NORTHCOTT NUMBERS FOR THE HOUSE AND THE WEIL HEIGHT

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In memory of Professor Andrzej Schinzel, 1937–2021.

Abstract. For an algebraic number α and $\gamma \in \mathbb{R}$, let $\overline{\alpha}$ be the house, $h(\alpha)$ be the (logarithmic) Weil height, and $h_\gamma(\alpha) = (\deg \alpha)^\gamma h(\alpha)$ be the γ-weighted (logarithmic) Weil height of α. Let $f : \overline{\mathbb{Q}} \to [0, \infty)$ be a function on the algebraic numbers $\overline{\mathbb{Q}}$, and let $S \subset \overline{\mathbb{Q}}$. The Northcott number $\mathcal{N}_f(S)$ of S, with respect to f, is the infimum of all $X \geq 0$ such that $\{\alpha \in S; f(\alpha) < X\}$ is infinite. This paper studies the set of Northcott numbers $\mathcal{N}_f(\mathcal{O})$ for subrings of $\overline{\mathbb{Q}}$ for the house, the Weil height, and the γ-weighted Weil height. We show:

(1) Every $t \geq 1$ is the Northcott number of a ring of integers of a field w.r.t. the house $\overline{\cdot}$.
(2) For each $t \geq 0$ there exists a field with Northcott number in $[t, 2t]$ w.r.t. the Weil height $h(\cdot)$.
(3) For all $0 \leq \gamma \leq 1$ and $\gamma' < \gamma$ there exists a field K with $\mathcal{N}_{h_{\gamma'}}(K) = 0$ and $\mathcal{N}_{h_\gamma}(K) = \infty$.

For (1) we provide examples that satisfy an analogue of Julia Robinson's property (JR), examples that satisfy an analogue of Vidaux and Videla's isolation property, and examples that satisfy neither of those. Item (2) concerns a question raised by Vidaux and Videla due to its direct link with decidability theory via the Julia Robinson number. Item (3) is a strong generalisation of the known fact that there are fields that satisfy the Lehmer conjecture but which are not Bogomolov in the sense of Bombieri and Zannier.

1. Introduction

In this article we investigate the spectrum of Northcott numbers of subrings of the algebraic numbers $\overline{\mathbb{Q}}$ for the house and the Weil height. The Northcott number with respect to the Weil height was introduced by Vidaux and Videla [28], and refines the concept of the Northcott property which goes back to Northcott [15, 16] but was formally defined by Bombieri and Zannier [3]. Northcott numbers for various other height functions have been around implicitly and explicitly in the Bogomolov property, the Lehmer conjecture, the Schinzel–Zassenhaus conjecture (now Dimitrov's theorem), the Julia Robinson property, and the Julia Robinson number. To unify all these concepts under the umbrella of Northcott numbers we start with the following obvious generalisation.

Definition 1 (Northcott number)**.** *For a subset S of the algebraic numbers $\overline{\mathbb{Q}}$ and $f : \overline{\mathbb{Q}} \to [0, \infty)$ we set*

$$\mathcal{N}_f(S) = \inf\{t \in [0, \infty); \#\{\alpha \in S; f(\alpha) < t\} = \infty\},$$

with the usual interpretation $\inf \emptyset = \infty$. We call $\mathcal{N}_f(S) \in [0, \infty]$ the Northcott number *of S (with respect to f). If $\mathcal{N}_f(S) = \infty$ then we say that S has the Northcott property (with respect to f).*

Throughout this introduction ring always means not the zero ring. Next we give some background on the relevant results that use the house $\overline{\cdot}$ of an algebraic number (i.e., the maximum modulus of its conjugates over $\mathbb{Q}$).

In 1959, Julia Robinson [19] showed the undecidability of the first order theory of any number field, extending the case $\mathbb{Q}$ dealt with in her Ph.D. dissertation. A few years later she began [20, 21] to investigate decidability questions for certain rings of totally real algebraic integers of infinite degree. To this end she introduced the following property, nowadays called property (JR). Let $\mathcal{O}$ be a ring of totally real algebraic integers, and let $\mathcal{O}^+ \subset \mathcal{O}$ be its subset of totally positive elements. The ring $\mathcal{O}$ has property (JR) if the following holds

$$\#\{\alpha \in \mathcal{O}^+; \overline{\alpha} < \mathcal{N}_{\overline{\cdot}}(\mathcal{O}^+)\} = \infty.$$

As usual, $x < \infty$ is true for all $x \in \mathbb{R}$ by convention. Note that the Northcott property implies the property (JR). Let $\mathbb{N} = \{1, 2, 3, \ldots\}$ be the set of positive integers, and set $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

Robinson showed that the semi-ring $(\mathbb{N}_0, 0, 1, +, \cdot)$ is first order definable in $\mathcal{O}$ for any ring $\mathcal{O}$ of totally real algebraic integers with property (JR) (not necessarily the ring of integers of a field, as pointed out by Vidaux and Videla [27]). She then proved that the rings of integers $\mathcal{O}_K$ of the maximal totally real extension K of $\mathbb{Q}$, and of $K = \mathbb{Q}(\sqrt{n}; n \in \mathbb{N})$ both have property (JR): the former since the infimum 4 in the definition of $\mathcal{N}_{\overline{\cdot}}(\mathcal{O}_K^+)$ is attained, and the latter since it has the Northcott property. Hence, both have undecidable first order theory. Since the aforementioned field is a pro-2 extension of $\mathbb{Q}$, it follows from a result of Videla [29], that its ring of integers is first order definable in this field, and thus the field inherits the undecidability from its ring of integers.

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A question that arose from Robinson's work, explicitly proposed by Vidaux and Videla [27, Question 1.5], is, which numbers can be realised as Northcott numbers[1] $\mathcal{N}_{\lceil\cdot\rceil}(\mathcal{O}_K^+)$. Important progress on this question was made by Gillibert and Ranieri [11] who proved that all numbers of the form $[2\sqrt{2n}] + 2\sqrt{2n}$ or $8n$, with $n \geq 1$ odd and square-free, are of this type. Further results on the distribution of the Northcott numbers $\mathcal{N}_{\lceil\cdot\rceil}(\mathcal{O}^+)$ were obtained by Castillo [4], and Castillo, Vidaux, and Videla [5].

Another question, explicitly proposed by Robinson herself, is, if in fact the ring of integers $\mathcal{O}_K$ of every totally real field K has property (JR). Gillibert and Ranieri [11] noted that all their examples do have property (JR).

Vidaux and Videla [27, Definition 1.2] introduced a related condition which they call isolation property, and which also allows, by the same strategy as for the property (JR), to define the semi-ring $(\mathbb{N}_0, 0, 1, +, \cdot)$ by a first order formula in $\mathcal{O}$. A ring $\mathcal{O}$ of totally real algebraic integers has the isolation property if it does not have property (JR), and if there exists $M > \mathcal{N}_{\lceil\cdot\rceil}(\mathcal{O}^+)$ such that for all $\epsilon > 0$ we have

$$\#\{\alpha \in \mathcal{O}^+; \mathcal{N}_{\lceil\cdot\rceil}(\mathcal{O}^+) + \epsilon \leq \lceil\alpha\rceil < M\} < \infty.$$

Since there are only finitely many totally real integers that assume a fixed house value t (in particular that assume the value $\mathcal{N}_{\lceil\cdot\rceil}(\mathcal{O}^+)$) it follows that the above cardinality gets arbitrarily large as ϵ gets small. Vidaux and Videla [27] have constructed rings of totally real algebraic integers that satisfy their isolation property but it is unknown if any of these is the ring of integers $\mathcal{O}_K$ of a field, and so Robinson's question also remains open. Nevertheless, examining decidability of subrings and subfields of $\overline{\mathbb{Q}}$ by Julia Robinson's strategy (and refinements thereof) is an active area of research. From the growing body of literature, we refer the reader to the work (and references therein) of Shlapentokh [23], Springer [25], as well as Martínez-Ranero, Utreras, and Videla [13].

Our first result shows that if we consider the full set of algebraic integers, and we do not restrict to totally real fields, then every real number $t \geq 1$ is a Northcott number with respect to the house. Furthermore, the analogous question to Julia Robinson's one can be answered in the negative, i.e., the infimum in the definition of $\mathcal{N}_{\lceil\cdot\rceil}(\mathcal{O}_K)$ is not always attained. Finally, we can also construct rings of integers $\mathcal{O}_K$ with given Northcott number that neither have the analogue of property (JR) nor the analogue of the isolation property[2].

Theorem 1. *Let $t > 1$ be a real number.*

(a) *There exists a field K of algebraic numbers such that its ring of integers $\mathcal{O}_K$ satisfies $\mathcal{N}_{\lceil\cdot\rceil}(\mathcal{O}_K) = t$ and $\#\{\alpha \in \mathcal{O}_K; \lceil\alpha\rceil < t\} = \infty$.*

(b) *There exists $M > t$ and a field K of algebraic numbers such that its ring of integers $\mathcal{O}_K$ satisfies $\mathcal{N}_{\lceil\cdot\rceil}(\mathcal{O}_K) = t$ and $\#\{\alpha \in \mathcal{O}_K; \lceil\alpha\rceil \leq t$ or $t + \epsilon \leq \lceil\alpha\rceil < M\} < \infty$ for all $\epsilon > 0$.*

(c) *There exists a field K of algebraic numbers such that its ring of integers $\mathcal{O}_K$ satisfies $\mathcal{N}_{\lceil\cdot\rceil}(\mathcal{O}_K) = t$ and $\#\{\alpha \in \mathcal{O}_K; \lceil\alpha\rceil \leq t\} < \infty$ and $\#\{\alpha \in \mathcal{O}_K; t + \epsilon \leq \lceil\alpha\rceil < M\} = \infty$ for all $M > t$ and all small enough $\epsilon > 0$.*

Since $\lceil\alpha\rceil \geq 1$ for every non-zero algebraic integer there is no ring of algebraic integers $\mathcal{O}$ for which $\#\{\alpha \in \mathcal{O}; \lceil\alpha\rceil < 1\} = \infty$. But, by our method, it is easy to construct fields K whose ring of integers have Northcott number $t = 1$, and that satisfy either selection of the remaining two properties.

The proof of Theorem 1 comes in two steps. First we construct a ring with prescribed Northcott number (and the additional topological features), and then we prove that the constructed ring is integrally closed (in its field of fractions). For the latter we exploit a criterion of Dedekind, demanding our construction to satisfy certain congruence constraints. The Siegel–Walfisz theorem about the distribution of primes in residue classes ensures that we can satisfy these congruence conditions.

The original problems considered by Robinson, and by Vidaux and Videla (restricting to $\mathcal{O}_K^+$ for totally real fields K) are more difficult than those we address in Theorem 1. However, it is conceivable that the methods in this paper are also useful to address these original questions.

Our construction of rings with prescribed Northcott number relies on our next result. Consider a sequence $(\xi_i)_i$ of algebraic integers, let $\mathcal{O}_0$ be a ring, containing 1, of algebraic integers, $\mathcal{O}_i = \mathcal{O}_0[\xi_1, \ldots, \xi_i]$, and let $\mathcal{O} = \bigcup_{i\geq 1} \mathcal{O}_i = \mathcal{O}_0[\xi_1, \xi_2, \xi_3, \ldots]$. Let K_i be the field of fractions of $\mathcal{O}_i$, and set $d_i = [K_{i-1}(\xi_i) : K_{i-1}]$. For a subfield $K \subset \overline{\mathbb{Q}}$ and an algebraic number ξ let $M_{\xi,K} \in K[x]$ be the monic minimal polynomial of ξ over K. We introduce a new quantity $\eta(K, \xi)$ which measures the largest root of $\sigma(M_{\xi,K})$ and how equidistributed the normalised roots[3] of $\sigma(M_{\xi,K})$ on the unit circle are for each field homomorphism $\sigma : K \to \mathbb{C}$. The definition of $\eta(K, \xi)$ is given in Section 4. We always consider lim inf as element of the extended real number line $\mathbb{R} \cup \{\pm\infty\}$.

[1] Vidaux and Videla call $\mathcal{N}_{\lceil\cdot\rceil}(\mathcal{O}^+)$ the Julia Robinson number of the ring $\mathcal{O}$.

[2] It seems natural to impose the additional condition $\mathcal{N}_{\lceil\cdot\rceil}(\mathcal{O})$ is attained only for finitely many elements of $\mathcal{O}$ for the analogue of the isolation property in the non totally real case, since this condition automatically holds only in the totally real case.

[3] We say d complex points are (perfectly) equidistributed on a circle (of radius R) if they are pairwise distinct, all lie on the circle, and the arc-length between neighboring points is $2\pi R/d$. By "normalised" we mean scaled by the reciprocal of the largest of their moduli.

Theorem 2. *Suppose that $\mathcal{N}_{\overline{\square}}(\mathcal{O}_i) = \infty$, $d_i > 1$, and that $M_{\xi_i,K_{i-1}} \in \mathcal{O}_{i-1}[x]$ for all $i \in \mathbb{N}$. Then*

$$\mathcal{N}_{\overline{\square}}(\mathcal{O}) \geq \liminf_{i\to\infty} \eta(K_{i-1}, \xi_i).$$

Since $d_i > 1$ the ξ_i are pairwise distinct, and thus we also have the trivial upper bound

$$\mathcal{N}_{\overline{\square}}(\mathcal{O}) \leq \liminf_{i\to\infty} \overline{|\xi_i|}.$$

The simplest application of Theorem 2 is when $M_{\xi_i,K_{i-1}} = M_{\xi_i,\mathbb{Q}}$ and the conjugates over $\mathbb{Q}$ are perfectly equidistributed on a circle $|z| = t_i$, e.g., if they are of the form $\xi_j^{(i)} = t_i\zeta_{d_i}^j$ $(1 \leq j \leq d_i)$. In Section 6 we explain this and other applications, including a more sophisticated result (Corollary 13), that requires the full strength of Theorem 2.

The Northcott number $\mathcal{N}_{\overline{\square}}(\mathcal{O}_K)$ is also related to the invariant $c_1(K)$ for fields $K \subset \overline{\mathbb{Q}}$ introduced by Gaudron and Rémond in their investigations of the Siegel property for fields. This invariant is often difficult to determine; however, they show [10, Lemme 5.4] that $c_1(K) \geq \mathcal{N}_{\overline{\square}}(\mathcal{O}_K)$ provided K has infinite degree over $\mathbb{Q}$.

They also provide an example [10, Exemple 4.6] of a field K that has infinitely many elements of bounded Weil height but whose ring of integers has only finitely many elements of bounded house, i.e., $\mathcal{N}_h(K) < \infty = \mathcal{N}_{\overline{\square}}(\mathcal{O}_K)$. Their proof of $\mathcal{N}_{\overline{\square}}(\mathcal{O}_K) = \infty$ relies on the (perfect) orthogonality relations of the roots of unity, and could be adapted to handle the aforementioned simplest case $\xi_j^{(i)} = t_i\zeta_{d_i}^j$. Their method, has the advantage that it can deal with integral elements in $\mathbb{Q}[\xi_1, \xi_2, \xi_3, \ldots]$ but, in contrast to ours, it seems restricted to the perfectly equidistributed case, and cannot provide results such as Corollary 13 of Section 6.

The next height function we consider is the classical logarithmic absolute Weil height $h(\cdot)$. Again, we first give some background, and then we state our result.

The origin of the Northcott property goes back to two seminal papers of D.G. Northcott [15, 16] from 1949 and 1950, in which he showed that there are only finitely many algebraic numbers of bounded degree and bounded Weil height $h(\cdot)$, and proved the finiteness of the number of preperiodic points of bounded degree under non-linear algebraic endomorphisms of projective varieties defined over $\overline{\mathbb{Q}}$.

The Northcott property (with respect to $h(\cdot)$) is well known to have many diophantine applications, and thus it is natural to refine this concept via the Northcott number as done by Vidaux and Videla [28]. Indeed, it is often enough to know that the Northott number of a specified set is a sufficiently large finite number. For instance, to show that the non-linear polynomial $f \in K[x]$ has only finitely many preperiodic points in the field $K \subset \overline{\mathbb{Q}}$ it suffices to know that $\mathcal{N}_h(K) > 2c_f$ where $h(f(\alpha)) \geq \deg f \cdot h(\alpha) - c_f$. Even more concretely, for the polynomial $f_n = x^{2n} - x^{2n-1} + \cdots - x + 1$ one can take[4] $c_{f_n} = 2\log 2$.

On the opposite end, the first and the last author [18] have recently proved an arithmetic Bertini-type result for which fields with prescribed arithmetic features and sufficiently small Northcott number are needed.

These observations raise the question which numbers can be realised as the Northcott number of a field or a ring of integers of a subfield of $\overline{\mathbb{Q}}$. A similar question was raised by Vidaux and Videla [28, Question 6].

Question 1 (Vidaux, Videla 2016)**.** *Which real numbers can be realised as Northcott number (with respect to the absolute logarithmic Weil height) of a ring extension of $\mathbb{Q}$?*

Interestingly, Vidaux and Videla's motivation for the above Question 1 comes from their earlier question about the spectrum of the Julia Robinson numbers (i.e., the spectrum of the Northcott numbers $\mathcal{N}_{\overline{\square}}(\mathcal{O}_K^+)$ for totally real fields K), and the fact that $\overline{|\alpha|} \geq h(\alpha)$ for every non-zero algebraic integer. Given their motivation it seems to us equally natural to propose the analogous question for the house $\overline{\square}$ — a question that is completely answered by Theorem 1.

However, back to the Weil height $h(\cdot)$. To the best of our knowledge, there are currently only two possible "values" known as Northcott numbers for subrings of $\overline{\mathbb{Q}}$, namely 0 (attained, e.g., by $\overline{\mathbb{Q}}$) and ∞ (attained, e.g., by any number field). Here we show that the set of values cannot be sparse.

Theorem 3. *Let $t \geq 0$. There exists a field $L \subset \overline{\mathbb{Q}}$ satisfying*

$$t \leq \mathcal{N}_h(L) \leq \mathcal{N}_h(\mathcal{O}_L) \leq 2t.$$

More precisely, every field L generated over a number field K by any sequence of roots p_i^{1/d_i} that converge to $\exp(2t)$, and where p_i and d_i are primes and the p_i are strictly increasing, satisfies the conclusion.

For the aforementioned example $f_n = x^{2n} - x^{2n-1} + \cdots - x + 1 \in \mathbb{Z}[x]$, we conclude from Theorem 3 that f_n has only finitely many preperiodic points in L (with a bound independent of n), provided $t > 4\log 2$.

[4]Note that with $g_1 = y^n$ and $g_2 = y^n f_n(x/y)$ we have $x^{n+1} = -yg_1 + (x+y)g_2$ and $y^{n+1} = yg_1$. From this it is routine to compute c_{f_n}.

Finally, let us mention that Gaudron and Rémond's Siegel property [10] is also related to $\mathcal{N}_h(K)$. For instance, they show [10, Corollaire 1.2] that if K is a Siegel field of infinite degree over $\mathbb{Q}$ then $\mathcal{N}_h(K) < \infty$.

Our last result is concerned with Northcott numbers for differently normalised Weil heights. Many results around the Lehmer conjecture can be expressed in terms of the Northcott number of a suitably normalised Weil height. For example, writing $\mu \subset \overline{\mathbb{Q}}$ for the set of roots of unity and $\deg \alpha = [\mathbb{Q}(\alpha) : \mathbb{Q}]$, Dobrowolski's Theorem states that $\mathcal{N}_f(\overline{\mathbb{Q}}\backslash\mu) > 0$ for[5] $f(\alpha) = \left(\frac{\log^+ \deg \alpha}{\log^+ \log \deg \alpha}\right)^3 (\deg \alpha) h(\alpha)$. Let us now restrict ourselves to the case where

$$f(\alpha) = h_\gamma(\alpha) = (\deg \alpha)^\gamma h(\alpha) \quad \text{for some } \gamma \in \mathbb{R}.$$

Lehmer's conjecture itself states that $\mathcal{N}_{h_1}(\overline{\mathbb{Q}}\backslash\mu) > 0$. And the Bogomolov property for a set $S \subset \overline{\mathbb{Q}}$, also introduced by Bombieri and Zannier [3], can be rephrased as $\mathcal{N}_{h_0}(S\backslash\mu) > 0$. In analogy the first author and Pengo [17] say the set S has the Lehmer property if $\mathcal{N}_{h_1}(S\backslash\mu) > 0$. Generalising both properties we say a set $S \subset \overline{\mathbb{Q}}$ is γ-Bogomolov if $\mathcal{N}_{h_\gamma}(S\backslash\mu) > 0$, and we say S is γ-Northcott if $\mathcal{N}_{h_\gamma}(S) = \infty$ (i.e., S has the Northcott property with respect to $h_\gamma(\cdot)$). Note that by Dobrowolski's Theorem the field $\overline{\mathbb{Q}}$ (and hence each of its subsets) is γ-Bogomolov for every $\gamma > 1$.

Amoroso's Theorem 1.3 in [1] shows that the field $\mathbb{Q}(\zeta_3, 2^{1/3}, \zeta_{3^2}, 2^{1/3^2}, \zeta_{3^3}, 2^{1/3^3}, \ldots)$, where ζ_d denotes a primitive d-th root of unity, is 1-Bogomolov but not 0-Bogomolov. Another example, as we explain now, of such a field is $\mathbb{Q}^{tr}(\sqrt{-1})$, where $\mathbb{Q}^{tr}$ denotes the maximal totally real extension of $\mathbb{Q}$. By a result of Schinzel [22, Theorem 2] we have $(\deg \alpha)h(\alpha) \geq \log(\frac{1+\sqrt{17}}{4})/2$ for every unit α in the ring of integers of $\mathbb{Q}^{tr}(\sqrt{-1})$. This implies that $\mathbb{Q}^{tr}(\sqrt{-1})$ is 1-Bogomolov. But from Example 5.3 in [2], the sequence

$$\left(\left(\frac{2+\sqrt{-1}}{2-\sqrt{-1}}\right)^{1/k}\right)_{k\geq 1}$$

has all its elements in $\mathbb{Q}^{tr}(\sqrt{-1})$ which shows that this field is not 0-Bogomolov.

This raises the question whether for every $\gamma \leq 1$ and $\epsilon > 0$ there exists a field K that is γ-Bogomolov (or even γ-Northcott) but not $(\gamma - \epsilon)$-Bogomolov. Our next result answers this question in the affirmative.

Theorem 4. *Let $0 \leq \gamma \leq 1$, and $\epsilon > 0$. Choose sequences of primes $(d_i)_{i\in\mathbb{N}}$ and $(p_i)_{i\in\mathbb{N}}$ such that $d_{i+1} \geq 2d_i$, and $d_i^{1-\gamma+\epsilon/2} \leq \log p_i \leq \log(2) + d_i^{1-\gamma+\epsilon/2}$ for all $i \in \mathbb{N}$. Then $\mathbb{Q}(p_i^{1/d_i}; i \in \mathbb{N})$ is γ-Northcott but not $(\gamma-\epsilon)$-Bogomolov.*

While the proofs of Theorem 3 and of Theorem 4 rely on a method from [30], the proof of Theorem 1 is essentially different and is based on an equidistribution argument. However, it turns out that both methods are particularly easy to apply for fields of the shape $\mathbb{Q}(p_i^{1/d_i}; i \in \mathbb{N})$ for certain primes p_i and d_i, and this is the reason that all fields constructed in these three theorems are of this type.

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2. Definitions and basic properties of Northcott numbers

In this section, we introduce some notation, and collect some basic results about Northcott numbers.

We write $|\cdot|$ for the usual absolute value on $\mathbb{C}$, and for the maximum norm on $\mathbb{C}^d$ we set

$$\|\boldsymbol{\alpha}\| = \max_{1\leq i\leq d} |\alpha_i|.$$

For a field K of characteristic 0 we denote the set of field homomorphisms $\sigma : K \to \mathbb{C}$ by

$$\mathrm{Hom}(K) = \{\sigma : K \to \mathbb{C};\ \text{field homomorphism}\}.$$

The house of an algebraic number α, written $\overline{|\alpha|}$, is defined as

$$\overline{|\alpha|} = \max_{\sigma\in\mathrm{Hom}(\overline{\mathbb{Q}})} |\sigma(\alpha)|.$$

Next we define the Weil height. To this end let K be a number field, and let M_K denote the set of places of K, that is, equivalence classes of absolute values. For each place $v \in M_K$ we let $|\cdot|_v$ be the unique representative of v that extends one of the canonical absolute values[6] on $\mathbb{Q}$. For $v \in M_K$, let K_v abbreviate the completion of

[5] For $x \in \mathbb{R}$ we write $\log^+ x = \log\max\{x, \exp(1)\}$.

[6] I.e., for $x \in \mathbb{Q}$, we have $|x|_v = \max\{-x, x\}$ if v is archimedean, and otherwise $|p|_v = p^{-1}$ if v lies above the prime p.

K with respect to $|\cdot|_v$. The (logarithmic) Weil height of $\alpha \in K$ is given by

$$h(\alpha) = \frac{1}{[K:\mathbb{Q}]} \sum_{v \in M_K} [K_v : \mathbb{Q}_v] \log \max\{1, |x|_v\}.$$

The product formula implies that $h(\alpha)$ does not depend on the ambient field K, and hence $h(\cdot)$ extends to a function on $\overline{\mathbb{Q}}$. Further, we have $h(\sigma(\alpha)) = h(\alpha)$ for any field homomorphism $\sigma : \overline{\mathbb{Q}} \to \overline{\mathbb{Q}}$, and $h(\alpha+\beta) \leq h(\alpha) + h(\beta) + \log 2$, and $h(\alpha\beta) \leq h(\alpha) + h(\beta)$ for all $\alpha, \beta \in \overline{\mathbb{Q}}$. More generally, suppose we have a function $f : \overline{\mathbb{Q}} \to [0, \infty)$ and that there exists a continuous function $F : \mathbb{R}^2 \to [0, \infty)$ such that for any field homomorphism $\sigma : \overline{\mathbb{Q}} \to \overline{\mathbb{Q}}$, and all algebraic numbers α, β the following is true

$$(f1) \qquad f(\sigma(\alpha)) = f(\alpha),$$
$$(f2) \qquad f(\alpha+\beta) \leq F(f(\alpha), f(\beta)),$$
$$(f3) \qquad f(\alpha\beta) \leq F(f(\alpha), f(\beta)).$$

With $F(x,y) = \max\{xy, x+y+\log 2\}$ the properties (f1), (f2), and (f3) are satisfied for the Weil height $h(\cdot)$ and for the house $\boxdot$. Furthermore, note that for each non-zero algebraic integer α we have

$$h(\alpha) \leq \log \boxdot\alpha\boxdot.$$

Dvornicich and Zannier observed that the proof of Northcott's Theorem yields a more general statement, which we state here in an even slightly more general form.

Lemma 5 (Dvornicich and Zannier [8, Thm. 2.1])**.** *Suppose f from Definition 1 satisfies $(f1)$, $(f2)$, and $(f3)$. Let K be a subfield of $\overline{\mathbb{Q}}$, and $U \subset K$. Let $S \subset \overline{\mathbb{Q}}$ be a set of roots of monic irreducible polynomials in $K[x]$ with coefficients in U and uniformly bounded degree. If U has the Northcott property with respect to f, then S has the Northcott property with respect to f as well.*

Proof. The following is a straightforward adaptation of the proof of [8, Thm. 2.1]. For the sake of completeness, we provide the details. Let $X > 0$, and $\alpha \in S$ an element with $f(\alpha) < X$. If β is a conjugate of α over K, then $(f1)$ implies $f(\beta) = f(\alpha) \leq X$. Let E be an integer such that $[K(\gamma) : K] \leq E$ for any element $\gamma \in S$. Denote the monic minimal polynomial of α over K by $M_{\alpha,K}(x) = a_0 + a_1 x + \cdots + a_{d-1}x^{d-1} + x^d$. By assumption, $a_i \in U$ for any $0 \leq i < d$. Next we will exploit that each a_i is an elementary symmetric function in the conjugates of α (over K). To this end, we first observe that there are at most $d \leq E$ conjugates of α over K. By using the properties $(f2)$ and $(f3)$, and the fact that a continuous function attains its maximum on a compact set, we infer that $f(a_i)$ is bounded from above in terms of X, E and the function $F(\cdot,\cdot)$ for all $0 \leq i < d$. Since U has the Northcott property with respect to f, there are only finitely many such $(a_0, \ldots, a_{d-1}) \in U^d$. Hence the number of $\alpha \in S$ with $f(\alpha) \leq X$ is finite for any $X > 0$. □

The most important case is when $K = \mathbb{Q}$ and $U = \mathbb{Q}$ or $U = \mathbb{Z}$ respectively, from which it follows that each number field has the Northcott property with respect to $h(\cdot)$, and the ring of integers of each number field has the Northcott property with respect to the house $\boxdot$. We point out two further immediate consequences of Lemma 5 for the Weil height and the house.

Remark 1. *Suppose that $K \subset L$ are fields of algebraic numbers and that $[L:K]$ is finite. We have*

(a) $\mathcal{N}_h(K) = \infty$ *if and only if* $\mathcal{N}_h(L) = \infty$,
(b) $\mathcal{N}_{\boxdot}(\mathcal{O}_K) = \infty$ *if and only if* $\mathcal{N}_{\boxdot}(\mathcal{O}_L) = \infty$.

However, the Northcott number with respect to the Weil height and the house is not preserved under finite extensions in general. Indeed, $\log \mathcal{N}_{\boxdot}(\mathcal{O}_{\mathbb{Q}^{tr}}) \geq \mathcal{N}_h(\mathbb{Q}^{tr}) \geq \frac{1}{2}\log((1+\sqrt{5})/2))$ *by Schinzel's result* [22, Theorem 2] *but $\mathbb{Q}^{tr}(\sqrt{-1})$ contains infinitely many roots of unity, and hence,* $\log \mathcal{N}_{\boxdot}(\mathcal{O}_{\mathbb{Q}^{tr}(\sqrt{-1})}) = \mathcal{N}_h(\mathbb{Q}^{tr}(\sqrt{-1})) = 0$.

Next we describe a general characterisation of the Northcott number of a set that is represented as a union of an infinite nested sequence of sets. Let $f : \overline{\mathbb{Q}} \to [0, \infty)$. For each set $S \subset \overline{\mathbb{Q}}$ we set

$$\delta_f(S) = \inf\{f(\alpha); \alpha \in S\}.$$

Let $A_0 \subset A_1 \subset A_2 \subset \cdots$ be a nested sequence of subsets of $\overline{\mathbb{Q}}$, and we set $A = \cup_{i \geq 0} A_i$. The next lemma shows that this quantity, capturing the relative behaviour of the height function at each step, determines the Northcott number of A under fairly mild assumptions.

Lemma 6. *Suppose that $\mathcal{N}_f(A_i) = \infty$ for all $i \in \mathbb{N}_0$. We have*[7]

$$\mathcal{N}_f(A) = \liminf_{i \to \infty} \delta_f(A_i \backslash A_{i-1}).$$

[7]Of course, if the sequence $(A_i)_i$ becomes stationary, so that $\delta_f(A_i \backslash A_{i-1}) = \infty$ for all large enough i, then the right hand-side is interpreted as ∞.

Proof. Since $\mathcal{N}_f(A_i) = \infty$ for all $i \geq 0$ we conclude that $\mathcal{N}_f(A) \leq \liminf \delta_f(A_i \backslash A_{i-1})$. To prove that $\mathcal{N}_f(A) \geq \liminf \delta_f(A_i \backslash A_{i-1})$ we can assume that $\mathcal{N}_f(A) < \infty$. There exists a sequence $(\alpha_i)_i \subset A$ of pairwise distinct elements with $(f(\alpha_i))_i$ converges to $\mathcal{N}_f(A)$. For α_i we set $\iota = \iota(\alpha_i) = \min\{l; \alpha_i \in A_l\}$ so that $\alpha_i \in A_\iota \backslash A_{\iota-1}$. Hence,

$$f(\alpha_i) \geq \delta_f(A_\iota \backslash A_{\iota-1}). \tag{2.1}$$

Since $\mathcal{N}_f(A_i) = \infty$ for all i we infer that $\iota \to \infty$ as $i \to \infty$. As the left hand side in (2.1) tends to $\mathcal{N}_f(A)$ the claim drops out. □

3. A lower bound for the maximum of a unitary complex polynomial on given points

For $d \in \mathbb{N}$ let

$$\zeta_d = \exp\left(\frac{2\pi i}{d}\right)$$

denote a primitive d^{th}-root of unity.

Definition 2 (Finite discrepancy)**.** *For $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_d) \in \mathbb{C}^d$ we set*

$$D(\boldsymbol{\xi}) = \inf_{\phi} \max_{1 \leq j \leq d} \min_{1 \leq i \leq d} |\xi_i - \phi(\zeta_d^j)|, \tag{3.2}$$

where ϕ runs over all rotations about the origin (i.e., $\phi(z) = uz$ for some $u \in \mathbb{T} = \{c \in \mathbb{C}; |c| = 1\}$).

Note that $D(\boldsymbol{\xi})$ is invariant under permutation of the entries of $\boldsymbol{\xi}$ - a fact that will be used in the sequel without further notice. The *multiplicity*, on the other hand, is important: $D(\xi) = ||\xi| - 1|$ but $D(\xi, \xi) = \sqrt{1 + |\xi|^2}$. Also, note that $D(\boldsymbol{\xi}) = 0$ if and only if $\xi_1, \ldots, \xi_d$ are perfectly equidistributed on the unit circle, i.e., if $\{\xi_1, \ldots, \xi_d\} = \{\phi(\zeta_d), \ldots, \phi(\zeta_d^d)\}$ for some rotation about the origin ϕ.

Lemma 7 (ℓ^2-lower bound)**.** *Let $\boldsymbol{\xi} \in \mathbb{C}^d$ and suppose $B(x) = b_0 + b_1 x + \cdots + b_n x^n \in \mathbb{C}[x]$ has degree strictly less than d. We have*

$$\max_{1 \leq i \leq d} |B(\xi_i)| \geq \left(1 - d^{3/2} D(\boldsymbol{\xi}) \max_{1 \leq i \leq d} \{1, |\xi_i|\}^{d-2}\right) \sqrt{\sum_{0 \leq i \leq n} |b_i|^2}.$$

If $n = 0$ then we can omit the first factor.

Proof. Note that the statement is trivially true for $d = 1$. So we can assume $d \geq 2$. First, let us assume that $\xi_i = \zeta_d^i$ for $1 \leq i \leq d$. We observe that

$$\left(\max_{1 \leq i \leq d} |B(\zeta_d^i)|\right)^2 \geq \frac{1}{d} \sum_{1 \leq i \leq d} |B(\zeta_d^i)|^2.$$

Since

$$|B(\zeta_d^i)|^2 = \sum_{0 \leq k,l \leq n} b_k \zeta_d^{ik} \overline{b_l \zeta_d^{il}} = \sum_{0 \leq k,l \leq n} b_k \overline{b_l} \zeta_d^{i(k-l)},$$

we conclude that

$$\left(\max_{1 \leq i \leq d} |B(\zeta_d^i)|\right)^2 \geq \frac{1}{d} \sum_{0 \leq k,l \leq n} b_k \overline{b_l} \sum_{1 \leq i \leq d} \zeta_d^{i(k-l)}.$$

The inner-most sum vanishes unless $d \mid k - l$. Because $n < d$ this can only occur if $k = l$, in which case the inner-most sum equals d. Using this and taking the square-root completes the proof in this case.

Now we note that the same estimate holds true if, $D(\boldsymbol{\xi}) = 0$ i.e., if, after relabeling, $\xi_i = u\zeta_d^i$ for some fixed u on the unit circle. Indeed, $B(\xi_i) = B(u\zeta_d^i) = \tilde{B}(\zeta_d^i)$ with $\tilde{B}(x) = b_0 + b_1 ux + \cdots + b_n u^n x^n$, and the coefficient vectors of B and $\tilde{B}$ both have the same ℓ^2-norm.

Next we consider the general case. Using the trivial estimate and the previous special case we obtain

$$\begin{aligned}\max_{1 \leq i \leq d} |B(\xi_i)| &\geq \sup_{\phi} \left(\max_{1 \leq j \leq d} |B(\phi(\zeta_d^j))| - \max_{1 \leq j \leq d} \min_{1 \leq i \leq d} |B(\xi_i) - B(\phi(\zeta_d^j))|\right) \\ &\geq \sqrt{\sum_{0 \leq i \leq n} |b_i|^2} - \inf_{\phi} \max_{1 \leq j \leq d} \min_{1 \leq i \leq d} |B(\xi_i) - B(\phi(\zeta_d^j))|.\end{aligned}$$

Hence, it suffices to show

$$\inf_{\phi} \max_{1 \leq j \leq d} \min_{1 \leq i \leq d} |B(\xi_i) - B(\phi(\zeta_d^j))| \leq D(\boldsymbol{\xi}) d^{3/2} \max_{1 \leq i \leq d} \{1, |\xi_i|\}^{d-2} \sqrt{\sum_{0 \leq i \leq n} |b_i|^2}. \tag{3.3}$$

Let γ denote the (complex) line segment connecting a point $\zeta \in \mathbb{T}$ with an arbitrary point ξ. We use the parametrisation $\gamma(\tau) = (1-\tau)\zeta + \tau\xi$ with $\tau \in [0,1]$. By using the complex line integral of B along γ, we write $B(\xi) - B(\zeta) = \int_\gamma B'(\tau)\,\mathrm{d}\tau$. Hence,

$$|B(\xi) - B(\zeta)| \le |\xi - \zeta| \max_{\tau\in[0,1]} |B'(\gamma(\tau))|. \tag{3.4}$$

Further, as $|\zeta| \le \max\{1, |\xi|\}$, we have $|\gamma(\tau)| \le \max\{1, |\xi|\}$ for any $\tau \in [0,1]$. By the Cauchy–Schwarz inequality,

$$|B'(\gamma(\tau))| \le n \sqrt{\sum_{0\le i<n} |\gamma(\tau)|^{2i}} \sqrt{\sum_{0\le i<n} |b_i|^2} \le d^{3/2} \max\{1,|\xi|\}^{d-2} \sqrt{\sum_{0\le i\le n} |b_i|^2}. \tag{3.5}$$

Combining (3.4) and (3.5), with $\xi = \xi_i$ and $\zeta = \phi(\zeta_d^j)$, while using $\max\{1,|\xi_i|\} \le \max_{1\le i\le d}\{1,|\xi_i|\}$ the inequality (3.3) drops out. This proves the lemma. □

Next we derive a corollary. Recall that $\|\cdot\|$ denotes the maximum norm on $\mathbb{C}^d$.

Corollary 8. *Let $\boldsymbol{\xi} \in \mathbb{C}^d \setminus \{\mathbf{0}\}$ and suppose that $B(x) = b_0 + b_1 x + \cdots + b_n x^n \in \mathbb{C}[x]$ has degree strictly less than d. We have*

$$\max_{1\le i\le d} |B(\xi_i)| \ge \left(1 - d^{3/2} D\left(\frac{\boldsymbol{\xi}}{\|\boldsymbol{\xi}\|}\right)\right) \sqrt{\sum_{0\le i\le n} |b_i \cdot \|\boldsymbol{\xi}\|^i|^2},$$

If $n = 0$ then we can omit the first factor.

Proof. Apply Lemma 7 with $\boldsymbol{\xi}$ and $B(x) = b_0 + b_1 x + \cdots + b_n x^n$ replaced by $\frac{\boldsymbol{\xi}}{\|\boldsymbol{\xi}\|}$ and $\tilde{B}(x) = B(\|\boldsymbol{\xi}\|x)$. □

Decomposing $B(x) = \sum_{j\in J} b_j x^j + \sum_{k\notin J} b_k x^k$ and applying Lemma 7 or Corollary 8 to $\sum_{j\in J} b_j x^j$ sometimes allows to produce non-trivial results, even when $\deg B \ge d$. Let us record here only the special case when all ξ_i lie on the unit circle.

Corollary 9. *Let $\boldsymbol{\xi} \in \mathbb{T}^d$, and suppose $B(x) = b_0 + b_1 x + \cdots + b_n x^n \in \mathbb{C}[x]$. Then for every $I \subset \{1,2,\dots,d\}$ and each non-empty $J \subset \{0,1,\dots,n\}$ with $\max_{j\in J} j - \min_{j\in J} j < \#I$ we have*

$$\max_{1\le i\le d} |B(\xi_i)| \ge \left(1 - (\#I)^{3/2} D_I\right) \sqrt{\sum_{j\in J} |b_j|^2} - \sum_{k\notin J} |b_k|.$$

where $D_I = D((\xi_i)_{i\in I})$.

4. Application to Galois orbits and lower bounds for the house

In this section we apply the results of Section 3 to a diophantine setting. Throughout this section let K be subfield of $\overline{\mathbb{Q}}$, $\xi \in \overline{\mathbb{Q}}\setminus\{0\}$, and $d = [K(\xi) : K]$. Let us write

$$M_{\xi,K} = (x - \xi_1)\cdots(x - \xi_d) \in K[x]$$

for the monic minimal polynomial of ξ over K with conjugates $\xi_1, \dots, \xi_d$ over K. Recall that

$$\mathrm{Hom}(K) = \{\sigma : K \to \mathbb{C};\ \text{field homomorphism}\}.$$

For each $\sigma \in \mathrm{Hom}(K)$ let

$$\tau_{1,\sigma}, \dots, \tau_{d,\sigma} \in \mathrm{Hom}(K(\xi))$$

be the d extensions of σ from K to $K(\xi)$, and set

$$\boldsymbol{\tau}_\sigma(\xi) = (\tau_{1,\sigma}(\xi), \dots, \tau_{d,\sigma}(\xi)).$$

So the components of $\boldsymbol{\tau}_\sigma(\xi)$ are precisely the d distinct roots of the irreducible polynomial $\sigma(M_{\xi,K}) \in \sigma(K)[x]$.

Lemma 10. *Let $\mathcal{O}$ be a subring of $\mathcal{O}_K$, and let ξ be a non-zero algebraic integer. Suppose that $P(x) = a_0 + a_1 x + \cdots + a_n x^n \in \mathcal{O}[x]$ is of degree $1 \le n < d$, and let $\alpha = P(\xi)$. Then we have*

$$|\overline{\alpha}| \ge \max_{\sigma\in\mathrm{Hom}(K)} \left(1 - d^{3/2} D\left(\frac{\boldsymbol{\tau}_\sigma(\xi)}{\|\boldsymbol{\tau}_\sigma(\xi)\|}\right)\right) |\sigma(a_n)|\,\|\boldsymbol{\tau}_\sigma(\xi)\|^n$$

Proof. We have

$$|\overline{\alpha}| \ge \max_{\sigma\in\mathrm{Hom}(K)} \max_{\substack{\tau\in\mathrm{Hom}(K(\xi))\\ \tau|_K=\sigma}} |\tau(P(\xi))| = \max_{\sigma\in\mathrm{Hom}(K)} \max_{\substack{\tau\in\mathrm{Hom}(K(\xi))\\ \tau|_K=\sigma}} |\sigma(P)(\tau(\xi))|\,.$$

Due to Corollary 8, the right hand side is at least

$$\max_{\sigma\in\mathrm{Hom}(K)} \left(1 - d^{3/2} D\left(\frac{\boldsymbol{\tau}_\sigma(\xi)}{\|\boldsymbol{\tau}_\sigma(\xi)\|}\right)\right) \left(\sum_{0\le k\le n} \left|\sigma(a_k)\|\boldsymbol{\tau}_\sigma(\xi)\|^k\right|^2\right)^{1/2}. \tag{4.6}$$

The second factor is, trivially, at least $|\sigma(a_n)|\,\|\boldsymbol{\tau}_\sigma(\xi)\|^n$, which proves the claim. □

We now introduce a new invariant for the tuple (K,ξ).

Definition 3. *Let*

$$\eta(K,\xi)=\min_{\sigma\in\mathrm{Hom}(K)}\min\{\|\boldsymbol{\tau}_\sigma(\xi)\|,\|\boldsymbol{\tau}_\sigma(\xi)\|^{d-1}\}\left(1-d^{3/2}D\left(\frac{\boldsymbol{\tau}_\sigma(\xi)}{\|\boldsymbol{\tau}_\sigma(\xi)\|}\right)\right). \tag{4.7}$$

Remark 2. *The quantity $\eta(K,\xi)$ can also be expressed in terms of the minimal polynomial $M_{\xi,K}$. For $M=(x-\alpha_1)\cdots(x-\alpha_d)\in\mathbb{C}[x]\backslash\mathbb{C}$, not a monomial, define*

$$\eta_0(M)=\min\{\|\boldsymbol{\alpha}\|,\|\boldsymbol{\alpha}\|^{d-1}\}\left(1-d^{3/2}D\left(\frac{\boldsymbol{\alpha}}{\|\boldsymbol{\alpha}\|}\right)\right),$$

where $\boldsymbol{\alpha}=(\alpha_1,\dots,\alpha_d)\in\mathbb{C}^d\setminus\{\mathbf{0}\}$. We have

$$\eta(K,\xi)=\min_{\sigma\in\mathrm{Hom}(K)}\eta_0(\sigma(M_{\xi,K})),$$

where $\sigma(M_{\xi,K})$ is understood to be applied coefficient-wise. This shows that $\eta(K,\xi)=\eta(L,\xi)$ whenever $M_{\xi,K}=M_{\xi,L}$. In particular, if $[K(\xi):K]=[\mathbb{Q}(\xi):\mathbb{Q}]$ then

$$\eta(K,\xi)=\eta(\mathbb{Q},\xi). \tag{4.8}$$

Our next result is key to determine the Northcott number of various examples, including those leading to Theorem 1. Roughly speaking, $\eta(\cdot,\cdot)$ will provide a lower bound for the house of "new elements" in extensions as required to apply the general Lemma 6.

Proposition 11 (Key Proposition)**.** *Let $\mathcal{O}$ be a subring of $\mathcal{O}_K$, let ξ be an algebraic integer but not in $\mathcal{O}_K$, and suppose that $M_{\xi,K}\in\mathcal{O}[x]$. Then we have*

$$\delta_{\overline{\cdot}}(\mathcal{O}[\xi]\backslash\mathcal{O})\geq\eta(K,\xi).$$

Proof. Suppose $\alpha\in\mathcal{O}[\xi]\backslash\mathcal{O}$. Since $M_{\xi,K}\in\mathcal{O}[x]$ we can choose a polynomial $P\in\mathcal{O}[x]$ of degree $1\leq n<d$ such that $\alpha=P(\xi)=a_0+a_1\xi+\cdots+a_n\xi^n$. From Lemma 10 we conclude that

$$\overline{|\alpha|}\geq\max_{\sigma\in\mathrm{Hom}(K)}\left(1-d^{3/2}D\left(\frac{\boldsymbol{\tau}_\sigma(\xi)}{\|\boldsymbol{\tau}_\sigma(\xi)\|}\right)\right)|\sigma(a_n)|\,\|\boldsymbol{\tau}_\sigma(\xi)\|^n.$$

As $a_n\neq0$ is an algebraic integer there exists $\sigma\in\mathrm{Hom}(K)$ such that $|\sigma(a_n)|\geq1$. Noticing that $\|\boldsymbol{\tau}_\sigma(\xi)\|^n\geq\min\{\|\boldsymbol{\tau}_\sigma(\xi)\|,\|\boldsymbol{\tau}_\sigma(\xi)\|^{d-1}\}$ completes the proof. □

For any eligible choices of K the hypothesis ensures that the minimal polynomials of ξ over K, and over the field of fractions of $\mathcal{O}$ are identical. Hence $\eta(K,\xi)$ is independent of the particular choice of K.

5. Proof of Theorem 2

With Proposition 11 at hand, we can now easily prove Theorem 2. Recall that $(\xi_i)_i$ is a sequence of algebraic integers, $\mathcal{O}_0$ is a ring (containing 1) of algebraic integers,

$$\mathcal{O}_i=\mathcal{O}_0[\xi_1,\dots,\xi_i],\ \text{and}\ \mathcal{O}=\bigcup_{i\geq1}\mathcal{O}_i=\mathcal{O}_0[\xi_1,\xi_2,\xi_3,\dots].$$

The field K_i is the field of fractions of $\mathcal{O}_i$, and by hypothesis $M_{\xi_i,K_{i-1}}\in\mathcal{O}_{i-1}[x]$. Moreover, $d_i=[K_{i-1}(\xi_i):K_{i-1}]>1$, and thus $\xi_i\notin\mathcal{O}_{K_{i-1}}$. From Proposition 11 we conclude that

$$\delta_{\overline{\cdot}}(\mathcal{O}_{i-1}[\xi_i]\backslash\mathcal{O}_{i-1})\geq\eta(K_{i-1},\xi_i).$$

Applying Lemma 6 with $A_i=\mathcal{O}_i$ proves Theorem 2.

6. Applications of the equidistribution method

In this section we discuss a few special cases of Theorem 2. Recall that by Northcott's Theorem (cf. Lemma 5) one has $\mathcal{N}_{\overline{\cdot}}(\mathcal{O})=\infty$ for any ring of algebraic integers $\mathcal{O}$, whose field of fractions has finite degree over $\mathbb{Q}$. We will use this fact without further notice.

Corollary 12. *Let $(p_i)_{i\in\mathbb{N}}$, and $(d_i)_{i\in\mathbb{N}}$ be two sequences of primes, and suppose the primes d_i are pairwise distinct. Let $\xi_i=p_i^{1/d_i}$ be any d_i-th root, $\mathcal{O}_i=\mathbb{Z}[\xi_1,\dots,\xi_i]$, K_i be the field of fractions of $\mathcal{O}_i$, and let $\mathcal{O}=\cup_{i\geq1}\mathcal{O}_i$. Then $\eta(K_{i-1},\xi_i)=\eta(\mathbb{Q},\xi_i)=\overline{|\xi_i|}$, and $\mathcal{N}_{\overline{\cdot}}(\mathcal{O})=\liminf_{i\to\infty}\overline{|\xi_i|}$.*

Proof. Let $\liminf_{i\to\infty}\overline{|\xi_i|}=t\in[1,\infty]$. Since the ξ_i are pairwise distinct, we get $\mathcal{N}_{\overline{\cdot}}(\mathcal{O})\leq t$. We derive, by Eisenstein's criterion and the tower-law, that $d_i=[K_{i-1}(\xi_i):K_{i-1}]=[\mathbb{Q}(\xi_i):\mathbb{Q}]$. Hence, $M_{\xi_i,K_{i-1}}=M_{\xi_i,\mathbb{Q}}\in\mathcal{O}_{i-1}[x]$, and $\eta(K_{i-1},\xi_i)=\eta(\mathbb{Q},\xi_i)$. As the d_i conjugates $\overline{|\xi_i|}\,\zeta_{d_i}^j$ of ξ_i are perfectly equidistributed on the circle of radius $\overline{|\xi_i|}$ about the origin, we have $\eta(\mathbb{Q},\xi_i)=\overline{|\xi_i|}$. Thus, Theorem 2 implies $\mathcal{N}_{\overline{\cdot}}(\mathcal{O})\geq t$. □

Next we consider a generalisation of Corollary 12. To reduce clutter we define for each non-zero algebraic number ξ the unordered tuple of normalised conjugates

$$\boldsymbol{c}_\xi = \left(\frac{\xi_1}{\overline{|\xi|}}, \dots, \frac{\xi_d}{\overline{|\xi|}}\right),$$

where $d = [\mathbb{Q}(\xi) : \mathbb{Q}]$ and $\xi_1, \dots, \xi_d$ are the conjugates of ξ. We remind the reader that $D(\boldsymbol{c}_\xi)$ is well-defined as the finite discrepancy $D(\cdot)$ is indifferent to the order of the components. Before stating the result let us informally discuss what we would get by applying Theorem 2 in the most naive way.

Let $(\gamma_i)_i$ be a sequence of algebraic integers of degree m_i, let $\xi_i = \gamma_i^{1/n_i}$ be a n_i-th root of γ_i, and suppose that $[K_{i-1}(\xi_i) : K_{i-1}] = m_i n_i > 1$. Applying Theorem 2 yields

$$\mathcal{N}_{\overline{\cdot}}(\mathcal{O}) \geq \liminf_{i\to\infty} \eta(K_{i-1}, \xi_i) = \liminf_{i\to\infty} \eta(\mathbb{Q}, \xi_i) = \liminf_{i\to\infty} \overline{|\xi_i|}\left(1 - (m_i n_i)^{3/2} D\left(\boldsymbol{c}_{\xi_i}\right)\right).$$

If $\lim_{i\to\infty}(m_i n_i)^{3/2} D\left(\boldsymbol{c}_{\xi_i}\right) = 0$ then we conclude that $\mathcal{N}_{\overline{\cdot}}(\mathcal{O}) = \liminf_{i\to\infty} \overline{|\xi_i|}$. In general the best shot at the former equation we have is Lemma 15 (further below in this section), which ensures the required condition provided

$$\lim_{i\to\infty} m_i^{3/2} n_i^{1/2} D\left(\boldsymbol{c}_{\gamma_i}\right) = 0.$$

Hence, we need the normalised conjugates $\boldsymbol{c}_{\gamma_i}$ of γ_i to converge very rapidly to a perfect equidistribution as i gets large.

However, Theorem 2 allows us to consider the conjugates of γ_i over K_{i-1} and of $\xi_i = \gamma_i^{1/n_i}$ over $K_{i-1}(\gamma_i)$ separately. Now the conjugates of ξ_i over $K_{i-1}(\gamma_i)$ are perfectly equidistributed and those of γ_i over K_{i-1} are much easier to control than those of ξ_i over $\mathbb{Q}$. In this way we can relax the above condition to

$$m_i^{3/2} D\left(\boldsymbol{c}_{\gamma_i}\right) \leq 1 - \overline{|\gamma_i|}^{1/n_i - 1}$$

for all sufficiently large i. Here is the precise result.

Corollary 13. *Let $(\gamma_i)_i$ be a sequence of algebraic integers of respective degree m_i with conjugates $\gamma_1^{(i)}, \dots, \gamma_{m_i}^{(i)}$, and with smallest (complex) absolute value $s_i = \min_{1\leq j\leq m_i} |\gamma_j^{(i)}|$. Let $(n_i)_i$ be a sequence of integers > 1, and choose a n_i-th root γ_i^{1/n_i} for each $i \in \mathbb{N}$. Suppose that the following four properties hold*

- *$[\mathbb{Q}(\gamma_1^{1/n_1}, \dots, \gamma_i^{1/n_i}) : \mathbb{Q}] = m_1 n_1 \cdots m_i n_i$ for each $i \in \mathbb{N}$.*
- *$D(\boldsymbol{c}_{\gamma_i}) \leq m_i^{-3/2}(1 - \overline{|\gamma_i|}^{1/n_i - 1})$ for all but finitely many i.*
- *$s_i \geq 1$ for all but finitely many i.*
- *$\lim_{i\to\infty} \left(\dfrac{s_i}{\overline{|\gamma_i|}}\right)^{1/n_i} = 1$.*

With $\mathcal{O} = \mathbb{Z}[\gamma_i^{1/n_i}; i \in \mathbb{N}]$, we have

$$\mathcal{N}_{\overline{\cdot}}(\mathcal{O}) = \liminf_{i\to\infty} \overline{|\gamma_i|}^{1/n_i}.$$

Proof. Let $i_0 \geq 1$ be an index such that the second and third condition is satisfied for all $i \geq i_0$, and set $\mathcal{O}_0 = \mathbb{Z}[\gamma_i^{1/n_i}; i \leq i_0]$. For $i \in \mathbb{N}$ set $\xi_{2i-1} = \gamma_{i+i_0}$ and $\xi_{2i} = \gamma_{i+i_0}^{1/n_{i+i_0}}$. Since $n_{i+i_0} > 1$ the ξ_{2i} are pairwise distinct. Thus $\mathcal{N}_{\overline{\cdot}}(\mathcal{O}) \leq \liminf_{i\to\infty} \overline{|\xi_{2i}|}$.

To prove the reversed inequality we let $\mathcal{O}_i = \mathcal{O}_0[\xi_1, \dots, \xi_i]$, and we write K_i for the field of fractions of $\mathcal{O}_i$. The first condition implies that the minimal polynomial $M_{\xi_{2i-1}, K_{2i-2}}$ has coefficients in $\mathbb{Z}$, and that

$$M_{\xi_{2i}, K_{2i-1}} = x^{n_{i+i_0}} - \gamma_{i+i_0} = x^{n_{i+i_0}} - \xi_{2i-1}.$$

Hence, $M_{\xi_i, K_{i-1}} \in \mathcal{O}_{i-1}[x]$ for all $i \in \mathbb{N}$, and $\xi_{2i} \notin \mathcal{O}_{K_{2i-1}}$ as $n_{i+i_0} > 1$. Moreover, if $\xi_{2i-1} \in \mathcal{O}_{K_{2i-2}}$ then $m_{i+i_0} = 1$ and thus $\xi_{2i-1} \in \mathbb{Z} \subset \mathcal{O}_0$. Hence, we can assume without loss of generality that ξ_i is not in $\mathcal{O}_{K_{i-1}}$. Thus we can apply Theorem 2, and it suffices to check that $\liminf_{i\to\infty} \eta(K_{i-1}, \xi_i) \geq \liminf_{i\to\infty} \overline{|\xi_{2i}|}$. Note that the first condition implies $[K_{2i-2}(\xi_{2i-1}) : K_{2i-2}] = [\mathbb{Q}(\xi_{2i-1}) : \mathbb{Q}]$. Hence,

$$\eta(K_{2i-2}, \xi_{2i-1}) = \eta(\mathbb{Q}, \xi_{2i-1}) = \overline{|\gamma_{i+i_0}|}\left(1 - m_{i+i_0}^{3/2} D\left(\boldsymbol{c}_{\gamma_{i+i_0}}\right)\right) \geq \overline{\left|\gamma_{i+i_0}^{1/n_{i+i_0}}\right|} = \overline{|\xi_{2i}|},$$

using the second hypothesis.

Finally, note that by the first hypothesis each $\sigma \in \mathrm{Hom}(K_{2i-1})$ maps γ_{i+i_0} to a conjugate $\gamma_k^{(i+i_0)}$, and thus the n_{i+i_0} extensions $\tau_{j,\sigma}$ of σ map ξ_{2i} to the n_{i+i_0} perfectly equidistributed complex numbers ${\gamma_k^{(i+i_0)}}^{1/n_{i+i_0}} \zeta_{n_{i+i_0}}^l$ with $1 \leq l \leq n_{i+i_0}$. Therefore, using the third condition, we have for all $i \in \mathbb{N}$ that

$$\eta(K_{2i-1}, \xi_{2i}) = \min\left\{s_{i+i_0}, s_{i+i_0}^{n_{i+i_0}-1}\right\}^{1/n_{i+i_0}} = s_{i+i_0}^{1/n_{i+i_0}} = \left(\frac{s_{i+i_0}}{\overline{|\gamma_{i+i_0}|}}\right)^{1/n_{i+i_0}} \overline{|\gamma_{i+i_0}|}^{1/n_{i+i_0}} = \left(\frac{s_{i+i_0}}{\overline{|\gamma_{i+i_0}|}}\right)^{1/n_{i+i_0}} \overline{|\xi_{2i}|},$$

and thus by the fourth hypothesis $\liminf_{i\to\infty} \eta(K_{2i-1}, \xi_{2i}) \geq \liminf_{i\to\infty} \overline{|\xi_{2i}|}$.

□

Next, note that Theorem 2 requires that all $\mathcal{O}_i$ have the Northcott property (with respect to the $\boxed{\cdot}$) but nonetheless $\mathcal{O}_0$ may be an infinite ring extension of $\mathbb{Z}$. J. Robinson has shown that the ring of integers of the composite field of all real quadratic number fields has the Northcott property. Various other examples have been established in [3, 6, 7, 30]. Theorem 2 allows to extend the Northott property of these rings $\mathcal{O}_0$ to certain bigger rings, or alternatively, to create ring extensions of these $\mathcal{O}_0$ with prescribed finite Northcott number.

Corollary 14. *Let $\mathcal{O}_0$ be a ring of algebraic integers, and let $(\xi_i)_i$ be pairwise distinct algebraic integers. Let $\mathcal{O}_i = \mathcal{O}_0[\xi_1, \dots, \xi_i]$, let K_i be the field of fractions of $\mathcal{O}_i$, set $\mathcal{O} = \mathcal{O}_0[\xi_1, \xi_2, \xi_3, \dots]$, and suppose that $[K_{i-1}(\xi_i) : K_{i-1}] = [\mathbb{Q}(\xi_i) : \mathbb{Q}] = d_i > 1$. If $\mathcal{N}_{\boxed{\cdot}}(\mathcal{O}_{K_0}) = \infty$, then*

$$\liminf_{i\to\infty} \boxed{\xi_i} \geq \mathcal{N}_{\boxed{\cdot}}(\mathcal{O}) \geq \liminf_{i\to\infty} \boxed{\xi_i} \left(1 - d_i^{3/2} D\left(\boldsymbol{c}_{\xi_i}\right)\right).$$

In particular, if n_i are rational integers, and the elements $\xi_i = n_i^{1/d_i}$ have degree $d_i > 1$ over K_{i-1}, then $\mathcal{N}_{\boxed{\cdot}}(\mathcal{O}) = \liminf_{i\to\infty} |n_i^{1/d_i}|$.

Proof. First note that by Lemma 5 (cf. Remark 1) we conclude $\mathcal{N}_{\boxed{\cdot}}(\mathcal{O}_{K_i}) = \infty$, and thus $\mathcal{N}_{\boxed{\cdot}}(\mathcal{O}_i) = \infty$. Now note that

$$\eta(K_{i-1}, \xi_i) = \eta(\mathbb{Q}, \xi_i) = \boxed{\xi_i} \left(1 - d_i^{3/2} D\left(\boldsymbol{c}_{\xi_i}\right)\right),$$

and $M_{\xi_i, K_{i-1}} = M_{\xi_i, \mathbb{Q}}$. Hence, the claim follows from Theorem 2. □

We end this section with some simple estimates for the finite discrepancy, one of which (Lemma 15) has already been used in the paragraph after the proof of Corollary 12 to motivate Corollary 13. We believe they might also be useful for further applications of Theorem 2. We first introduce a slight refinement of the finite discrepancy. For $\boldsymbol{\xi} \in \mathbb{C}^d$ and $u \in \mathbb{T}$ we set

$$D_u(\boldsymbol{\xi}) = \max_{1\leq j\leq d} \min_{1\leq i\leq d} |\xi_i - u\zeta_d^j|,$$

so that

$$D(\boldsymbol{\xi}) = \inf_{u\in\mathbb{T}} D_u(\boldsymbol{\xi}). \tag{6.9}$$

For fixed $\boldsymbol{\xi} \in \mathbb{C}^d$ the function $D_u(\boldsymbol{\xi})$ is continuous in u. Since $\mathbb{T}$ is compact if follows that there exists $u \in \mathbb{T}$ such that

$$D(\boldsymbol{\xi}) = D_u(\boldsymbol{\xi}). \tag{6.10}$$

Lemma 15. *Let $n, m \in \mathbb{N}$. Let $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_m) \in \mathbb{C}^m$, and suppose that $D(\boldsymbol{\alpha}) \leq 1/2$. Then we have*

$$D\left(\left(\left(\alpha_k\right)^{1/n} \zeta_n^\ell\right)_{\substack{1\leq k\leq m\\ 1\leq \ell\leq n}}\right) \leq \frac{2}{n} D(\boldsymbol{\alpha}).$$

Note that the left hand-side is independent of the particular choice of the n-th root.

Proof. By (6.10) we can choose $u \in \mathbb{T}$ such that $D_u(\boldsymbol{\alpha}) = D(\boldsymbol{\alpha})$. By (6.9) it suffices to prove that

$$D_{u^{1/n}}\left(\left(\left(\alpha_k\right)^{1/n} \zeta_n^\ell\right)_{\substack{1\leq k\leq m\\ 1\leq \ell\leq n}}\right) \leq \frac{2}{n} D_u(\boldsymbol{\alpha}). \tag{6.11}$$

After reordering the coordinates of $\boldsymbol{\alpha}$, there exists for each $1 \leq k \leq m$ a real number δ satisfying $0 \leq \delta \leq D_u(\boldsymbol{\alpha}) \leq \frac{1}{2}$, and a real number φ such that

$$\alpha_k = \zeta_m^k u + \delta e^{i\varphi}.$$

Thus we obtain

$${\alpha_k}^{1/n} = (\zeta_m^k u + \delta e^{i\varphi})^{1/n} = \zeta_{mn}^{k+sm}\, u^{1/n}\, (1 + \delta e^{i\varphi'})^{1/n},$$

for some integer $1 \leq s \leq n$ and some real number φ'. The generalised binomial theorem implies for any complex number z, with $|z| < 1$, the Taylor expansion

$$(1+z)^{1/n} = 1 + \sum_{r\geq 1} \binom{1/n}{r} z^r \quad \text{where} \quad \binom{1/n}{r} = \frac{\frac{1}{n}\left(\frac{1}{n}-1\right)\dots\left(\frac{1}{n}-(r-1)\right)}{r!} \text{ for } r \geq 1.$$

Further,

$$\left|\binom{1/n}{r}\right| = \frac{\frac{1}{n}}{1} \frac{\left(1-\frac{1}{n}\right)}{2} \cdots \frac{(r-1)-\frac{1}{n}}{r} \leq \frac{1}{n} \quad (r \geq 1).$$

Consequently,

$$\left|(1 + \delta e^{i\varphi'})^{1/n} - 1\right| \leq \frac{1}{n} \sum_{r\geq 1} \delta^r = \frac{1}{n} \frac{\delta}{(1-\delta)} \leq \frac{2\delta}{n}.$$

Finally, note that for any fixed $1 \le s \le n$, the expression $\zeta_{mn}^{k+sm} u^{1/n} \zeta_n^{\ell} = \zeta_{mn}^{k+(\ell+s)m} u^{1/n}$, for $1 \le k \le m$ and $1 \le \ell \le n$ ranges over all elements $\zeta_{nm}^{r} u^{1/n}$ for $1 \le r \le mn$. This leads to (6.11), hence gives the result. □

Lemma 16. *Let $n, m \in \mathbb{N}$ be coprime, let $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_m) \in \mathbb{C}^m$, $\boldsymbol{\beta} = (\beta_1, \dots, \beta_n) \in \mathbb{C}^n$. Then we have*

$$D\left((\alpha_k \beta_l)_{\substack{1 \le k \le m \\ 1 \le l \le n}}\right) \le (1 + D(\boldsymbol{\alpha}))(1 + D(\boldsymbol{\beta})) - 1.$$

Proof. By (6.10) we can choose $u, u' \in \mathbb{T}$ such that $D_u(\boldsymbol{\alpha}) = D(\boldsymbol{\alpha})$ and $D_{u'}(\boldsymbol{\beta}) = D(\boldsymbol{\beta})$. By (6.9) it suffices to prove that

$$D_{uu'}((\alpha_k \beta_\ell)_{\substack{1 \le k \le m \\ 1 \le \ell \le n}}) \le (1 + D_u(\boldsymbol{\alpha}))(1 + D_{u'}(\boldsymbol{\beta})) - 1. \tag{6.12}$$

After reordering the coordinates of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ we can write $\alpha_k = \zeta_m^k u + \delta_k e^{i\varphi_k}$ and $\beta_\ell = \zeta_n^\ell u' + \delta'_\ell e^{i\varphi'_\ell}$ for certain non-negative real δ_k, δ'_ℓ and φ_k, φ'_ℓ satisfying $0 \le \delta_k \le D_u(\boldsymbol{\alpha})$, $0 \le \delta'_\ell \le D_{u'}(\boldsymbol{\beta})$.

Further

$$(\zeta_m^k u + \delta_k e^{i\varphi_k})(\zeta_n^\ell u' + \delta'_\ell e^{i\varphi'_\ell}) = \zeta_{mn}^{kn+\ell m} uu' + \epsilon_{k,\ell}$$

where $|\epsilon_{k,\ell}| \le \delta_k + \delta'_\ell + \delta_k \delta'_\ell \le (1 + D_u(\boldsymbol{\alpha}))(1 + D_{u'}(\boldsymbol{\beta})) - 1$. By coprimality of m and n, the exponent $kn + \ell m$ represents all residue classes modulo mn as $1 \le k \le m$ and $1 \le \ell \le n$. This proves (6.12), and hence the lemma. □

7. From arbitrary rings of algebraic integers to integrally closed subrings

In this section we record two classical results that provide a strategy to show that a ring of algebraic numbers is integrally closed, i.e., is the ring of integers of a field.

In the first step we need to identify when a given algebraic integer θ of K generates the ring of integers $\mathcal{O}_K$ over $\mathbb{Z}$. There are at least three such criteria in the literature - Dedekind's, Uchida's and Lüneburg's criterion (cf. [26]). For our purpose Dedekind's classical criterion is well suited (cf. [14, Theorem 1.1]).

Lemma 17 (Dedekind's criterion)**.** *Let θ be an algebraic integer, and let $M_{\theta,\mathbb{Q}}$ be its minimal polynomial over $\mathbb{Q}$. Let $K = \mathbb{Q}(\theta)$, and let q be a rational prime. Let $\overline{M_{\theta,\mathbb{Q}}}$ be the reduction of $M_{\theta,\mathbb{Q}}$ modulo q and $\overline{M_{\theta,\mathbb{Q}}} = \varphi_1^{e_1} \dots \varphi_k^{e_k}$ be the decomposition of $\overline{M_{\theta,\mathbb{Q}}}$ into irreducible factors over the ring of polynomials $\mathbb{F}_q[x]$ over the field with q elements. Let $\mu_i, g \in \mathbb{Z}[x]$ be such that*

$$M_{\theta,\mathbb{Q}} = \mu_1^{e_1} \dots \mu_k^{e_k} + qg$$

and $\overline{\mu}_i = \varphi_i$ for all $i \le k$. The following are equivalent:

1. *The prime q does not divide $[\mathcal{O}_K : \mathbb{Z}[\theta]]$.*
2. *For all $i \le k$, either $e_i = 1$, or $\varphi_i \nmid \overline{g}$ in $\mathbb{F}_q[x]$.*

Next we require a criterion to factor the ring of integers of a compositum into a product of rings of integers. If $\mathcal{O}$ is an order of a number field K then we write $\Delta_{\mathcal{O}}$ for the discriminant of that order, and just Δ_K if $\mathcal{O} = \mathcal{O}_K$ is the maximal order. For subfields K and F of $\overline{\mathbb{Q}}$ we write KF for their composite field. The next statement can be found in [9, Proposition III.2.13].

Lemma 18. *If F, K are number fields and linearly disjoint over $\mathbb{Q}$, with $(\Delta_F, \Delta_K) = 1$, then $\mathcal{O}_{KF} = \mathcal{O}_K \cdot \mathcal{O}_F$ where the right hand side denotes the smallest subring of $\mathcal{O}_{\overline{\mathbb{Q}}}$ containing $\mathcal{O}_K$ and $\mathcal{O}_F$.*

We will apply these general criteria to rings of the form $\mathbb{Z}[p_1^{1/d_1}, p_2^{1/d_2}, p_3^{1/d_3}, \dots]$, where d_i and p_i are suitably chosen primes. To this end we first need to compute the modulus of the discriminant $\Delta_{\mathbb{Z}[p_i^{1/d_i}]}$.

Lemma 19. *Let $n \in \mathbb{N}$, suppose $x^d - n \in \mathbb{Z}[x]$ is irreducible, and let $\theta \in \overline{\mathbb{Q}}$ be one of its roots. We have $\left|\Delta_{\mathbb{Z}[\theta]}\right| = d^d n^{d-1}$.*

Proof. Note that the minimal polynomial $M_{\theta,\mathbb{Q}}$ of θ is given by $\prod_{i \le d} \left(x - \zeta_d^i \theta\right)$. Thus

$$\left|\Delta_{\mathbb{Z}[\theta]}\right| = \left| \prod_{\substack{1 \le i,j \le d \\ i \ne j}} (\zeta_d^j \theta - \zeta_d^i \theta) \right| = |\theta^{d(d-1)}| \left| \prod_{\substack{1 \le i,j \le d \\ i \ne j}} (1 - \zeta_d^{i-j}) \right|. \tag{7.13}$$

Using the basic identity

$$\prod_{i<d} \left(x - \zeta_d^i\right) = \frac{x^d - 1}{x - 1} = \sum_{k=0}^{d-1} x^k,$$

and that for any fixed $i \le d$ the relation $\{\zeta_d^{i-j};\, j \le d,\, j \ne i\} = \{\zeta_d^j;\, j \le d-1\}$ holds, we see that

$$\prod_{\substack{1\le i,j\le d\\ i\ne j}} (1-\zeta_d^{i-j}) = \left(\prod_{i<d}(1-\zeta_d^i)\right)^d = d^d.$$

The latter in conjunction with (7.13) shows that $\left|\Delta_{\mathbb{Z}[\theta]}\right| = n^{d-1}d^d$. □

Next, we apply Dedekind's criterion in our setting. It shows that if p and d are distinct primes and the Fermat quotient $\frac{p^{d-1}-1}{d}$ is not divisible by d then the field $\mathbb{Q}(p^{1/d})$ is monogenic.

Lemma 20. *Let p and d be odd primes. Put $\theta = p^{1/d}$ and $K = \mathbb{Q}(\theta)$. If $d^2 \nmid p^d - p$ then $\mathcal{O}_K = \mathbb{Z}[\theta]$.*

Proof. It is well-known that $\Delta_{\mathbb{Z}[\theta]} = \Delta_K[\mathcal{O}_K : \mathbb{Z}[\theta]]^2$. Hence, in order to rule out that $[\mathcal{O}_K : \mathbb{Z}[\theta]] > 1$, it suffices by Lemma 19 to check that $p, d \nmid [\mathcal{O}_K : \mathbb{Z}[\theta]]$ as $x^d - p$ is irreducible in $\mathbb{Z}[x]$ by Eisenstein's criterion. We use Lemma 17 and its notation, with $M_{\theta,\mathbb{Q}}$ as above. Next we study the two critical cases $q = p$ and $q = d$ separately.

Case $q = p$: Then $\overline{M_{\theta,\mathbb{Q}}} = x^d = \varphi^d$ with $\mu = x$, and $g = -1$. So $\varphi \nmid \overline{g}$.

Case $q = d$: Then $\overline{M_{\theta,\mathbb{Q}}} = \overline{x^d - p} = (x-p)^d = \varphi^d$ with $\mu = x - p$, and

$$g = \frac{x^d - p - (x-p)^d}{d} = \frac{p^d - p}{d} - \sum_{i=1}^{d-1} \frac{\binom{d}{i}}{d} x^{d-i}(-p)^i.$$

So $\varphi \mid \overline{g}$ if and only if $\overline{g}(p) = 0$. Now $\overline{g}(p) = \frac{p^d-p}{d} - 0 = \frac{p^d-p}{d} \ne 0$. □

Next we want to show that for given $T > 1$ and each prime d_i sufficiently large, there exists a prime p_i such that the sequence p_i^{1/d_i} converges to T in a prescribed way (i.e. from above or from below). But we also want the fields $\mathbb{Q}(p_i^{1/d_i})$ to be monogenic, and so we need that the Fermat quotients $\frac{p_i^{d_i-1}-1}{d_i}$ are not divisible by d_i. We use the observation $\frac{(d-1)^{d-1}-1}{d} \equiv 1 \pmod d$, for any odd prime d, which follows from a straightforward computation (see the proof of the next lemma). Thus, to achieve the required monogenicity it suffices to have

$$p_i \equiv d_i - 1 \pmod{d_i^2}. \tag{7.14}$$

Lemma 21. *Let d be an odd prime. We have $\frac{(d-1)^{d-1}-1}{d} \equiv 1 \pmod d$.*

Proof. First, note that

$$(d-1)^{d-1} = \sum_{0\le j\le d-1} \binom{d-1}{j} d^j (-1)^{d-1-j} = (-1)^{d-1} + (d-1)d(-1)^{d-2} + d\sum_{2\le j\le d-1} \binom{d-1}{j} d^{j-1}(-1)^{d-1-j}.$$

Because d is odd, the first two terms simplify to $1 - d(d-1)$. Hence,

$$\frac{(d-1)^{d-1}-1}{d} = 1 + d\left(-1 + \sum_{2\le j\le d-1} \binom{d-1}{j} d^{j-2}(-1)^{d-1-j}\right)$$

which implies the claim. □

A straightforward application of the Siegel–Walfisz Theorem, see [12, Cor. 5.29], guarantees the existence of the required primes p_i in the right interval, and the prescribed residue class.

Lemma 22. *Fix $T > 1$. If d is a sufficiently large prime in terms of T, then there exist a prime p in the interval $(T^d, 2T^d)$ and one prime p in the interval $(T^d/2, T^d)$ such that in both cases $p \equiv d - 1 \pmod{d^2}$.*

Proof. Let $\pi(x; q, a)$ denote the number of primes $p \le x$ that solve the congruence $p \equiv a \pmod q$. Let φ denote Euler's totient function, and let

$$\mathrm{Li}(x) = \int_2^x \frac{\mathrm{d}\tau}{\log \tau}.$$

The Siegel–Walfisz theorem states for any $N > 1$ and $q \ge 1$ we have

$$\pi(x; q, a) = \frac{\mathrm{Li}(x)}{\varphi(q)} + O_N(x(\log x)^{-N})$$

uniformly for all $1 \le a \le q$ with $(a, q) = 1$ for any $x \ge 2$. We will only check that

$$\pi(2T^d; d^2, d-1) - \pi(T^d; d^2, d-1) \ge 2.$$

Here the 2 ensures that we have an element in the open interval $(T^d, 2T^d)$. The case $\pi(T^d; d^2, d-1) - \pi(T^d/2; d^2, d-1) \geq 2$ is shown similarly. Thus, specialising $x = T^d$ and $N = 4$, we infer that

$$\begin{aligned}\pi(2x; d^2, d-1) - \pi(x; d^2, d-1) &= \frac{\mathrm{Li}(2x)}{\varphi(d^2)} - \frac{\mathrm{Li}(x)}{\varphi(d^2)} + O\left(\frac{x}{(\log x)^4}\right) \\ &= \frac{1}{\varphi(d^2)} \int_x^{2x} \frac{\mathrm{d}\tau}{\log \tau} + O\left(\frac{x}{(\log x)^4}\right).\end{aligned}$$

Because $d = (\log x)/\log T$, we have

$$\frac{1}{\varphi(d^2)} \int_x^{2x} \frac{\mathrm{d}\tau}{\log \tau} > \frac{1}{\left(\frac{\log x}{\log T}\right)^2} \frac{x}{\log(2x)} > (\log T)^2 \frac{x}{(1+\log x)^3}.$$

Since d is large (and hence x as well), this implies $\pi(2x; d^2, d-1) - \pi(x; d^2, d-1) \geq 2$. □

We can now deduce:

Lemma 23. *Let $(p_i, d_i)_i$ be a sequence where both components are prime, $\min\{p_{i+1}, d_{i+1}\} > \max\{p_i, d_i\}$ for all i, and such that $\mathcal{O}_{\mathbb{Q}(\xi_i)} = \mathbb{Z}[\xi_i]$ with $\xi_i = p_i^{1/d_i}$. Then with $K = \cup_{i\geq 1}\mathbb{Q}(\xi_1, \ldots, \xi_i)$ and $\mathcal{O}_i = \mathbb{Z}[\xi_1, \ldots, \xi_i]$ we have $\mathcal{O}_K = \bigcup_{i\geq 1} \mathcal{O}_i$.*

Proof. Set $K_i = \mathbb{Q}(\xi_1, \ldots, \xi_i)$. The only primes that ramify in K_{i-1} are $p_1, d_1, \ldots, p_{i-1}, d_{i-1}$. Hence we have $(\Delta_{K_{i-1}}, \Delta_{\mathbb{Q}(\xi_i)}) = 1$. Moreover, K_{i-1} and $\mathbb{Q}(\xi_i)$ are linearly disjoint over $\mathbb{Q}$ since their degrees $d_1 \cdots d_{i-1}$ and d_i are coprime. We conclude from Lemma 18 that $\mathcal{O}_{K_i} = \mathcal{O}_{K_{i-1}} \cdot \mathbb{Z}[\xi_i]$, and hence by induction $\mathcal{O}_{K_i} = \mathbb{Z}[\xi_1, \ldots, \xi_i] = \mathcal{O}_i$. This proves that $\mathcal{O}_K = \cup_{i\geq 1}\mathcal{O}_i$. □

8. Proof of Theorem 1

We use the specific construction given in Corollary 12 with additional constraints on the primes p_i and d_i as required to ensure the specific properties.

(a): Let $(d_i)_i$ be a sequence of strictly increasing primes. Combining Lemma 20, the congruence condition (7.14), and Lemma 22, we see that for each i large enough there exists a prime $p_i \in (t^{d_i}/2, t^{d_i})$ such that $\mathcal{O}_{\mathbb{Q}(p_i^{1/d_i})} = \mathbb{Z}[p_i^{1/d_i}]$. We select a subsequence of $(p_i, d_i)_i$ to ensure $\min\{p_{i+1}, d_{i+1}\} > \max\{p_i, d_i\}$ for all i. Set $\xi_i = p_i^{1/d_i}$, $\mathcal{O}_i = \mathbb{Z}[\xi_1, \ldots, \xi_i]$, and $\mathcal{O} = \cup_i \mathcal{O}_i$. The ξ_i are pairwise distinct, and $2^{-1/d_i}t < \boxed{\xi_i} < t$. Thus $\#\{\alpha \in \mathcal{O}; \boxed{\alpha} < t\} = \infty$, and it follows from Corollary 12 that $\mathcal{N}_{\boxed{\cdot}}(\mathcal{O}) = t$. Finally, by Lemma 23 we see that $\mathcal{O}_K = \mathcal{O}$. This proves the existence of a field K with $\mathcal{N}_{\boxed{\cdot}}(\mathcal{O}_K) = t$ and satisfying (a).

(b): To construct a field K with $\mathcal{N}_{\boxed{\cdot}}(\mathcal{O}_K) = t$ and (b) we proceed in the same manner but choose the primes $p_i \in (t^{d_i}, 2t^{d_i})$, so that by Corollary 12 $\eta(K_{i-1}, \xi_i) = \boxed{\xi_i} \in (t, 2^{1/d_i}t)$. It follows from Proposition 11 that $\#\{\alpha \in \mathcal{O}_K; \boxed{\alpha} \leq t\} < \infty$.

Now suppose $\epsilon > 0$ and $M - t < (t-1)/2$, and suppose $\alpha \in \mathcal{O}$ with $t + \epsilon < \boxed{\alpha} < M$. Clearly, there are only finitely many such α in $\mathcal{O}_0$, so we can assume $\alpha \notin \mathcal{O}_0$. Thus $\alpha \in \mathcal{O}_i \backslash \mathcal{O}_{i-1}$ for some $i \in \mathbb{N}$. Note that $d_i = [\mathbb{Q}(\xi_i) : \mathbb{Q}]$. Hence, there exists $P = \sum_k a_k x^k \in \mathcal{O}_{i-1}[x]$ of degree $1 \leq n < d_i$ such that $\alpha = P(\xi_i)$. Since $M_{\xi_i, K_{i-1}}(x) = M_{\xi_i, \mathbb{Q}}(x) = x^{d_i} - p_i \in \mathbb{Q}[x]$ it follows that the d_i distinct roots of $\sigma(M_{\xi_i, K_{i-1}})(x)$ are perfectly equidistributed on the circle $|z| = \boxed{\xi_i} = p_i^{1/d_i}$, and therefore $D\left(\frac{\boldsymbol{\tau}_\sigma(\xi_i)}{\|\boldsymbol{\tau}_\sigma(\xi_i)\|}\right) = 0$ for all $\sigma \in \mathrm{Hom}(K_{i-1})$. By Lemma 10 we conclude that $\boxed{\alpha} \geq \boxed{\xi_i}^n > t^n$, and thus $t^n < t + (t-1)/2$. This forces $n < 2$ and thus $n = 1$. So $\alpha = a_1\xi_i + a_0$. Again, by Lemma 10 we get that $\boxed{\alpha} \geq |\sigma(a_1)|\boxed{\xi_i}$ for all $\sigma \in \mathrm{Hom}(K_{i-1})$, and thus $\boxed{\alpha} \geq \boxed{a_1} \cdot \boxed{\xi_i}$. It follows that $M > \boxed{\alpha} > \boxed{a_1} \cdot \boxed{\xi_i} > \boxed{a_1} \cdot t$, and thus $\boxed{a_1} < M/t < 1 + (t-1)/(2t) < 1 + (t-1)/2 < t$. Hence there are only finitely many possibilities for $a_1 \in \mathcal{O}$, in particular, there is an element with smallest house > 1. We can assume that M/t is below the smallest house value > 1 for such elements a_1. This forces $\boxed{a_1} = 1$, which in turn implies that all archimedian absolute values are equal to 1 since a_1 is integral.

Next note that the sectors bounded by the d_i rays starting at 0 and joining the conjugates $\boxed{\xi_i}\zeta_{d_i}^j$ partition the complex plane. Consider the sector that contains $\sigma(a_0)$ where $\sigma \in \mathrm{Hom}(K_{i-1})$ is such that $\boxed{a_0} = |\sigma(a_0)|$. Due to the coprimality of the degrees d_i there exists an extension $\tau \in \mathrm{Hom}(K_i)$ of σ that sends $a_1\xi_i$ to a conjugate that lies in the same sector as $\sigma(a_0)$. It follows that $\boxed{\alpha} \geq |\tau(a_1\xi_i + a_0)| \geq |\tau(a_1\xi_i)| + |\cos(2\pi/d_i)\sigma(a_0)|$. We can assume that $d_i > 6$ and thus $\cos(2\pi/d_i) \geq 1/2$. Hence we conclude $M > \boxed{\alpha} \geq \boxed{\xi_i} + \boxed{a_0}/2$. Moreover, we can assume that $M < t + 1/2$. As $\boxed{\xi_i} > t$ and a_0 is integral we conclude that $a_0 = 0$. Thus $t + \epsilon < \boxed{\alpha} = \boxed{\xi_i} \leq 2^{1/d_i}t$, and thus i is bounded in terms of ϵ. Hence, there are only finitely many choices of α, and this proves part (b).

(c): And finally, to construct a field K with the third property we take a sequence $(T_i)_i$ of real numbers that converges to t from above. For each T_i we construct a sequence $(p_{ij}, d_{ij})_j$ with $p_{ij} \in (T_i^{d_{ij}}, 2T_i^{d_{ij}})$ and $\mathcal{O}_{\mathbb{Q}(p_{ij}^{1/d_{ij}})} = \mathbb{Z}[p_{ij}^{1/d_{ij}}]$. A slightly modified Cantor diagonalisation argument allows us to construct a sequence

$(p_m, d_m)_m$ that contains infinitely many elements from each sequence $(p_{ij}, d_{ij})_j$, and satisfies $\min\{p_{m+1}, d_{m+1}\} > \max\{p_m, d_m\}$ for all m. With $\xi_m = p_m^{1/d_m}$ we conclude from Lemma 23 that $\mathcal{O}_K = \mathcal{O} = \mathbb{Z}[\xi_m; m \in \mathbb{N}]$. Now by Corollary 12 we get $\mathcal{N}_{\lceil\cdot\rceil}(\mathcal{O}_K) = t$, and by Proposition 11 we see that $\lceil\alpha\rceil > t$ for all $\alpha \in \mathcal{O}_K \backslash \mathbb{Z}$. And, finally, since each T_i is a limit point of $(\lceil\xi_m\rceil)_m$ we see that $\#\{\alpha \in \mathcal{O}_K; t+\epsilon \leq \lceil\alpha\rceil < M\} = \infty$ for all $M > t$ and all $\epsilon > 0$ sufficiently small. This finishes the proof of Theorem 1.

9. Proofs of Theorem 3 and Theorem 4

Theorem 3 and Theorem 4 both use Lemma 6 applied to a common setting, with the sets A_i defined as follows. Let $(p_i)_i$ and $(d_i)_i$ be sequences of prime numbers. Let p_i^{1/d_i} be any choice of a d_i-th root of p_i. We set $A_i = K_i$ where $K_0 = \mathbb{Q}$ and $K_{i+1} = K_i(p_i^{1/d_i})$, and we write $K = \cup_i K_i$.

Lemma 24. *Let $\gamma \geq 0$, and recall that $h_\gamma(\alpha) = (\deg\alpha)^\gamma h(\alpha)$. Then $\mathcal{N}_{h_\gamma}(K_i) = \infty$ for all $i \in \mathbb{N}_0$. Moreover, if $p_i \notin \{d_1, p_1, \ldots, d_{i-1}, p_{i-1}\}$ then*

$$\delta_{h_\gamma}(K_i \backslash K_{i-1}) \geq d_i^\gamma \left(\frac{\log p_i}{2d_i} - \frac{\log d_i}{2(d_i-1)} \right).$$

Proof. First note that $\mathcal{N}_{h_\gamma}(K_i) = \infty$ by Northcott's Theorem. Next let us prove the inequality. Since only primes in $\{d_1, p_1, \ldots, d_{i-1}, p_{i-1}\}$ can ramify in K_{i-1} we conclude that p_i is unramified in K_{i-1}, and hence $x^{d_i} - p_i$ is an Eisenstein polynomial in $\mathcal{O}_{K_{i-1}}[x]$. Thus, $[K_i : K_{i-1}] = d_i$ is prime, and we conclude that $K_{i-1}(\alpha) = K_i$ for any $\alpha \in K_i \backslash K_{i-1}$. An inequality of Silverman [24, Theorem 2] (see also [30, (5)]) implies that

$$h(\alpha) \geq \frac{\log(N_{K_{i-1}/\mathbb{Q}}(D_{K_i/K_{i-1}}))}{2[K_{i-1} : \mathbb{Q}]d_i(d_i-1)} - \frac{\log d_i}{2(d_i-1)},$$

where $N_{K_{i-1}/\mathbb{Q}}(\cdot)$ denotes the norm and $D_{K_i/K_{i-1}}$ denotes the relative discriminant. A straightforward calculation shows (see [30, Proof of Theorem 4]) that $p_i^{[K_{i-1}:\mathbb{Q}](d_i-1)}$ divides $N_{K_{i-1}/\mathbb{Q}}(D_{K_i/K_{i-1}})$. Hence,

$$h(\alpha) \geq \frac{\log p_i}{2d_i} - \frac{\log d_i}{2(d_i-1)}.$$

Finally, we note that $\deg(\alpha) \geq [K_{i-1}(\alpha) : K_{i-1}] = d_i$, and hence

$$\delta_{h_\gamma}(K_i \backslash K_{i-1}) \geq d_i^\gamma \left(\frac{\log p_i}{2d_i} - \frac{\log d_i}{2(d_i-1)} \right).$$

□

We are now ready for the proof of Theorem 3:

Proof of Theorem 3:
Let $(p_i)_i$ and $(d_i)_i$ be sequences of primes and the p_i strictly increasing such that $\log(p_i)/d_i$ converges to $2t$. It follows that $p_i \notin \{d_1, p_1, \ldots, d_{i-1}, p_{i-1}\}$ for all sufficiently large i. Set $K_0 = \mathbb{Q}$ and $K_i = K_{i-1}(p_i^{1/d_i})$ as at the beginning of this section. Applying Lemma 24 with $\gamma = 0$ we conclude that $\liminf \delta_h(K_i \backslash K_{i-1}) \geq t$, and hence by Lemma 6 we get $\mathcal{N}_h(K) \geq t$. For the remaining inequality we note that p_i^{1/d_i} are all integral (and all distinct for sufficiently large i) with height $\log(p_i)/d_i$ converging to $2t$, and thus we immediately get $\mathcal{N}_h(K) \leq \mathcal{N}_h(\mathcal{O}_K) \leq 2t$. □

For the proof of Theorem 4 we need the following two lemmas.

Lemma 25. *Let $0 \leq \gamma \leq 1$, and let $(p_i)_i$ and $(d_i)_i$ be sequences of prime numbers such that ${d_i}^{\gamma-1}(\log p_i - \log d_i) \to \infty$ as $i \to \infty$, and $p_i \notin \{d_1, p_1, \ldots, d_{i-1}, p_{i-1}\}$ for all $i > i_0$. Then $L = \mathbb{Q}(p_i^{1/d_i}; i \in \mathbb{N})$ is γ-Northcott.*

Proof. Since

$$d_i^\gamma \left(\frac{\log p_i}{2d_i} - \frac{\log d_i}{2(d_i-1)} \right) = \frac{1}{2} \left(\frac{\log p_i - \log d_i}{d_i^{1-\gamma}} - \frac{\log d_i}{d_i^{2-\gamma}(1-1/d_i)} \right) \geq \frac{1}{2} \left(\frac{\log p_i - \log d_i}{d_i^{1-\gamma}} - 1 \right),$$

it follows from Lemma 24 that for $i > i_0$ we have

$$\delta_{h_\gamma}(K_i \backslash K_{i-1}) \geq \frac{1}{2} \left(\frac{\log p_i - \log d_i}{d_i^{1-\gamma}} - 1 \right) \to \infty.$$

Hence, the claim follows from Lemma 6. □

Lemma 26. *Let $0 \leq \gamma \leq 1$, $\epsilon > 0$, and let $(p_i)_i$ and $(d_i)_i$ be sequences of prime numbers such that ${d_i}^{\gamma-1}(\log p_i - \log d_i) \to \infty$ and ${d_i}^{\gamma-\epsilon-1} \log p_i \to 0$ as $i \to \infty$, and $p_i \notin \{d_1, p_1, \ldots, d_{i-1}, p_{i-1}\}$ for all $i > i_0$. Then $L = \mathbb{Q}(p_i^{1/d_i}; i \in \mathbb{N})$ is γ-Northcott but not $(\gamma-\epsilon)$-Bogomolov.*

Proof. Since $h_{\gamma-\epsilon}(p_i^{1/d_i}) = (\deg p_i^{1/d_i})^{\gamma-\epsilon} h(p_i^{1/d_i}) = {d_i}^{\gamma-\epsilon-1} \log p_i \to 0$ the claim follows immediately from Lemma 25. □

Proof of Theorem 4:
Theorem 4 now follows easily from Lemma 26 as all sequences of primes $(d_i)_i$ with $d_{i+1} \geq 2d_i$, and $(p_i)_i$ with $e^{d_i^{1-\gamma+\epsilon/2}} \leq p_i \leq 2e^{d_i^{1-\gamma+\epsilon/2}}$ satisfy the required conditions of Lemma 26. □

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