a prime ideal of $\mathcal{O}_L$ lying above 2. From the reductions $\mathbb{Z}_2 \to \mathbb{F}_2$ and $\mathcal{O}_L \to \mathcal{O}_L/\mathfrak{P} = \mathbb{F}_2$ we get a ring homomorphism

$$\mathbb{Z}_2 \otimes_{\mathbb{Z}} \mathcal{O}_L \to \mathbb{F}_2.$$

As under our hypothesis on k and n there are clearly no solutions of (4) in units of $\mathbb{F}_2$, this shows that there can be no such solutions in units of $\mathbb{Z}_2 \otimes_{\mathbb{Z}} \mathcal{O}_L$, and hence $\mathcal{X}_n(\mathbb{Z}_2) = \emptyset$.

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Email address: `frei@math.tugraz.at`

Email address: `martin.widmer@tugraz.at`

Institute of Analysis and Number Theory, Graz University of Technology, Steyrergasse 30/II, 8010 Graz, Austria

Email address: `volker.ziegler@sbg.ac.at`

University of Salzburg, Hellbrunnerstrasse 34/I, A-5020 Salzburg, Austria