WEAKLY ADMISSIBLE LATTICES, PRIMITIVE LATTICE POINTS, DIOPHANTINE APPROXIMATION, AND O-MINIMALITY

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Abstract. We generalise M. M. Skriganov's notion of (weak) admissibility for lattices to include standard lattices occurring in Diophantine approximation and algebraic number theory, and we prove estimates for the number of lattice points in sets such as aligned boxes which, in certain cases, improve on Skriganov's celebrated counting results. We establish a criterion under which our error term is sharp and we provide examples in dimensions 2 and 3 using continued fractions. We also establish a similar counting result for primitive lattice points, and apply the latter to the classical problem of Diophantine approximation with primitive points as studied by Chalk, Erdős and others. Finally, we use o-minimality to describe large classes of sets to which our counting results apply.

1. Introduction

In this article we generalise Skriganov's notion of (weak) admissibility for lattices to include standard lattices occurring in Diophantine approximation and algebraic number theory (e.g. ideal lattices), and we prove a sharp estimate for the number of lattice points in sets such as aligned boxes. Our result applies when the lattice is weakly admissible, whereas Skriganov's result requires the dual lattice to be weakly admissible (in his stronger sense). If both of them are weakly admissible then our error term is better provided the lattice is not admissible and the box is sufficiently distorted, which will be made precise later. Our error term also has a good dependence on the geometry of the lattice which allows us to apply a Möbius inversion to get a similar estimate for primitive lattice points. The motivation for this comes from classical results due to Chalk and Erdős [2] as well as more recent work of Dani, Laurent, and Nogueira [3, 4] on inhomogeneous Diophantine approximation by primitive points. Finally, we use o-minimality, a notion from model theory, to describe large classes of sets to which our counting results apply.

Let $\mathcal{S} = (\boldsymbol{m}, \boldsymbol{\beta})$ where $\boldsymbol{m} = (m_1, \ldots, m_n) \in \mathbf{N}^n$, $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_n) \in (0, \infty)^n$, and $n \in \mathbf{N} = \{1, 2, 3, \ldots\}$. We write $\underline{\mathbf{x}} = (\mathbf{x}_1, \ldots, \mathbf{x}_n) \quad (\mathbf{x}_i \in \mathbf{R}^{m_i})$ for the elements in

$$\mathbf{E} := \mathbf{R}^{m_1} \times \cdots \times \mathbf{R}^{m_n},$$

and $|\cdot|$ will be used to denote the Euclidean norm. We set

$$N := \dim \mathbf{E} = \sum_{i=1}^{n} m_i,$$

$$t := \sum_{i=1}^{n} \beta_i,$$

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and we will always assume that $N > 1$. We write

$$\mathrm{Nm}_{\boldsymbol{\beta}}(\underline{\mathbf{x}}) := \prod_{i=1}^{n} |\mathbf{x}_i|^{\beta_i}$$

for the multiplicative $\boldsymbol{\beta}$-norm on $\mathbf{E}$. Let $C \subset \mathbf{E}$ be a coordinate-tuple subspace, i.e.,

$$C = \{\underline{\mathbf{x}} \in \mathbf{E}; \mathbf{x}_i = \mathbf{0} \text{ (for all } i \in I)\}$$

where $I \subset \{1, \ldots, n\}$. We fix such a pair $(\mathcal{S}, C)$, and for $\Gamma \subset \mathbf{E}$ and $\varrho > 0$ we define the quantities

$$\nu(\Gamma, \varrho) := \inf\{\mathrm{Nm}_{\boldsymbol{\beta}}(\underline{\mathbf{x}})^{1/t}; \underline{\mathbf{x}} \in \Gamma \backslash C, |\underline{\mathbf{x}}| < \varrho\},$$
$$\mathrm{Nm}_{\boldsymbol{\beta}}(\Gamma) := \lim_{\varrho \to \infty} \nu(\Gamma, \varrho).$$

As usual we always interpret $\inf \emptyset = \infty$ and $\infty > x$ for all $x \in \mathbf{R}$. The above quantities in the special case when $C = \{\underline{\mathbf{0}}\}$ and $m_i = \beta_i = 1$ (for all $1 \leq i \leq n$) were introduced by Skriganov in [9, 10]. By a lattice in $\mathbf{R}^N$ we always mean a lattice of rank N.

Definition 1. *Let Λ be a lattice in $\mathbf{R}^N$. We say Λ is* weakly admissible *for $(\mathcal{S}, C)$ if $\nu(\Lambda, \varrho) > 0$ for all $\varrho > 0$. We say Λ is* admissible *for $(\mathcal{S}, C)$ if $\mathrm{Nm}_{\boldsymbol{\beta}}(\Lambda) > 0$.*

Note that weak admissibility for a lattice in $\mathbf{R}^N$ depends only on the choice of C whereas admissibility depends on C and $\boldsymbol{\beta}$. Also notice that a lattice Λ in $\mathbf{R}^N$ is weakly admissible (or admissible) in the sense of Skriganov [10] if and only if Λ is weakly admissible (or admissible) for $(\mathcal{S}, C)$ with $C = \{\underline{\mathbf{0}}\}$ and $m_i = \beta_i = 1$ (for all $1 \leq i \leq n$). Let us give some examples to illustrate that our notion of weak admissibility captures new interesting cases not covered by Skriganov's notion of weak admissibility.

Example 1. *Let $\Theta \in Mat_{r \times s}(\mathbf{R})$ be a matrix with r rows and s columns and consider*[1]

$$\Lambda = \begin{bmatrix} I_r & \Theta \\ \mathbf{0} & I_s \end{bmatrix} \mathbf{Z}^{r+s} = \{(\mathbf{p} + \Theta\mathbf{q}, \mathbf{q}); (\mathbf{p}, \mathbf{q}) \in \mathbf{Z}^r \times \mathbf{Z}^s\}. \tag{1.1}$$

We take $n = 2$, $m_1 = r$, $m_2 = s$ and $C = \{(\mathbf{x}_1, \mathbf{x}_2); \mathbf{x}_2 = \mathbf{0}\}$. Then the lattice Λ is weakly admissible for $(\mathcal{S}, C)$ (for every choice of $\boldsymbol{\beta}$) if $\mathbf{p} + \Theta\mathbf{q} \neq \mathbf{0}$ for every $\mathbf{q} \neq \mathbf{0}$. If $\boldsymbol{\beta} = (1, \beta)$ then Λ is admissible for $(\mathcal{S}, C)$ if we have

$$|\mathbf{p} + \Theta\mathbf{q}||\mathbf{q}|^{\beta} \geq c_\Lambda \tag{1.2}$$

for every $(\mathbf{p}, \mathbf{q})$ with $\mathbf{q} \neq \mathbf{0}$ and some fixed $c_\Lambda > 0$. The above lattice Λ naturally arises when considering Diophantine approximations for the matrix Θ (cf. Corollary 1.2). Recall that the matrix Θ is called badly approximable if (1.2) holds true with $\beta = s/r$. W. M. Schmidt [8] has shown that the Hausdorff dimension of the set of badly approximable matrices is full, i.e., rs.

Another example comes from the Minkowski-embedding of, e.g., an ideal in a number field.

Example 2. *Suppose K is a number field with r real and s pairs of complex conjugate embeddings. Let $\sigma : K \to \mathbf{R}^r \times \mathbf{C}^s$ be the Minkowski-embedding and identify $\mathbf{C}$ in the usual way with $\mathbf{R}^2$. Set n=r+s, $C = \{\underline{\mathbf{0}}\}$, $m_i = \beta_i = 1$ for $1 \leq i \leq r$ and $m_i = \beta_i = 2$ for $r+1 \leq i \leq r+s$. Now let $\mathfrak{A} \subset K$ be a free $\mathbf{Z}$-module of rank $N = r+2s$. Then $\Lambda = \sigma\mathfrak{A}$ is admissible in $(\mathcal{S}, C)$. In particular, this generalises the examples of Skriganov for totally real number fields to arbitrary number fields K. Unlike in Skriganov's setting we can also consider cartesian products of such modules $\mathfrak{A}_j$ by using the embedding $\sigma : K^p \to \mathbf{R}^{pr} \times \mathbf{C}^{sp}$ that sends a tuple $\boldsymbol{\alpha}$ to $(\sigma_1(\boldsymbol{\alpha}), \ldots, \sigma_{r+s}(\boldsymbol{\alpha}))$. Now m_i is p if σ_i is real and $2p$ otherwise while n and β_i remain unchanged. Again we get that $\Lambda = \sigma(\mathfrak{A}_1 \times \cdots \times \mathfrak{A}_p)$ is an admissible lattice in $(\mathcal{S}, C)$.*

[1] Despite the row notation we treat the vectors as column vectors.

Now we introduce the sets in which we count the lattice points. Essentially these are the sets that can be transformed into "nice" sets by maps of the form $\phi(\underline{\mathbf{x}}) = (\theta_1\mathbf{x}_1, \ldots, \theta_n\mathbf{x}_n)$ with $\theta_1^{\beta_1}\cdots\theta_n^{\beta_n} = 1$.

For $\boldsymbol{Q} = (Q_1, \ldots, Q_n) \in (0,\infty)^n$ we consider the $\boldsymbol{\beta}$-weighted geometric mean

$$\overline{Q} = \left(\prod_{i=1}^{n} Q_i^{\beta_i}\right)^{1/t},$$

and we assume throughout this note that

$$Q_i \le \overline{Q} \text{ (for all } i \notin I). \tag{1.3}$$

We set

$$Q_{max} := \max_{1\le i\le n} Q_i,$$
$$Q_{min} := \min_{1\le i\le n} Q_i.$$

For $\kappa > 0$ and $M \in \mathbf{N}$ we introduce the family of sets

$$\mathcal{F}_{\kappa,M} := \{S \subset \mathbf{R}^N; \partial(\phi(S)) \in \text{ Lip}(N, M, \kappa\cdot \text{diam}(\phi(S)))\ \forall \phi \in \text{GL}_N(\mathbf{R})\}.$$

Here $\text{GL}_N(\mathbf{R})$ denotes the group of invertible $N\times N$-matrices with real entries, $\text{diam}(\cdot)$ denotes the diameter, $\partial(\cdot)$ denotes the topological boundary, and the notation Lip-$(\cdot,\cdot,\cdot)$ is explained in Definition 2 Section 2.

It is an immediate consequence of [15, Theorem 2.6] that every bounded convex set in $\mathbf{R}^N$ lies in $\mathcal{F}_{\kappa,M}$ for $\kappa = 16N^{5/2}$ and $M = 1$. We will also show (Proposition 7.1) that if $Z \subset \mathbf{R}^{d+N}$ is definable in an o-minimal structure and each fiber $Z_T = \{\underline{\mathbf{x}}; (T,\underline{\mathbf{x}}) \in Z\} \subset \mathbf{R}^N$ is bounded then each fiber Z_T lies in $\mathcal{F}_{\kappa_Z,M_Z}$ for certain constants κ_Z and M_Z depending only on Z but not on T. This result provides another rich source of interesting examples, and might be of independent interest.

For all $1 \le i \le n$ let $\pi_i : \mathbf{E} \to \mathbf{R}^{m_i}$ be the projection defined by $\pi_i(\underline{\mathbf{x}}) = \mathbf{x}_i$. We fix values κ and M, and we assume throughout this article that $Z_{\boldsymbol{Q}} \subset \mathbf{R}^N$ is such that for all $1 \le i \le n$

(1) $Z_{\boldsymbol{Q}} \in \mathcal{F}_{\kappa,M}$,
(2) $\pi_i(Z_{\boldsymbol{Q}}) \subset B_{\mathbf{y}_i}(Q_i)$ for some $\mathbf{y}_i \in \mathbf{R}^{m_i}$.

Here $B_{\mathbf{y}_i}(Q_i)$ denotes the closed Euclidean ball in $\mathbf{R}^{m_i}$ about $\mathbf{y}_i$ of radius Q_i. As is well known (see, e.g., [11]) $\partial(Z_{\boldsymbol{Q}}) \in$ Lip(N, M, L) implies that $Z_{\boldsymbol{Q}}$ is measurable. For $\Gamma \subset \mathbf{E}$ we introduce the quantities

$$\lambda_1(\Gamma) := \inf\{|\underline{\mathbf{x}}|; \underline{\mathbf{x}} \in \Gamma\backslash\underline{\mathbf{0}}\},$$

and

$$\mu(\Gamma,\varrho) := \min\{\lambda_1(\Gamma\cap C), \nu(\Gamma,\varrho)\}.$$

If $\mu(\Gamma,\varrho) = \infty$ then we interpret $1/\mu(\Gamma,\varrho)$ as 0. Finally, we introduce the error term

$$\mathcal{E}_\Lambda(Z_{\boldsymbol{Q}}) := \left| |Z_{\boldsymbol{Q}} \cap \Lambda| - \frac{\text{Vol}Z_{\boldsymbol{Q}}}{\det\Lambda}\right|.$$

Our first result is a sharp upper bound for $\mathcal{E}_\Lambda(Z_{\boldsymbol{Q}})$.

Theorem 1.1. *Suppose Λ is a weakly admissible lattice for $(\mathcal{S}, C)$ and define $c_1 := M((1+\kappa)N^{2N})^N$. Then we have*

$$\mathcal{E}_\Lambda(Z_{\boldsymbol{Q}}) \le c_1 \inf_{0<B\le Q_{max}} \left(\frac{\overline{Q}}{\mu(\Lambda,B)} + \frac{Q_{max}}{B}\right)^{N-1}.$$

Considering suitable homogeneously expanding parallelepipeds it is clear that the error term cannot be improved in this generality. However, the situation becomes more interesting when we restrict the sets $Z_{\boldsymbol{Q}}$ to aligned boxes.

Theorem 1.2. *Suppose $2 \leq n \leq 3$, $m_i = \beta_i = 1$ $(1 \leq i \leq n)$ (hence $N = n$) and $C = \{\underline{\mathbf{x}}; \mathbf{x}_n = \mathbf{0}\}$. Then there exists a lattice Λ, weakly admissible for $(\mathcal{S}, C)$, and a sequence of aligned boxes $Z_{\boldsymbol{Q}} = [-Q_1, Q_1] \times \cdots \times [-Q_n, Q_n]$, increasingly distorted (i.e., $\overline{Q}/Q_{max}$ tends to zero), whose volume $(2\overline{Q})^N$ tends to infinity such that*

$$\mathcal{E}_\Lambda(Z_{\boldsymbol{Q}}) \geq c_{abs} \inf_{0<B\leq Q_{max}} \left(\frac{\overline{Q}}{\mu(\Lambda, B)} + \frac{Q_{max}}{B} \right)^{N-1},$$

where $c_{abs} > 0$ is an absolute constant.

Higher dimensional examples may require a better understanding of parametric geometry of numbers for multiple parameters.

Let us now assume that $m_i = \beta_i = 1$ $(1 \leq i \leq n)$ and that $C = \{\underline{\mathbf{0}}\}$, and also that Λ is unimodular. In this setting Skriganov [10, Theorem 6.1] proved error estimates for homogeneously expanding aligned boxes (and more generally certain polyhedrons), provided the dual lattice $\Lambda^\perp$ (with respect to the standard inner product) is weakly admissible. As shown in [12] his method also leads to results for arbitrarily aligned boxes, again, provided $\Lambda^\perp$ is weakly admissible, of the form[2]

$$\mathcal{E}_\Lambda(Z_{\boldsymbol{Q}}) \ll_N \frac{1}{\nu(\Lambda^\perp, (\overline{Q}/Q_{min})^*)^N} \inf_{\varrho \geq \gamma_N} \left(\frac{\overline{Q}^{N-1}}{\sqrt{\varrho}} + \frac{r^{N-1}}{\nu(\Lambda^\perp, 2^r \overline{Q}/Q_{min})^N} \right), \tag{1.4}$$

where γ_N denotes the Hermite constant, $r = N^2 + N \log(\varrho/\nu(\Lambda^\perp, \varrho\overline{Q}/Q_{min}))$, and $(\overline{Q}/Q_{min})^* = \max\{\overline{Q}/Q_{min}, \gamma_N\}$.

Now in general, even if Λ and $\Lambda^\perp$ are both weakly admissible, there is no way to bound $\nu(\Lambda, \cdot)$ in terms of $\nu(\Lambda^\perp, \cdot)$. However, if $\nu(\Lambda, \cdot) = \nu(\Lambda^\perp, \cdot)$ then[3] we can directly compare our result with Skriganov's. Using that $\overline{Q}/Q_{min} \geq (Q_{max}/\overline{Q})^{1/(N-1)}$ we find the following crude lower bound for the right hand-side of (1.4)

$$\nu(\Lambda, (Q_{max}/\overline{Q})^{1/(N-1)})^{-2N}. \tag{1.5}$$

Choosing $B = Q_{max}/\overline{Q}$ we see that the error term in Theorem 1.1 is bounded from above by

$$c_1(2\overline{Q})^{N-1}\nu(\Lambda, Q_{max}/\overline{Q})^{-(N-1)}. \tag{1.6}$$

In particular, if $N = 2$ then our error term is better whenever $\nu(\Lambda, Q_{max}/\overline{Q})^{-3} > c_1(\mathrm{Vol} Z_{\boldsymbol{Q}})^{1/2}$, so if the box is sufficiently distorted in terms of $\nu(\Lambda, \cdot)$ and the volume of the box. (Also note that for $\nu(\Lambda, Q_{max}/\overline{Q})^{-1} = o(\overline{Q})$ as $\overline{Q}$ tends to infinity we still get asymptotics.)

Another significant difference between our error term and Skriganov's results concerns the dependence on the lattice. If we replace Λ by $k\Lambda$ then the error term in (1.4) increases by a factor k^N. (WE CAN'T DO THAT, Λ MUST BE UNIMODULAR! BUT USE (1.5) AND (1.6) TO COMPARE!) On the other hand $\mu(k\Lambda, B) = k\mu(\Lambda, B)$ and hence our error term in Theorem 1.1 gets significantly smaller for $k\Lambda$. This improvement

[2] In the above setting our definition of $\nu(\cdot, \cdot)$ is the N-th root of Skriganov's and the one in [12].

[3] This identity of the $\nu(\cdot, \cdot)$-functions holds true if, e.g., $\Lambda = A\mathbf{Z}^N$ with a symplectic matrix A. In particular, the identity holds for any unimodular lattice of rank 2. See [12] for this and more general results.

allows us to sieve for coprimality, and thus to prove asymptotics for the number of primitive lattice points.

Let Λ be a lattice in $\mathbf{R}^N$. We say $\underline{\mathbf{x}} \in \Lambda$ is primitive if $\underline{\mathbf{x}}$ is not of the form $k\underline{\mathbf{y}}$ for some $\underline{\mathbf{y}} \in \Lambda$ and some integer $k > 1$. We write

$$\Lambda^* := \{\underline{\mathbf{x}} \in \Lambda; \underline{\mathbf{x}} \text{ is primitive}\}.$$

To state our next result let $T : [0, \infty) \to [1, \infty)$ be monotonic increasing and an upper bound for the divisor function, i.e.,

$$T(k) \geq \sum_{d|k} 1$$

for all $k \in \mathbf{N}$. Finally, $\zeta(\cdot)$ denotes the Riemann zeta function.

Theorem 1.3. *Suppose Λ is a weakly admissible lattice for $(\mathcal{S}, C)$. Then there exists a constant $c_2 = c_2(N, \kappa, M)$, depending only on N, κ, M, such that*

$$\left| |Z_{\boldsymbol{Q}} \cap \Lambda^*| - \frac{\mathrm{Vol} Z_{\boldsymbol{Q}}}{\zeta(N) \det \Lambda} \right| \leq c_2 \left(\left(\frac{\overline{Q}}{\mu} + 1 \right)^{N-1} + \left(\frac{\overline{Q}}{\mu} + 1 \right) T\left(H\right) \right)$$

where

$$H = N^{2N+2}(\overline{Q} + |\phi(\underline{\mathbf{y}})|) \left(\frac{1}{\mu} + \frac{1}{\overline{\overline{Q}}} \right),$$

$\mu = \mu(\Lambda, Q_{max})$, and $|\phi(\underline{\mathbf{y}})|$ is the Euclidean norm of $(\overline{Q}\mathbf{y}_1/Q_1, \ldots, \overline{Q}\mathbf{y}_n/Q_n) \in \mathbf{E}$.

Note that for every $a > 2$ there is a $b = b(a) \geq \exp(\exp(1))$ such that for $x \geq b$ we can take $T(x) = a^{\frac{\log x}{\log \log x}}$. We use $\overline{Q} + |\phi(\underline{\mathbf{y}})| \leq \overline{Q}(1 + |\underline{\mathbf{y}}|/Q_{min})$ and $1/\mu + 1/\overline{Q} \leq 2/\mu$ to obtain the following corollary.

Corollary 1.1. *Suppose Λ is a weakly admissible lattice for $(\mathcal{S}, C)$ and $a > 2$. Then there exists a constant $c_3 = c_3(a, N, \kappa, M, |\underline{\mathbf{y}}|)$, depending only on a, N, κ, M and $|\underline{\mathbf{y}}|$ such that for all $\overline{Q} \geq b\mu$ we have*

$$\left| |Z_{\boldsymbol{Q}} \cap \Lambda^*| - \frac{\mathrm{Vol} Z_{\boldsymbol{Q}}}{\zeta(N) \det \Lambda} \right| \leq c_3 \left(\left(\frac{\overline{Q}}{\mu} \right)^{N-1} + a^{\frac{\log(\eta\overline{Q}/\mu)}{\log\log(\eta\overline{Q}/\mu)}} \left(\frac{\overline{Q}}{\mu} \right) \right)$$

where $\mu = \mu(\Lambda, Q_{max})$, and $\eta = 1 + |\underline{\mathbf{y}}|/Q_{min}$.

Next we consider applications to Diophantine approximation. Let $\Theta \in \mathrm{Mat}_{r\times s}(\mathbf{R})$ be a matrix with r rows and s columns and suppose that $\varphi : [1, \infty) \to (0, 1]$ is a non-increasing function such that

$$|\mathbf{p} + \Theta\mathbf{q}||\mathbf{q}|^{\beta} \geq \varphi(|\mathbf{q}|) \tag{1.7}$$

for every $(\mathbf{p}, \mathbf{q})$ with $\mathbf{q} \neq \mathbf{0}$. Let $\mathbf{y}$ be in $\mathbf{R}^r$, $Q \geq 1$ and let $0 < \epsilon \leq 1$. We consider the system

$$\mathbf{p} + \Theta\mathbf{q} - \mathbf{y} \in [0, \epsilon]^r \tag{1.8}$$

$$\mathbf{q} \in [0, Q]^s. \tag{1.9}$$

Let $N^*_{\Theta,\mathbf{y}}(\epsilon, Q)$ be the number of $(\mathbf{p}, \mathbf{q}) \in \mathbf{Z}^{r+s}$ that satisfy the above system and have coprime coordinates, i.e., $\gcd(p_1, \ldots, p_r, q_1, \ldots, q_s) = 1$. In the one-dimensional case $r = s = 1$ Chalk and Erdős [2] proved in 1959 that if Θ is an irrational number and $\epsilon = \epsilon(\mathbf{q}) = (1/\mathbf{q})(\log \mathbf{q}/ \log\log \mathbf{q})^2$ then (1.8) has infinitely many coprime solutions, i.e., $N^*_{\Theta,\mathbf{y}}(\epsilon, Q)$ is unbounded as Q tends to infinity. No improvements or generalisations to higher dimensions have been obtained since.

The following corollary follows straightforwardly from Corollary 1.1, and we leave the proof to the reader. We suppose $\epsilon = \epsilon(Q)$ is a function of Q, and that $\epsilon \cdot Q^\beta$ tends to infinity as Q tends to infinity.

Corollary 1.2. *Suppose $a > 2$. Then, as Q tends to infinity, we have*

$$N^*_{\Theta,\mathbf{y}}(\epsilon, Q) = \frac{\epsilon^r Q^s}{\zeta(r+s)} + O(u^{r+s-1} + ua^{\frac{\log\delta}{\log\log\delta}})$$

where $u = \left(\frac{\epsilon Q^\beta}{\varphi(Q)}\right)^{1/(1+\beta)}$, *and* $\delta = \left(\frac{1}{\varphi(Q)}\left(\frac{Q}{\epsilon}\right)^\beta\right)^{1/(1+\beta)}$.

Corollary 1.2 also implies new results on how quickly ϵ can decay so that (1.8) still has infinitely many coprime solutions. As an example let us suppose that Θ is a badly approximable matrix so that in (1.7) we can choose $\beta = s/r$ and $\varphi(\cdot)$ to be constant. A straightforward computation shows that if $a > 2^{(r+s)/(r(r+s-1))}$ and $\epsilon = \epsilon(Q) = Q^{-s/r}a^{\log Q/\log\log Q}$ then $N^*_{\Theta,\mathbf{y}}(\epsilon, Q)$ tends to infinity as Q does. In particular, if $\epsilon = \epsilon(|\mathbf{q}|_\infty) = |\mathbf{q}|_\infty^{-s/r}a^{\log|\mathbf{q}|_\infty/\log\log|\mathbf{q}|_\infty}$ then (1.8) has infinitely many coprime solutions[4]. To the best of the author's knowledge this is the first such result apart from Erdős and Chalk's result in dimension 1.

A similar simple calculation shows that Corollary 1.2 in conjunction with the classical Khintchine Groshev Theorem implies that the same holds true not only for badly approximable matrices Θ but for almost[5] every $\Theta \in \mathrm{Mat}_{r\times s}(\mathbf{R})$.

Finally, we mention a connection to a question of Dani, Laurent and Nogueira. Suppose $\epsilon : [1,\infty) \to (0,1]$ and $Q^{s-1}\epsilon(Q)^r$ is non-increasing. Dani, Laurent and Nogueira conjecture[6] that if $\sum_{j\in\mathbf{N}} j^{s-1}\epsilon(j)^r = \infty$ then for almost every $\Theta \in \mathrm{Mat}_{r\times s}(\mathbf{R})$ there exist infinitely many coprime solutions of (1.8), where again we interpret $\epsilon = \epsilon(|\mathbf{q}|_\infty)$ as a function evaluated at $|\mathbf{q}|_\infty$. We cannot prove this conjecture, but as mentioned before our result shows at least that we have infinitely many such solutions for almost every Θ if $\epsilon(Q) \gg Q^{-s/r}a^{\log Q/\log\log Q}$ and, $a > 2^{(r+s)/(r(r+s-1))}$. DOUBLECHECK ABOVE CLAIMS!!!

2. Counting lattice points

Let $D \geq 2$ be an integer. Let Λ be a lattice of rank D in $\mathbf{R}^D$. Recall that $B_P(R)$ denotes the closed Euclidean ball about P of radius R. We define the successive minima $\lambda_1(\Lambda), \ldots, \lambda_D(\Lambda)$ of Λ as the successive minima in the sense of Minkowski with respect to the Euclidean unit ball. That is

$$\lambda_i = \inf\{\lambda; B_0(\lambda) \cap \Lambda \text{ contains } i \text{ linearly independent vectors}\}.$$

Definition 2. *Let M be a positive integers and let L be a non-negative real number. We say that a set S is in $Lip(D, M, L)$ if S is a subset of $\mathbf{R}^D$, and if there are M maps $\phi_1, \ldots, \phi_M : [0,1]^{D-1} \longrightarrow \mathbf{R}^D$ satisfying a Lipschitz condition*

$$|\phi_i(\mathbf{x}) - \phi_i(\mathbf{y})| \leq L|\mathbf{x} - \mathbf{y}| \text{ for } \mathbf{x}, \mathbf{y} \in [0,1]^{D-1}, i = 1, \ldots, M$$

such that S is covered by the images of the maps ϕ_i.

For any set S we write

$$1^*(S) = \begin{cases} 1 & \text{if } S \neq \emptyset, \\ 0 & \text{if } S = \emptyset. \end{cases}$$

We will apply the following basic counting principle.

[4]Here $|\cdot|_\infty$ denotes the maximum norm.

[5]With respect to the Lebesgue measure.

[6]Their conjecture is more general. In fact, as kindly pointed out by Michel Laurent, the case $s > 1$ of this special case of their conjecture follows from a result of Khintchine.

Lemma 2.1. *Let Λ be a lattice in $\mathbf{R}^D$ with successive minima $\lambda_1, \ldots, \lambda_D$. Let S be a set in $\mathbf{R}^D$ such that the boundary ∂S of S is in Lip(D, M, L), and suppose $S \subset B_P(L)$ for some point P. Then S is measurable, and moreover,*

$$\left| |S \cap \Lambda| - \frac{\mathrm{Vol} S}{\det \Lambda} \right| \leq c_{lp}(D) M \left(\left(\frac{L}{\lambda_1} \right)^{D-1} + 1^*(S \cap \Lambda) \right),$$

where $c_{lp}(D) = D^{3D^2/2}$.

Proof. By [14, Theorem 5.4] the set S is measurable, and moreover,

$$\left| |S \cap \Lambda| - \frac{\mathrm{Vol} S}{\det \Lambda} \right| \leq D^{3D^2/2} M \max_{1 \leq j < D} \left\{ 1, \frac{L^j}{\lambda_1 \cdots \lambda_j} \right\}. \tag{2.1}$$

First suppose $L \geq \lambda_1$. Then the lemma follows immediately from (2.1). Next we assume $L < \lambda_1$. We distinguish two subcases. First suppose $S \cap \Lambda \neq \emptyset$. Then

$$\max_{1 \leq j < D} \left\{ 1, \frac{L^j}{\lambda_1 \cdots \lambda_j} \right\} = 1 = 1^*(S \cap \Lambda) \leq \left(\frac{L}{\lambda_1} \right)^{D-1} + 1^*(S \cap \Lambda).$$

Now suppose $S \cap \Lambda = \emptyset$. As $L < \lambda_1$ we get, using Minkowski's second Theorem,

$$\left| |S \cap \Lambda| - \frac{\mathrm{Vol} S}{\det \Lambda} \right| = \frac{\mathrm{Vol} S}{\det \Lambda} \leq \frac{(2L)^D}{\lambda_1 \cdots \lambda_D} \leq 2^D \left(\frac{L}{\lambda_1} \right)^{D-1}.$$

This proves the lemma. □

3. Proof of Theorem 1.1

Let $\theta_i = \overline{Q}/Q_i$ $(1 \leq i \leq n)$, and let ϕ be the automorphism of $\mathbf{E}$ defined by

$$\phi(\underline{\mathbf{x}}) := (\theta_1 \mathbf{x}_1, \ldots, \theta_n \mathbf{x}_n).$$

Set

$$\theta_{min} := \min_{1 \leq i \leq n} \theta_i = \overline{Q}/Q_{max}.$$

Note that by (1.3) we have

$$\theta_i \geq 1 \text{ (for all } i \notin I). \tag{3.1}$$

Moreover,

$$\prod_{i=1}^{n} \theta_i^{\beta_i} = 1,$$

and hence,

$$\mathrm{Nm}_{\boldsymbol{\beta}}(\phi \underline{\mathbf{x}}) = \mathrm{Nm}_{\boldsymbol{\beta}}(\underline{\mathbf{x}}). \tag{3.2}$$

Lemma 3.1. *We have $\partial \phi(Z_{\boldsymbol{Q}}) \in$ Lip(N, M, L) for $L = 2n^{1/2}\kappa\overline{Q}$.*

Proof. We have

$$\phi(Z_{\boldsymbol{Q}}) \subset \phi(B_{\mathbf{y}_1}(Q_1) \times \cdots \times B_{\mathbf{y}_n}(Q_n)) = B_{\theta_1 \mathbf{y}_1}(\overline{Q}) \times \cdots \times B_{\theta_n \mathbf{y}_n}(\overline{Q}),$$

and hence, $\phi(Z_{\boldsymbol{Q}}) \subset B_{\phi \underline{\mathbf{y}}}(n^{1/2}\overline{Q})$. As $Z_{\boldsymbol{Q}} \in \mathcal{F}_{\kappa, M}$ the claim follows. □

Lemma 3.2. *The set $Z_{\boldsymbol{Q}}$ is measurable and*

$$\left| |Z_{\boldsymbol{Q}} \cap \Lambda| - \frac{\mathrm{Vol} Z_{\boldsymbol{Q}}}{\det \Lambda} \right| \leq c_4 \left(\left(\frac{\overline{Q}}{\lambda_1(\phi\Lambda)} \right)^{N-1} + 1^*(\phi Z_{\boldsymbol{Q}} \cap \phi\Lambda) \right)$$

where $c_4 = (1 + 2n^{1/2}\kappa)^{N-1} M c_{lp}(N)$.

Proof. Since $|Z_{\boldsymbol{Q}} \cap \Lambda| = |\phi Z_{\boldsymbol{Q}} \cap \phi\Lambda|$ and $\mathrm{Vol} Z_{\boldsymbol{Q}} / \det \Lambda = \mathrm{Vol} \phi Z_{\boldsymbol{Q}} / \det \phi\Lambda$, this follows immediately from Lemma 2.1 and Lemma 3.1. $\square$

Lemma 3.3. *Let $B > 0$. Then we have*

$$\lambda_1(\phi\Lambda) \geq \min\{\lambda_1(\Lambda \cap C_I), \nu(\Lambda, B), \theta_{min} B\}.$$

Proof. By (3.1) we have $\theta_i \geq 1$ (for all $i \notin I$). Moreover, if $\underline{\mathbf{x}} \in \Lambda \cap C_I$ then $\mathbf{x}_i = \mathbf{0}$ (for all $i \in I$), and thus

$$|\phi(\underline{\mathbf{x}})|^2 = \sum_{\substack{1 \leq i \leq n \\ i \notin I}} |\theta_i \mathbf{x}_i|^2 \geq \sum_{\substack{1 \leq i \leq n \\ i \notin I}} |\mathbf{x}_i|^2 = |\underline{\mathbf{x}}|^2.$$

Hence, if $\underline{\mathbf{x}} \in \Lambda \cap C_I$ and $\underline{\mathbf{x}} \neq 0$ then $|\phi(\underline{\mathbf{x}})| \geq \lambda_1(\Lambda \cap C_I)$.

Now suppose that $\underline{\mathbf{x}} \in \Lambda \backslash C_I$. If $\underline{\mathbf{z}}$ is an arbitrary point in $\mathbf{E}$ then, by the weighted arithmetic geometric mean inequality, we have

$$|\underline{\mathbf{z}}|^2 = \sum_{i=1}^{n} |\mathbf{z}_i|^2 \geq \frac{1}{\max_i \beta_i} \sum_{i=1}^{n} \beta_i |\mathbf{z}_i|^2 \geq \frac{t}{\max_i \beta_i} \left(\prod_{i=1}^{n} |\mathbf{z}_i|^{2\beta_i} \right)^{\frac{1}{t}} \geq \mathrm{Nm}_{\boldsymbol{\beta}}(\underline{\mathbf{z}})^{2/t},$$

and thus

$$|\underline{\mathbf{z}}| \geq \mathrm{Nm}_{\boldsymbol{\beta}}(\underline{\mathbf{z}})^{1/t}. \tag{3.3}$$

Using (3.3) and (3.2) we conclude that

$$|\phi(\underline{\mathbf{x}})| \geq \mathrm{Nm}_{\boldsymbol{\beta}}(\phi \underline{\mathbf{x}})^{1/t} = \mathrm{Nm}_{\boldsymbol{\beta}}(\underline{\mathbf{x}})^{1/t}.$$

First suppose that $|\underline{\mathbf{x}}| < B$. Then we have by definition of $\nu(\cdot, \cdot)$

$$\mathrm{Nm}_{\boldsymbol{\beta}}(\underline{\mathbf{x}})^{1/t} \geq \nu(\Lambda, B),$$

and hence $|\phi(\underline{\mathbf{x}})| \geq \nu(\Lambda, B)$. Now suppose $|\underline{\mathbf{x}}| \geq B$. Then we have

$$|\phi(\underline{\mathbf{x}})| = \theta_{min} |(\theta_1 \mathbf{x}_1 / \theta_{min}, \ldots, \theta_n \mathbf{x}_n / \theta_{min})| \geq \theta_{min} |(\mathbf{x}_1, \ldots, \mathbf{x}_n)| = \theta_{min} |\underline{\mathbf{x}}| \geq \theta_{min} B.$$

This proves the lemma. $\square$

We can now easily finish the proof of Theorem 1.1. Since, $\theta_{min} Q_{max} = \overline{Q}$ we conclude $\lambda_1(\phi\Lambda) \geq \min\{\mu(\Lambda, B), B\overline{Q}/Q_{max}\}$. Thus, we have

$$\frac{\overline{Q}}{\lambda_1(\phi\Lambda)} \leq \frac{\overline{Q}}{\mu(\Lambda, B)} + \frac{Q_{max}}{B}. \tag{3.4}$$

The latter in conjunction with Lemma 3.2 and the fact $c_4 + 1 = (1 + 2n^{1/2}\kappa)^{N-1} M N^{3N^2/2} + 1 \leq M((1+\kappa)N^{2N})^N = c_1$ proves the theorem.

4. Preparations for the Möbius inversion

Recall that $T : [0, \infty) \to [1, \infty)$ is a monotonic increasing function that is an upper bound for the divisor function, i.e., $T(k) \geq \sum_{d|k} 1$ for all $k \in \mathbf{N}$. In this section D is a positive integer. For an endomorphism Ψ of $\mathbf{R}^D$ we write $\|\Psi\|$ for the (Euclidean) operator norm.

Lemma 4.1. *Let Λ be a lattice in $\mathbf{R}^D$ and let Ψ be in $GL_D(\mathbf{R})$ with $\Psi \mathbf{Z}^D = \Lambda$. Then*

$$|\{k \in \mathbf{N}; B_P(R) \backslash \{\mathbf{0}\} \cap k\Lambda \neq \emptyset\}| \leq T((R + |P|)\|\Psi^{-1}\|)(2R\|\Psi^{-1}\| + 1).$$

Proof. First assume $\Psi = id$ so that $\Lambda = \mathbf{Z}^D$. Suppose $v = (a_1, \ldots, a_D) \in \mathbf{Z}^D$ is non-zero, $kv \in B_P(R)$ and $P = (x_1, \ldots, x_D)$. Then ka_i lies in $[x_i - R, x_i + R]$ for $1 \leq i \leq D$. As $v \neq \mathbf{0}$ there exists an i with $a_i \neq 0$. We conclude that k is a divisor of some non-zero integer in $[x_i - R, x_i + R]$. There are at most $2R + 1$ integers in this interval, each of which of modulus at most $R + |P|$. Hence the number of possibilities for k is $\leq T(R + |P|)(2R + 1)$. This proves the lemma for $\Psi = id$. Next note that

$$|B_P(R)\backslash\{\mathbf{0}\} \cap k\Lambda| = |\Psi^{-1}B_P(R)\backslash\{\mathbf{0}\} \cap k\mathbf{Z}^D|.$$

Hence, the general case follows from the case $\Psi = id$ upon noticing $\Psi^{-1}B_P(R) \subset B_{\Psi^{-1}(P)}(R\|\Psi^{-1}\|)$, and $|\Psi^{-1}(P)| \leq \|\Psi^{-1}\||P|$. □

Next we estimate the operator norm $\|\Psi^{-1}\|$ for a suitable choice of Ψ.

Lemma 4.2. *Let Λ be a lattice in $\mathbf{R}^D$. There exists $\Psi \in GL_D(\mathbf{R})$ with $\Psi\mathbf{Z}^D = \Lambda$ and*

$$\|\Psi^{-1}\| \leq \frac{c_{on}(D)}{\lambda_1},$$

where $c_{on}(D) = D^{2D+1}$.

Proof. Any lattice Λ in $\mathbf{R}^D$ has a basis $v_1, \ldots, v_D$ with $\frac{|v_1|\cdots|v_D|}{|\det[v_1 \ldots v_D]|} \leq D^{2D}$, see, e.g., [14, Lemma 4.4]. Let Ψ be the map that sends the canonical basis $e_1, \ldots, e_n$ to $v_1, \ldots, v_n$. Now suppose Ψ^{-1} sends e_i to $(\varrho_1, \ldots, \varrho_n)$ then by Cramer's rule

$$|\varrho_j| = \left|\frac{\det[v_1 \ldots e_i \ldots v_D]}{\det[v_1 \ldots v_j \ldots v_D]}\right| \leq \frac{|\det[v_1 \ldots e_i \ldots v_D]|}{|v_1| \cdots |v_j| \cdots |v_D|} D^{2D}.$$

Now we apply Hadamard's inequality to obtain

$$\frac{|\det[v_1 \ldots e_i \ldots v_D]|}{|v_1| \cdots |v_j| \cdots |v_D|} \leq \frac{|v_1| \cdots |e_j| \cdots |v_D|}{|v_1| \cdots |v_i| \cdots |v_D|} = \frac{1}{|v_i|} \leq \frac{1}{\lambda_i}.$$

Next we use that for a $D \times D$ matrix $[a_{ij}]$ with real entries we have $\|[a_{ij}]\| \leq \sqrt{D}\max_{ij}|a_{ij}|$, and this proves the lemma. □

We combine the previous two lemmas.

Lemma 4.3. *Let Λ be a lattice in $\mathbf{R}^D$, and let $\lambda_1 = \lambda_1(\Lambda)$. Then*

$$\sum_{k=1}^{\infty} 1^*(B_P(R)\backslash\{\mathbf{0}\} \cap k\Lambda) \leq T\left(c_{on}(D)\left(\frac{R + |P|}{\lambda_1}\right)\right)\left(\frac{2c_{on}(D)R}{\lambda_1} + 1\right).$$

Proof. Note that $\sum_{k=1}^{\infty} 1^*(B_P(R)\backslash\{\mathbf{0}\} \cap k\Lambda) = |\{k \in \mathbf{N}; B_P(R)\backslash\{\mathbf{0}\} \cap k\Lambda \neq \emptyset\}|$. Hence, the lemma follows immediately from Lemma 4.1 and Lemma 4.2. □

5. Proof of Theorem 1.3

Set

$$R := n^{1/2}\overline{Q}.$$

Lemma 5.1. *Let $Z^*_{\boldsymbol{Q}} = Z_{\boldsymbol{Q}}\backslash\{\underline{\mathbf{0}}\}$. Then*

$$\left||Z^*_{\boldsymbol{Q}} \cap \Lambda| - \frac{\mathrm{Vol}Z_{\boldsymbol{Q}}}{\det\Lambda}\right| \leq c_5\left(\left(\frac{\overline{Q}}{\lambda_1(\phi\Lambda)}\right)^{N-1} + 1^*(B_{\phi(\underline{\mathbf{y}})}(R)\backslash\{\underline{\mathbf{0}}\} \cap \phi\Lambda)\right)$$

where $c_5 = (1 + 2n^{1/2}\kappa)^{N-1}(M+1)c_{lp}(N)$.

Proof. Lemma 3.1 implies that $\partial Z^*_{\boldsymbol{Q}} \in \mathrm{Lip}(N, M+1, L)$ with $L = 2n^{1/2}\kappa\overline{Q}$. As noted in the proof of the latter lemma we have $\phi(Z^*_{\boldsymbol{Q}}) \subset B_{\phi\underline{\mathbf{y}}}(R)\backslash\{\underline{\mathbf{0}}\}$. We conclude as in Lemma 3.2. □

For $\underline{\mathbf{x}} \in \Lambda \backslash \underline{\mathbf{0}}$ we define $\gcd(\underline{\mathbf{x}}) := d$ if $\underline{\mathbf{x}} = d\underline{\mathbf{x}}'$ for some $\underline{\mathbf{x}}' \in \Lambda$ but $\underline{\mathbf{x}} \neq k\underline{\mathbf{x}}'$ for all integers $k > d$ and all $\underline{\mathbf{x}}' \in \Lambda$. (An equivalent definition is $\gcd(A\underline{\mathbf{z}}) := \gcd(\underline{\mathbf{z}})$ where $\underline{\mathbf{z}} \in \mathbf{Z}^N$, $\gcd(\underline{\mathbf{z}}) := \gcd(z_1, \ldots, z_N)$, and $\Lambda = A\mathbf{Z}^N$.) Next we define

$$F(d) = \{\underline{\mathbf{x}} \in \Lambda \cap Z^*_{\boldsymbol{Q}}; \gcd(\underline{\mathbf{x}}) = d\}.$$

In particular, $\Lambda^* \cap Z_{\boldsymbol{Q}} = F(1)$. Then for $k \in \mathbf{N}$ we have the disjoint union

$$\bigcup_{k|d} F(d) = k\Lambda \cap Z^*_{\boldsymbol{Q}}.$$

If $\underline{\mathbf{x}} = k\underline{\mathbf{x}}'$ lies in $k\Lambda \cap Z^*_{\boldsymbol{Q}}$ then $k\phi\underline{\mathbf{x}}'$ lies in $k\phi\Lambda \cap B_{\phi(\underline{\mathbf{y}})}(R)$, and hence

$$k \leq \frac{R + |\phi(\underline{\mathbf{y}})|}{\lambda_1(\phi\Lambda)} \leq \frac{R + |\phi(\underline{\mathbf{y}})|}{\mu(\Lambda, Q_{max})} + \frac{R + |\phi(\underline{\mathbf{y}})|}{\overline{Q}} =: G$$

where for the second inequality we have applied Lemma 3.3. We use the Möbius function $\mu(\cdot)$ and the Möbius inversion formula to get

$$|\Lambda^* \cap Z_{\boldsymbol{Q}}| = |F(1)| = \sum_{k=1}^{\infty} \mu(k) \sum_{\substack{d \\ k|d}} |F(d)| = \sum_{k=1}^{[G]} \mu(k) \sum_{\substack{d \\ k|d}} |F(d)| = \sum_{k=1}^{[G]} \mu(k) |k\Lambda \cap Z^*_{\boldsymbol{Q}}|.$$

For the rest of this section we will write $g \ll h$ to mean there exists a constant $c = c(N, M, \kappa)$ such that $g \leq ch$. Applying Lemma 5.1 with Λ replaced by $k\Lambda$ yields

$$\left| |Z_{\boldsymbol{Q}} \cap \Lambda^*| - \frac{\mathrm{Vol} Z_{\boldsymbol{Q}}}{\zeta(N) \det \Lambda} \right| \ll$$
$$\sum_{k=1}^{[G]} \left(\frac{\overline{Q}}{k\lambda_1(\phi\Lambda)} \right)^{N-1} + \sum_{k=1}^{[G]} 1^*(B_{\phi(\underline{\mathbf{y}})}(R) \backslash \{\underline{\mathbf{0}}\} \cap k\phi\Lambda) + \sum_{k > G} \frac{\mathrm{Vol} Z_{\boldsymbol{Q}}}{k^N \det \Lambda}.$$

First we note that

$$\sum_{k > G} k^{-N} \leq \sum_{k \geq \max\{G, 1\}} k^{-N} \ll \max\{G, 1\}^{1-N} \leq \max\{\frac{R}{\lambda_1(\phi\Lambda)}, 1\}^{1-N},$$

and moreover,

$$\frac{\mathrm{Vol} Z_{\boldsymbol{Q}}}{\det \Lambda} = \frac{\mathrm{Vol} \phi Z_{\boldsymbol{Q}}}{\det \phi\Lambda} \leq \frac{\mathrm{Vol} B_{\underline{\mathbf{0}}}(R)}{\det \phi\Lambda} \ll \frac{R^N}{\lambda_1(\phi\Lambda)^N}.$$

Combining both with (3.4) yields

$$\sum_{k > G} \frac{\mathrm{Vol} Z_{\boldsymbol{Q}}}{k^N \det \Lambda} \ll \frac{R}{\lambda_1(\phi\Lambda)} \ll \frac{\overline{Q}}{\lambda_1(\phi\Lambda)} \leq \frac{\overline{Q}}{\mu(\Lambda, Q_{max})} + 1.$$

Next we note that by Lemma 4.3

$$\sum_{k=1}^{[G]} 1^*(B_{\phi(\underline{\mathbf{y}})}(R) \backslash \{\underline{\mathbf{0}}\} \cap k\phi\Lambda) \leq T\left(c_{on}(N) \frac{R + |\phi(\underline{\mathbf{y}})|}{\lambda_1(\phi(\Lambda))} \right) \left(\frac{2c_{on}(N)R}{\lambda_1(\phi(\Lambda))} + 1 \right).$$

Moreover,

$$\left(\frac{2c_{on}(N)R}{\lambda_1(\phi(\Lambda))} + 1 \right) \ll \frac{\overline{Q}}{\mu(\Lambda, Q_{max})} + 1,$$

and

$$\frac{R + |\phi(\underline{\mathbf{y}})|}{\lambda_1(\phi(\Lambda))} \leq \frac{R + |\phi(\underline{\mathbf{y}})|}{\mu(\Lambda, Q_{max})} + \frac{R + |\phi(\underline{\mathbf{y}})|}{\overline{Q}} = G.$$

Since $c_{on}(N)G < H$ we conclude that

$$\sum_{k=1}^{[G]} 1^*(B_{\phi(\underline{\mathbf{y}})}(R)\backslash\{\underline{\mathbf{0}}\} \cap k\phi\Lambda) \ll T(H)\left(\frac{\overline{Q}}{\mu(\Lambda, Q_{max})}+1\right).$$

Finally,

$$\sum_{k=1}^{[G]}\left(\frac{\overline{Q}}{k\lambda_1(\phi\Lambda)}\right)^{N-1} \ll \left(\frac{\overline{Q}}{\mu(\Lambda, Q_{max})}+1\right)^{N-1}\sum_{k=1}^{[G]} k^{1-N} \ll \left(\frac{\overline{Q}}{\mu(\Lambda, Q_{max})}+1\right)^{N-1}\mathcal{L}^*,$$

where

$$\mathcal{L}^* = \begin{cases} \max\{\log(G), 1\} & \text{if } N = 2, \\ 1 & \text{if } N > 2. \end{cases}$$

If $N > 2$ then $\mathcal{L}^* = 1$ and we are done. So suppose $N = 2$. Hence $c_{on}(N) = 32$. By assumption $T(x) \geq 1$ so that $\mathcal{L}^* \leq T(c_{on}(N)G)$ for $G \leq \exp(1)$. Now suppose $G > \exp(1)$. Since T is monotonic and $2^{[\log_2[32G]]} \leq 32G$ we have $T(32G) \geq [\log_2[32G]] + 1 \geq \log_2(32G-1) \geq \log G$. Thus, $\mathcal{L}^* \leq T(c_{on}(N)G) \leq T(H)$. This finishes the proof.

6. Lower bounds for the error term

The main goal of this section is to prove Theorem 1.2. Throughout this section we assume that $m_i = \beta_i = 1$ $(1 \leq i \leq n)$, so that $N = n = t$, and that Λ is weakly admissible for $(\mathcal{S}, C)$ but not admissible for $(\mathcal{S}, C)$. To simplify the notation we write $\mathrm{Nm}(\cdot) := \mathrm{Nm}_{\boldsymbol{\beta}}(\cdot)$ and $\nu(\cdot) := \nu(\Lambda, \cdot)$. After scaling we can assume that $\det\Lambda = 1$. Let $k \geq 1$ and $\{\underline{\mathbf{x}}_j\}_{j=1}^{\infty} = \{(x_{j1}, \ldots, x_{jn})\}_{j=1}^{\infty}$ be a sequence of pairwise distinct elements in $\Lambda\backslash C$ with

$$\mathrm{Nm}(\underline{\mathbf{x}}_j) \leq k\nu(|\underline{\mathbf{x}}_j|)^n.$$

We define

$$\begin{aligned} N_j &:= a\nu(|\underline{\mathbf{x}}_j|)^{-n}, \\ Z_{\boldsymbol{Q}_j} &:= N_j B_{\underline{\mathbf{x}}_j}, \\ c_j &:= \lambda_{n-1}(\Lambda, B_{\underline{\mathbf{x}}_j}), \end{aligned}$$

where $a > 0$ is a constant which will be specified later, $B_{\underline{\mathbf{x}}_j}$ denotes the box

$$B_{\underline{\mathbf{x}}_j} := [-|x_{j1}|, |x_{j1}|] \times \cdots \times [-|x_{jn}|, |x_{jn}|]$$

and $\lambda_j(\Lambda, B_{\underline{\mathbf{x}}_j})$ are the corresponding successive minima. For $1 \leq i \leq n$ we choose the minimal eligible values $Q_i = N_j|x_{ji}|$ for the set $Z_{\boldsymbol{Q}_j}$, so that[7]

$$\overline{Q} \leq (ak)^{\frac{1}{n}} N_j^{\frac{n-1}{n}}. \tag{6.1}$$

Note that N_j tends to infinity since Λ is not admissible. Thus $\overline{Q}/Q_{max}$ tends to zero. Once we have also specified the coordinate-tuple subspace C we also want that our sets $Z_{\boldsymbol{Q}_j}$ satisfy the condition (1.3).

Let $v_1, \ldots, v_{n-1}$ be linearly independent lattice points in $\lambda_{n-1}(\Lambda, B_{\underline{\mathbf{x}}_j})B_{\underline{\mathbf{x}}_j}$. Then the lattice points $\sum_{l=1}^{n-1} m_l v_l$ with $-N_j/(c_j n) \leq m_l \leq N_j/(c_j n)$ are all distinct and lie all in $Z_{\boldsymbol{Q}_j}$. Since $2[N_j/(c_j n)] + 1 \geq N_j/(c_j n)$, we conclude that

$$|Z_{\boldsymbol{Q}_j} \cap \Lambda| - \mathrm{Vol} Z_{\boldsymbol{Q}_j} \geq (N_j/(c_j n))^{n-1} - 2^n ak N_j^{n-1}.$$

[7]To simplify the notation we suppress the dependence on j.

We now make the crucial assumption that the $n-1$-th successive minimum c_j is uniformly bounded[8] in j, i.e., there exists a constant $c_\Lambda \geq 1$ such that

$$c_j \leq c_\Lambda \tag{6.2}$$

for all j. Taking $a := 1/(4k(2c_\Lambda n)^{n-1})$ we get

$$\mathcal{E}_\Lambda(Z_{\boldsymbol{Q}_j}) \geq |Z_{\boldsymbol{Q}_j} \cap \Lambda| - \mathrm{Vol} Z_{\boldsymbol{Q}_j} \geq ((c_\Lambda n)^{1-n} - 2^n ak)N_j^{n-1} \gg_{c_\Lambda,n} N_j^{n-1}. \tag{6.3}$$

Next we prove a general criterion for Λ under which we have

$$|Z_{\boldsymbol{Q}_j} \cap \Lambda| - \mathrm{Vol} Z_{\boldsymbol{Q}_j} \geq c \inf_{0<B\leq Q_{max}} \left(\frac{\overline{Q}}{\mu(\Lambda,B)} + \frac{Q_{max}}{B}\right)^{N-1} \tag{6.4}$$

with a certain constant $c > 0$.

Proposition 6.1. *Suppose that the condition (6.2) and*

$$\nu(|\underline{\mathbf{x}}_j|/\nu(|\underline{\mathbf{x}}_j|)^n) \gg \nu(|\underline{\mathbf{x}}_j|) \tag{6.5}$$

hold true. Then there exists $c = c(k, c_\Lambda, n) > 0$ such that (6.4) holds true for all j large enough.

Proof. We have $Q_{max} \leq N_j|\underline{\mathbf{x}}_j|$, and so ignoring the first few members of the sequence $\underline{\mathbf{x}}_j$, we can assume that $\mu(\Lambda, Q_{max}) \geq \nu(N_j|\underline{\mathbf{x}}_j|) = \nu(a|\underline{\mathbf{x}}_j|/\nu(|\underline{\mathbf{x}}_j|)^n) \gg \nu(|\underline{\mathbf{x}}_j|)$. Hence,

$$\inf_{0<B\leq Q_{max}} \left(\frac{\overline{Q}}{\mu(\Lambda,B)} + \frac{Q_{max}}{B}\right) \leq \left(\frac{\overline{Q}}{\mu(\Lambda,Q_{max})} + 1\right) \ll \left(\frac{\overline{Q}}{\nu(|\underline{\mathbf{x}}_j|)} + 1\right) \ll_k N_j.$$

This, in conjunction with (6.3), shows that (6.4) holds true. □

We give an example with $n = 2$ and $C = \{\underline{\mathbf{x}}; \mathbf{x}_2 = \mathbf{0}\}$. Let α be an irrational real number, and consider the lattice Λ given by the vectors $(p - q\alpha, q)$ with $p, q \in \mathbf{Z}$. Then Λ is weakly admissible for $(\mathcal{S}, C)$. To choose an appropriate α we consider its continued fraction expansion $\alpha = [a_0, a_1, a_2, \ldots]$. Using the recurrence relation $q_{j+1} = a_{j+1}q_j + q_{j-1}$ for the denominator q_j of the j-th convergent p_j/q_j (in lowest terms) we can define α by setting $a_0 = a_1 = 1$ (so that $q_0 = q_1 = 1$) and $a_{j+1} = [\log q_j] + 1$. A very crude estimate yields $a_j + \log a_j - 1 \leq a_{j+1} \leq 3a_j$. Put $\underline{\mathbf{x}}_j = (p_j - q_j\alpha, q_j) \in \Lambda\backslash C$ so that $|\underline{\mathbf{x}}_j| > |\underline{\mathbf{x}}_{j-1}|$, at least for j large enough. From the theory of continued fractions we know that for $\underline{\mathbf{x}} \in \Lambda\backslash C$ the inequality $\mathrm{Nm}(\underline{\mathbf{x}}) < 1/2$ implies that $\underline{\mathbf{x}} = c\underline{\mathbf{x}}_j$ for some non-zero integer c and $j \in \mathbf{N}$. We conclude that for all sufficiently large ϱ we have $\nu(\varrho)^2 = \mathrm{Nm}(\underline{\mathbf{x}}_j)$ for some j. Also by the theory of continued fractions we know that $1/(a_{j+1} + 2) < \mathrm{Nm}(\underline{\mathbf{x}}_j) < 1/a_{j+1}$. Since $a_j > a_{j-1} + 2$ we conclude $\mathrm{Nm}(\underline{\mathbf{x}}_{j-1}) < \mathrm{Nm}(\underline{\mathbf{x}}_{j-2})$ and thus $\mathrm{Nm}(\underline{\mathbf{x}}_{j-1}) = \nu(|\underline{\mathbf{x}}_j|)^2$ for j large enough; so we can take $k = 1$. We also easily find that $|\underline{\mathbf{x}}_j|/\nu(|\underline{\mathbf{x}}_j|)^2 \leq |\underline{\mathbf{x}}_{j+1}|$ for j large enough. It is now straightforward to verify (6.5). Moreover, for j large enough, (1.3) holds true and so $Z_{\boldsymbol{Q}_j}$ is an eligible set. Since $n = 2$ we automatically have (6.2) with $c_\Lambda = 1$. Hence we can apply Proposition 6.1. Finally, we note that $\mathrm{Vol} Z_{\boldsymbol{Q}_j} = 4N_j^2\mathrm{Nm}(\underline{\mathbf{x}}_j) = (2a)^2\mathrm{Nm}(\underline{\mathbf{x}}_{j-1})^{-2}\mathrm{Nm}(\underline{\mathbf{x}}_j) \geq 2^{-12}a_j^2/(a_{j+1} + 2)$ which tends to infinity.

We note that the previous example proves the case $n = 2$ of Theorem 1.2. Next we construct a lattice Λ in dimension $n = 3$ (and hence $N = 3$), weakly admissible for $(\mathcal{S}, C)$ with $C = \{\underline{\mathbf{x}}; \mathbf{x}_3 = \mathbf{0}\}$, for which (6.4) holds true. This example does not rely on Proposition 6.1. Let $\alpha = [a_0, a_1, a_2, \ldots]$ be a badly approximable real number, so that the partial quotients a_i are bounded. We set $a_M = \max a_i$, and we consider the lattice

$$\Lambda = \{(p_1 - q\alpha, p_2 - 2q\alpha, q); p_1, p_2, q \in \mathbf{Z}\}. \tag{6.6}$$

[8] Note that $\lambda_1(\Lambda, B_{\underline{\mathbf{x}}_j}) \leq 1$ by definition of the sequence $\underline{\mathbf{x}}_j$. On the other hand $\mathrm{Vol} B_{\underline{\mathbf{x}}_j}$ tends to zero, so that by Minkowski's second Theorem $\lambda_n(\Lambda, B_{\underline{\mathbf{x}}_j}) \to \infty$ as j tends to infinity.

Proposition 6.2. *Let Λ be given by (6.6). Then there exists $c = c(a_M) > 0$ depending only on a_M such that (6.4) holds true for all j large enough.*

Proof. In this proof we write $h \ll g$ to mean $h \leq cg$ for a constant $c = c(a_M)$ depending only on a_M. First we note that

$$\mathrm{Nm}(\underline{\mathbf{x}}) \gg |\underline{\mathbf{x}}|^{-1}$$

for every $\underline{\mathbf{x}} \in \Lambda \backslash C$. Hence,

$$\nu(\varrho) \gg \varrho^{-1/3}. \tag{6.7}$$

Now suppose p_j/q_j is the j-th convergent of α, and put $\underline{\mathbf{x}}_j = (p_j - q_j\alpha, 2p_j - 2q_j\alpha, q_j) \in \Lambda \backslash C$. Then, for j large enough, (1.3) holds true, and so $Z_{\boldsymbol{Q}_j}$ is an eligible set. Since

$$\mathrm{Nm}(\underline{\mathbf{x}}_j) \ll |\underline{\mathbf{x}}_j|^{-1},$$

we also conclude that there exists $k = k(a_M) \geq 1$ such that

$$\mathrm{Nm}(\underline{\mathbf{x}}_j) \leq k\nu(|\underline{\mathbf{x}}_j|)^3.$$

Since $q_{j+1} = a_{j+1}q_j + q_{j-1}$ we get $q_{j+1} \ll q_j$ and, by GIVE REFERENCE (FOR MAT412 THM.3), $|p_{j+1} - q_{j+1}\alpha| < |p_j - q_j\alpha|$. Furthermore, (p_j, q_j) and (p_{j+1}, q_{j+1}) are linearly independent, and thus $\underline{\mathbf{x}}_j$ and $\underline{\mathbf{x}}_{j+1}$ are linearly independent. Hence, we conclude

$$\lambda_2(\Lambda, B_{\underline{\mathbf{x}}_j}) \ll 1,$$

and thus, by virtue of (6.3), we get $\mathcal{E}_\Lambda(Z_{\boldsymbol{Q}_j}) \gg N_j^2$. Moreover, we have

$$|\underline{\mathbf{x}}_{j-1}| < |\underline{\mathbf{x}}_j| \ll |\underline{\mathbf{x}}_{j-1}| \tag{6.8}$$

and thus

$$\nu(|\underline{\mathbf{x}}_j|) \leq \mathrm{Nm}(\underline{\mathbf{x}}_{j-1})^{1/3} \ll |\underline{\mathbf{x}}_{j-1}|^{-1/3} \ll |\underline{\mathbf{x}}_j|^{-1/3}. \tag{6.9}$$

Combining (6.7), (6.8) and (6.9) implies that

$$\varrho^{-1/3} \ll \nu(\varrho) \ll \varrho^{-1/3}.$$

Therefore, we have

$$N_j \ll \nu(|\underline{\mathbf{x}}_j|)^{-3} \ll |\underline{\mathbf{x}}_j| \ll q_j \leq |\underline{\mathbf{x}}_j| \ll \nu(|\underline{\mathbf{x}}_j|)^{-3} \ll N_j.$$

Thus, $N_j^2 \ll Q_{max} = N_j q_j \ll N_j^2$, and due to (6.1), $\overline{Q} \ll N_j^{2/3}$. Hence, with $B = N_j$ we have

$$\frac{\overline{Q}}{\nu(B)} \ll \frac{Q_{max}}{B},$$

and thus for all j large enough

$$\inf_{0<B\leq Q_{max}} \left(\frac{\overline{Q}}{\mu(\Lambda, B)} + \frac{Q_{max}}{B}\right)^2 \ll \left(\frac{Q_{max}}{B}\right)^2 \ll N_j^2 \ll \mathcal{E}_\Lambda(Z_{\boldsymbol{Q}_j}).$$

Hence, we have shown that (6.4) holds true. Finally, we observe that $\mathrm{Vol} Z_{\boldsymbol{Q}_j} = 8N_j^3\mathrm{Nm}(\underline{\mathbf{x}}_j) \gg N_j^2$ which tends to infinity. □

7. $\mathcal{F}_{\kappa,M}$ - Families via o-minimality

In this section let $d \geq 1$ and $D \geq 2$ both be integers. For $Z \subset \mathbf{R}^{d+D}$ and $T \in \mathbf{R}^d$ we write $Z_T = \{x \in \mathbf{R}^D; (T,x) \in Z\}$ and call this the fiber of Z above T. For the convenience of the reader we quickly recall the definition of an o-minimal structure following [5]. For more details we refer to [16, 5] and [13].

Definition 3. *A structure (over $\mathbf{R}$) is a sequence $\mathcal{S} = (\mathcal{S}_n)_{n\in\mathbf{N}}$ of families of subsets in $\mathbf{R}^n$ such that for each n:*

(1) *$\mathcal{S}_n$ is a boolean algebra of subsets of $\mathbf{R}^n$ (under the usual set-theoretic operations).*
(2) *$\mathcal{S}_n$ contains every semi-algebraic subset of $\mathbf{R}^n$.*
(3) *If $A \in \mathcal{S}_n$ and $B \in \mathcal{S}_m$ then $A \times B \in \mathcal{S}_{n+m}$.*
(4) *If $\pi : \mathbf{R}^{n+m} \to \mathbf{R}^n$ is the projection map onto the first n coordinates and $A \in \mathcal{S}_{n+m}$ then $\pi(A) \in \mathcal{S}_n$.*

An o-minimal structure (over $\mathbf{R}$) is a structure (over $\mathbf{R}$) that additionally satisfies:

(5) *The boundary of every sets in $\mathcal{S}_1$ is finite.*

The archetypical example of an o-minmal structure is the family of all semialgebraic sets.

Following the usual convention, we say a set A is definable (in $\mathcal{S}$) if it lies in some $\mathcal{S}_n$. A map $f : A \to B$ is called definable if its graph $\Gamma(f) := \{(x, f(x)); x \in A\}$ is a definable set.

Proposition 7.1. *Suppose $Z \subset \mathbf{R}^{d+D}$ is definable in an o-minimal structure over $\mathbf{R}$, and assume further that all fibers Z_T are bounded sets. Then there exist constants κ_Z and M_Z depending only on Z (but independent of T) such that the fibers Z_T lie in $\mathcal{F}_{\kappa_Z,M_Z}$ for all $T \in \mathbf{R}^d$.*

Suppose the set Z is defined by the inequalities

$$f_1(T_1,\ldots,T_d,x_1,\ldots,x_D) \leq 0,\ldots,f_k(T_1,\ldots,T_d,x_1,\ldots,x_D) \leq 0, \tag{7.1}$$

where the f_i are certain real valued functions on $\mathbf{R}^{D+d}$. If all these functions f_i are definable in a common o-minmal structure then we can apply Proposition 7.1. This happens for instance if the f_i are restricted analytic functions[9] or polynomials in $z_1,\ldots,z_{d+D}$ and each $z_i \in \{x_i, \exp(x_i)\}$. For more details and examples we refer to [16, 6, 7].

For the proof of Proposition 7.1 we shall need the following lemma. We are grateful to Fabrizio Barroero for alerting us to Pila and Wilkies Reparametrization Lemma for definable families and its relevance for the lemma.

Lemma 7.1. *Suppose $Z \subset \mathbf{R}^{d+D}$ is definable in an o-minimal structure over $\mathbf{R}$, and assume further that all fibers Z_T are bounded sets. Then there exist constants κ_Z and M_Z depending only on Z such that the boundary ∂Z_T lies in Lip$(D, M_Z, \kappa_Z \cdot diam(Z_T))$ for every $T \in \mathbf{R}^d$.*

Proof. First note that if $|Z_T| \leq 1$ then ∂Z_T lies in Lip$(D,1,0)$. Hence, it suffices to prove the claim for those T with $|Z_T| \geq 2$. By replacing Z with the definable set $\{(T,x) \in Z; (\exists x,y \in Z_T)(x \neq y)\}$ we can assume that $|Z_T| \geq 2$ for all $T \in \pi(Z)$ where π is the projection to the first d coordinates. We use the existence of definable Skolem functions. By [13, Ch.6, (1.2) Proposition] there exists a definable map $f : \pi(Z) \to \mathbf{R}^d$ whose graph $\Gamma(f) \subset Z$. The proof of said (1.2) Proposition actually shows that there is an algorithmic way to construct the Skolem function f. We will use the fact that this choice of f is determined by Z and π and hence can be seen as part of the data of Z.

[9]By a restricted analytic function we mean a real valued function on $\mathbf{R}^n$, which is zero outside of $[-1,1]^n$, and is the restriction to $[-1,1]^n$ of a function, which is real analytic on an open neighborhood of $[-1,1]^n$.

Now we consider the set $Z' = \{(T,y); (T,x) \in Z, y = x - f(T)\}$. This set is again definable and each non-empty fiber contains the origin, i.e., $0 \in Z'_T$ for all $T \in \pi(Z)$. Next we scale the fibers and translate by the point $y_0 = (-1/2)(1,\ldots,1) \in \mathbf{R}^D$ to get a new definable set whose fibers all lie in $(0,1)^D$. We put $Z'' = \{(T,z); (T,y) \in Z', z = (3\cdot \mathrm{diam}(Z'_T))^{-1}y - y_0\}$ (recall that $\mathrm{diam}(Z'_T) = \mathrm{diam}(Z_T) > 0$ since Z_T has at least two points). We note that the graph of the function $T \to \mathrm{diam}(Z_T)$ from $\pi(Z)$ to $\mathbf{R}$ is given by

$$\{(T,t) \in \pi(Z) \times \mathbf{R}; \phi(T,t) \wedge \neg((\exists u \in \mathbf{R})(\phi(T,u) \wedge u < t)\}$$

where $\phi(T,t)$ stands for $(\forall x, y \in Z_T)(|x-y| \leq t)$. This shows that the aforementioned map is definable and hence, so is Z''. Also we have $Z''_T \subset (0,1)^D$ for all T. By [1, Lemma 3.15] the set $Z''' = \{(T,w); w \in \partial Z''_T\}$ is also definable. The fibers of a definable set are again definable (cf. [1, Lemma 3.1]), and hence by [13, Ch.4, (1.10) Corollary] we have $\dim \partial Z''_T \leq D-1$. From Pila and Wilkie's Reparameterization Lemma for definable families [5, 5.2. Corollary] we conclude that $\partial Z''_T$ lies in $\mathrm{Lip}(D, M_{Z'''}, \kappa_{Z'''})$ for all $T \in \mathbf{R}^d$ with certain constants $\kappa_{Z'''}$ and $M_{Z'''}$. Rescaling and retranslating gives $\partial Z_T \in \mathrm{Lip}(D, M_{Z'''}, \kappa_{Z'''} \cdot \mathrm{diam}(Z_T))$. Finally, we note that Z''' depends only on Z and f which itself can be seen as part of the data of Z, so that the constants $\kappa_{Z'''}$ and $M_{Z'''}$ may be chosen to depend only on Z. This completes the proof of the lemma. □

We can now prove Proposition 7.1. Consider the set

$$Z'''' := \{(\varphi, T, x); \varphi \in \mathrm{GL}_D(\mathbf{R}), x \in \varphi(Z_T)\}.$$

This set is definable in the given o-minimal structure, and we have $Z''''_{(\varphi,T)} = \varphi(Z_T)$. Applying Lemma 7.1 to the fibers $Z''''_{(\varphi,T)}$ we conclude that there exist constants $\kappa_{Z''''}$ and $M_{Z''''}$ such that $\partial\varphi(Z_T)$ lies in $\mathrm{Lip}(D, M_{Z''''}, \kappa_{Z''''} \cdot \mathrm{diam}(\varphi(Z_T)))$ for all $(\varphi, T) \in \mathrm{GL}_D(\mathbf{R}) \times \mathbf{R}^d$. Note that Z'''' depends only on Z so that $M_{Z''''}, \kappa_{Z''''}$ are depending only on Z, and this completes the proof of Proposition 7.1.

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