ASYMPTOTIC PROPERTIES OF RANDOM WALKS ON GRAPHS

Proposal for an FWF project by
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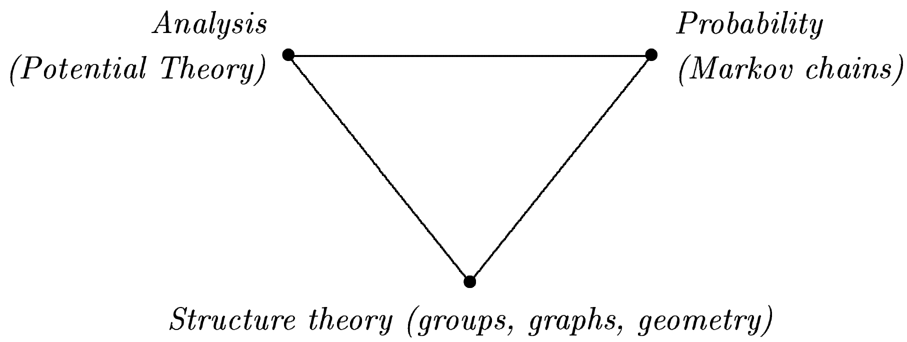
1. Introduction

In order to describe the general theme of the subject, let me quote from the preface to my book Woess (2000).

" 'Random walks' is a topic situated somewhere in between probability, potential theory, harmonic analysis, geometry, graph theory, and algebra. The beauty of the subject stems from this linkage, both in the way of thinking and in the methods employed, of different fields.

Let me briefly declare what - in my viewpoint - random walks are. These are time-homogeneous Markov chains whose transition probabilities are in some way (to be specified more precisely in each case) adapted to a given structure of the underlying state space. This structure may be geometric or algebraic; here it will be discrete and infinite. Typically, we shall use locally finite graphs to view the structure. This also includes groups via their Cayley graphs. From the probabilistic viewpoint, the question is what impact the particular type of structure has on various aspects of the behaviour of the random walk, such as transience/recurrence, decay and asymptotic behaviour of transition probabilities, rate of escape, convergence to a boundary at infinity and harmonic functions. Vice versa, random walks may also be seen as a nice tool for classifying, or at least describing the structure of graphs, groups and related objects."

Thus, this type of research is located within a triangle



With the present project, I propose to pursue in more detail several specific features of random walks, mainly on trees and tree-like structures of the following type: (A) trees with finitely many cone types, among which the so-called comb lattices, (B) a more general class of "context free" graphs, (C) transitive graphs with infinitely many ends, (D) products of certain trees, (E) the Diestel-Leader graphs. The specific features that I have in mind are: (a) asymptotic behaviour of transition probabilities, (b) asymptotic behaviour of Green and Martin kernels and the Martin boundary, (c) spatial behaviour of random walk trajectories, (d) internal diffusion limited aggregation.

2. State of the art

The mathematical theory of random walks on (infinite) discrete structures as it is understood today has its main roots in the articles by Pólya (1921) and Kesten (1959). The first studied recurence/transience of random walk on the integer lattices $\mathbb{Z}^d$, and the second founded the theory of random walks on groups, and in particular on free groups. In the last 20 years or so, work in this field has increased considerably, and the study has been widened from finitely generated groups to random walks on graphs in general – the latter subject was (re-)popularized mainly by the beautiful little book of Doyle and Snell (1984). It is also noteworthy that two major breakthroughs of the 1980ies were made possible by the structure-theoretical work of Gromov, namely the classification by Varopoulos (1986) of the recurrent finitely generated groups – based on the classification by Gromov (1981) of the groups with polynomial growth – and the complete description of the Martin boundary for random walks on hyperbolic graphs by Ancona (1988) – based on Gromov's (1987) theory of hyperbolic metric spaces. Another seminal contribution of the 1980ies was the fundamental work on Poisson boundaries of random walks on groups by Kaimanovich and Vershik (1983).

The 1990ies saw not only a significant increase, but also a significant diversification of the topics studied, as well as an increasing number of researchers or – typically small – research groups with different mathematical background (probability, analysis, combinatorics, ...) working on random walks all over the world. At a distance of 10 years or less, it is still hard to say which are the definite higlights of this period.

At the beginning of the 1990ies, Lalley (1993) settled an important question regarding finite range random walks on free groups, namely that the n-step return probabilities always behave like $\rho^n \cdot n^{-3/2}$. His compex-analysis-technique and efficient proof-strategy leads to a major extension of the much more straightforward nearest-neighbour-case that had been studied previously by Gerl and Woess (1986).

The analytic / functional analytic approach initiated by Varopoulos, whose first phase (until 1991) is exposed in the book by Varopoulos, Saloff-Coste and Coulhon (1992), was pursued further not only by Varopoulos (1996, etc.), but also extended towards significant new directions by his two (former) collaborators: the main topics are heat kernel estimates on manifolds and transition probability estimates on graphs and groups. Coulhon and Grigoryan (1998, 1999) and Coulhon, Grigoryan and Pittet (2001) have developped a strong machinery for giving upper and lower on- and off-diagonal estimates; Saloff-Coste (1995) and Pittet and Saloff-Coste (2000, 2001) [these are only a few selected references] have obtained very refined concrete on-diagonal estimates for random walks on specific classes of groups, in particular wreath-products, exhibiting a wealth of different possible behaviours.

A book that is quite important in the context of potential theoretic aspects of random walks is Soardi (1996), which deals with the correspondence with electric networks and with Dirichlet finite functions and their properties.

The 1990ies also saw the emergence of a very strong Israeli-US group (Benjamini, Lyons, Pemantle, Peres, Schramm) working (among other topics) on random walks, mostly from a much more probabilisitc viewpoint than the references of the last two paragraphs, but also in part with a strong potential theoretic component. Starting points where the papers by Lyons (1990) and Benjamini and Peres (1992) on random walks on trees, potential theoretic aspects and (in the first of the two) links with percolation. A few selected further highlights are the papers by Lyons, Pemantle and Peres (1996), where a surprising result on the rate of escape of biased ("homesick") random walk on certain groups is exhibited, and by Benjamini, Lyons, Peres and Schramm (2001), which contains a profound study of uniform spanning forest measures on infinite graphs and their links with harmonic Dirichlet functions, recurrence of random walks, amenability and harmonic measure; one of my personal favourites is Benjamini and Schramm (1996), which links the geometry of circle packings with random walks and harmonic functions on the associated graphs. More recent work of these authors goes more in the direction of percolation on (general, transitive) graphs, see Benjamini, Lyons, Peres and Schramm (1999) and the forthcoming book by Lyons with Peres (2002).

A very remarkable series of contributions in the 1990ies came from a single person, Kaimanovich, who – among other – extended and perfected his methods for determining the Poisson boundary of random walks on groups (and related structures). This boundary is a measure space that describes the set of limit points attained by the (transient) random walk at infinity, and provides an integral representation of bounded harmonic functions. See for example the papers Kaimanovich (1995, 2000) and by Kaimanovich and Masur (1996).

At the borderline between Random Walk and Geometric Group Theory, several highly interesting directions of work were opened by Grigorchuk, who became famous for the first construction of finitely generated groups with intermediate growth, Grigorchuk (1984). In particular, Bartholdi and Grigorchuk (2000) and Grigorchuk and Zuk (2001) have introduced innovative techniques for determining the spectra of random walk operators on groups which can be generated in terms of certain finite automata.

Let me remark that in this selection of recent work, I have not touched much of the study of "classical" random walks on the integer lattices $\mathbb{Z}^d$, where again one could present a long list of impressive contributions. For short, let me mention the books from the beginning of the 1990ies by Lawler (1991) and Révész (1990).

At the end of this selective and incomplete outline, that also indicates several of the actual directions of research, let me briefly come back to my own work in the last decade. For quite some time, one of my main concerns was topological boundary theory of random walks, see e.g. Cartwright, Soardi and Woess (1993) or Picardello and Woess (1994). Personally, I think that my best research paper of the 1990ies was Cartwright, Kaimanovich and Woess (1994), which gives a quite complete description of the behaviour in space (rate of escape, central limit theorem, boundary) of random walks on the "affine group" of a homogeneous tree. But probably my most important contribution was the book Woess (2000), in which I tried to give a comprehensive picture of central parts of the theory of random walks; many further references (up to 1999) can be found there.

In conclusion, I would like to mention the beautiful and prominently published introductory article by SALOFF-COSTE (2001).

3. MORE DETAILS

A. Trees with finitely many cone types.

Let T be a locally finite, infinite tree with root o. The root induces an orientation, where $x \xrightarrow{*} y$ for two vertices x, y if x lies on the geodesic from o to y. Given x, the cone C_x of T is the subtree rooted at x that is spanned by all y with $x \xrightarrow{*} y$. We say that T has *finitely many cone types* when the number of isomorphism classes of the rooted trees C_x, $x \in T$, is finite. A similar definition can be given for general rooted (connected) graphs, in paticular Cayley graphs of groups, and in the latter context the notion of cone types has led to the very popular theme of *automatic groups,* see the book by EPSTEIN WITH COAUTHORS (1992). The name "automatic groups" is motivated by the connection with automata and languages. The cone types are called *irreducible,* if each cone contains a sub-cone of every other type.

Various aspects of random walks on trees with irreducible cone types have been studied by NAGNIBEDA AND WOESS (2000) and also (simultaneously) by LALLEY (2000). Among other, NAGNIBEDA AND WOESS (2000) derive a criterion for transience / null recurrence / positive recurrence, a local limit theorem (giving the asymptotic behaviour of the return probabilities), a formula for the (linear) rate of escape and a central limit theorem for the distance from the root.

An interesting problem that I propose to study on these structures is *internal diffusion limited aggregation (IDLA).* This is the following process that has been "invented" by DIACONIS AND FULTON (1991) for random walks on $\mathbb{Z}$ and can be described on any (locally finite) graph with root o as follows. One grows a sequence of random "clouds" $C(n)$ which are connected subgraphs with n points containing the origin. One starts with $C(0) = \{o\}$, and if $C(n)$ is already given, then a random walk is started at o (independent of the past) and runs until it first hits a point outside $C(n)$. This point is then added to the cloud to obtain $C(n+1)$. This process has various theoretical and practical applications. For the simple random walk on $\mathbb{Z}^d$, it has been studied in two fundamental papers by LAWLER, BRAMSON AND GRIFFEATH (1992) and LAWLER (1995). The main question is whether there is a "limit shape" of the clouds, the answer being that on $\mathbb{Z}^d$ it is that of the Euclidean ball of the corresponding size, and to estimate the deviation from that shape. More recently, SALOFF-COSTE has been directing research regarding this process, see the papers by BLACHÈRE (2001a,b), where one one hand the results regarding $\mathbb{Z}^d$ are generalized and refined, and on the other hand, a first study of the analogous question for (Cayley graphs of) groups with polynomial growth is undertaken. Furthermore, BLACHÈRE (2001a) also has quite complete results for IDLA with respect to simple random walk on a homogeneous tree.

My principal interest here is to see if and how such results on IDLA can be obtained on structures that are not homogeneous under a transitive group action. Trees with finitely many (irreducible) cone types appear to be one of the simplest classes of graphs of this type, where a detailed study of Green functions (a basic tool) is feasible. For irreducible cone types, one should expect a rather regular behaviour of $C(n)$. The same cannot be expected when the cone types are not irreducible. Here, already the simplest class of trees of this type is quite puzzling, namely the so-called *comb lattices* or *natural spanning trees of* $\mathbb{Z}^d$. Random walks on these structures have been studied by Gerl (1996) and Cassi and Regina (1992), and more recently, my Ph.D. students Bertacchi and Zucca (2001) have shown that simple random walk on these graphs is "anomalous" in the sense that no form of the so-called Einstein relation for exponents associated with random walks can hold here; indeed, they give a uniform space-time estimate of transition probabilities that exhibits different exponents along the two main axes of the two-dimensional comb lattice. Thus, the comb lattices are the simplest "atypically behaving" objects, whence it will be very interesting to study various aspects of random walks on them in order to get a better understanding of what *is* to be considered "typical" or not.

The specific plan here is to start with a diploma (= master) thesis before the end of 2001, not to be included in the present application for financial support. Its topic will be, besides an exposition of previous results, an experimental study via computer simulation of IDLA on the structures indicated above and some other, similar and relatively simple structures (e.g., taking two graphs and glueing them together in one point). Regarding simulation, see the paper by Moore and Machta (1990). In the sequel, suggestions coming from these simulations should become the basis of a more profound research within the present project.

B. Context free graphs.

These graphs are a generalization of trees with finitely many cone types. In reality, they have been introduced prior to the work on automatic groups that also motivated the study of those trees, in a noteworthy, but (among mathematicians) too little known paper by Muller and Schupp (1985). Here, a cone is a subgraph with specified boundary that is obtained from an infinite, rooted, connected graph as a component after the deletion of a ball around the root. Cone types are isomorphism classes of such graphs, and the rest is quite analogous to the definition in **A** above. Such a graph arises as a Cayley graph of a finitely generated group if and only if the group is virtually free, see Woess (1987, 1989). However in the (typical) case when there is no group that acts transitively or with finitely many orbits, basically none of the aspects of random walks that have been mentioned in **A** have yet been studied on these graphs in general.

C. Free products and transitive graphs with infinitely many ends.

Let Γ_1 and Γ_2 be two groups, carrying probability measures μ_1 and μ_2, respectively. Consider the free product $\Gamma = \Gamma_1 * \Gamma_2$, containg each Γ_i as a subgroup, and a probability measure on Γ of the form $a_1 \cdot \mu_1 + a_2 \cdot \mu_2$, where $a_i > 0$ and $a_1 + a_2 = 1$. More than 15 years ago, I found a formula for expressing the Green function (resolvent)

$$G(z) = \sum_{n=0}^{\infty} \mu^{(n)}(id) z^n$$

in terms of the analogous functions associated with the μ_i, see Woess (1986). Precisely simultaneously, in principle the same was obtained by Cartwright and Soardi (1986), and shortly later also by Voiculescu (1986), who did this in a different, much wider C^*-algebraic setting, thus leading to the foundation of *free probability theory*.

In particular, regarding free products, WOESS (1986) provided practicable conditions for the two (or more) factors which yield for a very wide class of cases that the n-step return probabilities to the identity behave like

$$\mu^{(n)}(id) \sim const \cdot \rho(\mu)^n \cdot n^{-3/2} .$$

Here, $\rho(\mu)$ is the "spectral radius" of the random walk. The typical aspect is the term $n^{-3/2}$, already encountered for the free group, see §17.A and §19 of my book WOESS (2000) for details. On the other hand, CARTWRIGHT (1989) gave examples where one free product carries two measures, both supported by the simplest set of generators, where the above type of behaviour holds with terms $n^{-3/2}$ for one measure and $n^{-d/2}$ for the other ($d \neq 3$).

There is a question regarding a better explanation of this phenomenon. That is, one should prove the following conjecture.

Take a group Γ_0 and a probability measure μ_0, and two identical copies $\Gamma_i = \Gamma_0$, $\mu_i = \mu_0$. On the free product $\Gamma = \Gamma_1 * \Gamma_2$, consider $\mu = \frac{1}{2}(\mu_1 + \mu_2)$. Then, under the natural assumption of aperiodicity plus a "technical" condition in the formula for the resolvent – this is the condition $\Psi_0(\theta_0-) > 1/2$ in the terminology of (9.25) in WOESS (2000) – the following should hold.

$$\mu^{(n)}(id) \sim const \cdot \big(\rho(\mu)/\rho(\mu_0)\big)^n \mu_0^{(n)}(id) .$$

(while in the case $\Psi_0(\theta_0-) < 1/2$ it is known from WOESS (1986) that the typical $n^{-3/2}$-behaviour holds).

Free products are also a promising ground for gaining more understanding of IDLA. Much is known about IDLA on $\mathbb{Z}^d$, and the resolvent formula will be a useful tool for studying the same question on the free product $\mathbb{Z}^{d_1} * \mathbb{Z}^{d_2}$ when $d_1 = d_2$ and – even more interesting – $d_1 \neq d_2$. [By the way, there is the analogous question when one takes a copy of $\mathbb{Z}^{d_1}$ and one of $\mathbb{Z}^{d_2}$ and glues the two together at a single point.] Again, I propose that this question should first be studied experimentally in the context of a master thesis (not to be included in the present application for financial support).

Returning to the resolvent formula on free products, at a later stage VOICULESCU (1995) proved in his operator-theoretic setting a generalized formula which in our context corresponds to random walks on free products with amalgamation, apparently yielding practiable methods in the case when the amalgamating subgroup is finite. Here, it will be of great interest to give the correct interpretation of the method in terms of random walk and associated walk generating functions (Green functions). Then I propose to search for reasonable conditions under which one can still obtain the $n^{-3/2}$-behaviour of the return probabilities that was described above.

Free products, and free products with amalgamation over finite subgroups, have Cayley graphs that are tree-like in a certain sense: with exception of a "trivial" case, they have infinitely many ends. This means that such a graph can be split into an arbitrarily large number of connected components by removing a suitable finite piece. Thus, they are transitive graphs with infinitely many ends. However, in general, the structure of the latter graphs can be much more complicated than that of trees and is usually not that of a context-free graph in the sense of **B.** (This can be easily understood for free products: the tree-likeness depends on taking the product, but does not comprise the possibly complicated "inner" structure of the factors.) In general, transitive graphs with infinitely many ends need not be Cayley graphs of free products with amalgamation (nor of HNN-extensions) – indeed, they do not have to be Cayley graphs. However, there is still an analogy, as shown by DUNWOODY (1982) and DICKS AND DUNWOODY (1989). On the basis of this work, THOMASSEN AND WOESS (1993) have shown how one can decompose a (quasi-)transitive graph over a finite, connected piece into connected, (quasi-)transitive graphs. Here, again, one can attempt to use the "combinatorics of paths" to obtain an analogue of the resolvent formula for free products and exploit it for studying the asymptotic behaviour of return probabilities.

D. Products of certain trees.

Let me briefly return to the two-dimensional comb lattice $\mathcal{C}_2$ mentioned in **A** above.

A t-harmonic function is an eigenfunction of the transition operator with eigenvalue t. If one is interested in the positive t-harmonic functions ($t \geq 1$, because otherwise the do not exist), then the appropriate object to study is the *Martin boundary* $\mathcal{M}_t$. Let me refer to §24 in WOESS (2000) for an introduction to this topic in the terminology that is used here.

Since the random walk on $\mathcal{C}_2$ is recurrent, $\mathcal{M}_1$ is trivial (every positive harmonic function is constant). For $t > 1$, one knows from CARTIER (1972) that $\mathcal{M}_t$ coincides with the space of ends of the tree $\mathcal{C}_2$. Note that $\mathcal{C}_2$ is a spanning tree of $\mathbb{Z}^2$. Nevertheless, there is already a noteworthy difference between the Martin boundaries $\mathcal{M}_t(\mathcal{C}_2)$ – which is totally disconnected – and $\mathcal{M}_t(\mathbb{Z}^2)$ – which is ((homeomorphic with) the unit circle. It is still more interesting to descirbe the Martin boundary for (the simple random walk on) the Cartesian product $\mathcal{C}_2 \times \mathbb{Z}$, which is a subgraph of the three-dimensional grid $\mathbb{Z}^3$. In more "algebraic" terms, the so-called *minimal* Martin boundary is well understood on the basis of PICARDELLO AND WOESS (1992), but the whole Martin compactification (including the directions of convergence of the Martin kernel) is known only in very few cases of direct products besides $\mathbb{Z}^d$. For $\mathcal{C}_2 \times \mathbb{Z}$, one possible approach is to extend the uniform space-time asymptotics of BERTACCHI AND ZUCCA (2001) to arbitrary pairs of vertices, and to use this in the same way as PICARDELLO AND WOESS (1994), where the product of two homogeneous trees is considered. Computing the general space-time asymptotics on $\mathcal{C}_2$ seems relatively hard; the difficult situation is when neither starting nor end point lie on the principal axis.

Coming back to the Cartesian product of two homogeneous trees, PICARDELLO AND WOESS (1994) used "brute force" computations to obtain the Martin compactification for simple random walk. There is a method for determining the Martin compactification of the integer grids $\mathbb{Z}^d$ that I learnt from M. BABILLOT and that is exposed in §25.B of WOESS (2000) which I propose to use in this context. This is to use spherical harmonic analysis on each of the two factor trees in the Cartesian product (compare with SAWYER (1978)), and a Laplace transform technique involving the Weierstrass preparation theorem in order to get good asymptotic estimates of Green kernels. Indeed, this should yield results for a wider class of bi-spherical random walks on the product of two homogeneous trees.

E. The Diestel-Leader graphs.

These graphs are fascinating objects. They were "invented" by Diestel and Leader (2000) in an attempt to answer a question posed by myself about 12 years ago: is there a vertex-transitive graph that is not quasi-isometric to a Cayley graph of some finitely generated group ?

The graph $DL(p,q)$ is a subgraph of the direct product $\mathbb{T}_p \times \mathbb{T}_q$ of two homogeneous trees with degree $p+1$ and $q+1$, respectively, as follows. In each of the trees, select an end and consider the horocycle partition that it induces. Write $h(x)$ for the horocycle number of vertex x. The vertex set of $DL(p,q)$ is

$$\{x_1x_2 \in \mathbb{T}_p \times \mathbb{T}_q : h(x_1) + h(x_2) = 0\}.$$

Neighbourhood is the one induced by the direct product ($x_1x_2 \sim y_1y_2 \iff x_1 \sim y_1$ and $x_2 \sim y_2$).

This graph has one end. It is believed that it is not quasi-isometric to any Cayley graph when $p \neq q$. On the other hand, I learnt from R. MÖLLER (private comunication) that $DL(2,2)$ a Cayley graph of the *lamplighter group* $\mathbb{Z} \wr \mathbb{Z}_2$. In general, the Diestel-leader graphs also share some common geometric features with the Cayley graphs of the amenable Baumslag-Solitar groups, although $DL(p,q)$ is a non-amenable graph when $p \neq q$. Various aspects of random walks (e.g. asymptotic behaviour of transition probabilities, rate of escape, central limit theorem) on $DL(p,q)$ have been studied by BERTACCHI (2000). The Poisson boundary of random walks on $DL(p,q)$ has been described in detail by KAIMANOVICH AND WOESS (2000).

The following questions are of great interest: to find the minimal Martin boundary (i.e., the minimal positive harmonic functions), to describe the whole Martin compactification, and to understand how internal diffusion limited aggregation behaves on these graphs. I believe that each of these three problems is more difficult than the preceding one, and that the third and the second are very hard to tackle.

Altogether, the problems described here may be more than a group of two young and one older mathematicians may be able to attack successfully within three years. I believe we all know that the utmost details of one's mathematical research are not easily predictable in every detail in advance. In this sense, the above is a reservoir of questions that are circumscribed by the topic indicated in the title and description of this project – questions for which I believe to a have sufficient background to find a way towards solutions in together with two young collaborators.

4. THE LOCAL ENVIRONMENT

"Institut für Mathematik C" is one of the smaller parts of the Insitute of Mathematics at Graz University of Technology (TU Graz). I came back to Graz only two years ago, after 11 years at the University of Milan (Italy), where I first had been Associate Professor of Mathematical Analysis for 6 years and then Full Professor of Probability for 5 years.

My predecessor here in Graz has employed two assistants, Ao.Prof. O. LABACK (working in Applied Mathematics and Topology) and Ao.Prof. S. FRISCH (working in Ring Theory). Recently, Ao.Prof. D. PÜNGEL (who works in Complex Analysis) has moved from part D to part C of the Institute. All three are tenured.

Since my arrival, I have gradually started to build up a small research group. The fact that I share common interests with Ao.Prof. P. GRABNER of Institut für Mathematik A – see our paper GRABNER AND WOESS (1997) – has been very useful in this respect. He is running a big START project of the FWF, and we have been "sharing" two Ph.D. students, BERNHARD KRÖN and Mag. BERTRAN STEINSKY. They are employed in his project, an both Grabner and myself have been acting as their supervisors (while of course P. Grabner is the official advisor). Mag. B. Krön was a student at Salzburg where I met him during a stay as a visiting professor several years ago; he then came to Milano to write his Master Thesis under my supervision, after which I recommended him to P. Grabner (before coming here myself). He has been working on pure Graph Theory of inifnite graphs on one hand and on random walks on "fractal" graphs and their spectra on the other. He will take his Ph.D. in November 2001 and move to Vienna for a year with a postdoc fellowship under the supervision of Prof. KLAUS SCHMIDT. I hope that he will come back afterwards. Just a few months ago, B. Steinsky also came from Salzburg to Graz, upon suggestion by B. Krön. He is now working on random walks on a class of *radial* trees (where the lengths of the unbranched parts follow the powers of some integer ≥ 2), with the aim to show that the return probabilities exhibit logarithmic fluctuations similar to those of random walk on the Sierpiński graph.

Besides these two, Dipl.-Ing. ELMAR TEUFL is also very close to my research interests, although he is a "pure" student of P. Grabner, working on Brwonian motion on fractals in general and on the rate of escape of random walk on the Sierpiński graph in particular. Both B. Krön (past) and E. Teufl (past and present) hold (held) half-time temporary Assistant positions at Insitut für Mathematik C.

For a year, I had my former Ph.D. student from Milano, Dr. DANIELA BERTACCHI here with an Arge Alps Adria postdoc fellowship. She has been working on the "type problem on the average" for random walks on non-homogeneous graphs, improving and clarifying a classification proposed by BURIONI, CASSI AND VEZZANI (2000). D. Bertacchi has moved back to Milano in October 2001, where she has obtained an assistant (ricercatrice) position at my previous department at Università di Milano-Bicocca, but I expect that she will come to Graz quite often in the future.

Since July 2001, Institut für Mathematik C is hosting Dr. FRANZ LEHNER, who has been awarded a three year FWF fellowship within the Erwin Schrödinger Follow-up Program. Dr. Lehner has graduated in 1993 at Linz University, then obtained his Ph.D. in 1997 at Univ. Paris 6 under the direction of Prof. G. PISIER, after which he held several postdoc positions in France and Denmark, before seizing the possibility of returning to Austria. Dr. Lehner has been working on Spectral Theory within the framework of VOICULESCU's theory of Free Probability, whose rather close link with part of my research has been explained above. It can be expected that within the project proposed here there will be a strong interaction with Dr. Lehner, in particular regarding point **C** in §3 above.

Finally, the student who is expected to start working on his Diploma (= Master) thesis within the next weeks is Mr. WILFRIED HUSS. He will hold a half-time "Studienassistent" position at Insitut für Mathematik C starting with October 1, 2001.

Regarding the local infrastructure, let me mention that this is of course sufficient for hosting two researchers (desks, PC computer, access to libraries, etc.)

5. PROJECT DURATION

This project should start in September 2002 an have a duration of three years.

6. Personnel of the proposed project

The "fixed point" of this project is Dr. Sara Brofferio. She has written her Master Thesis (Tesi di Laurea) under my supervision at the University of Milan and was certainly one of the two best students that I had so far (the other being Mag. B. Krön). We have published a short note, Brofferio and Woess (2001), on random walks on the infinite permutation group. After finishing her studies in Milano, I suggested to her to go to Paris for Ph.D. There, she made the "DEA" in probability at Université de Paris 6. Since then she has been working on her Ph.D. Thesis under the supervision of Prof. Martine Babillot, working on difficult aspects of random walks on the affine and related groups that have applications, among other, to certain autoregressive models. See her paper Brofferio (2001). She will take her Ph.D. degree in late spring, 2002. Since I know her excellent abilities, I am very keen on collaborating with Sara Brofferio in the present Random Walk project after her Ph. D., and she would be interested in coming to Graz for a few years with a PostDoc fellowship.

Since I believe that such a project should also have an educative compenent, the second fellowship that I am applying for is proposed as a two years Ph.D. + one (final year) Post Doc. At the moment the "natural" candidate appears to be W. Huss in the first place. However, I am writing "N.N" in the application form: in this respect I would prefer to maintain some "degree of freedom" at the moment, since a different collaborator from "outside" would be welcome as well, also in the sense of some fruitful competition among those present.

7. International cooperations

My international contacts that will be significant for this project have in part already been mentioned.

Key names are L. Saloff-Coste (Cornell University) and V. A. Kaimanovich (Universté de Rennes). I have written joint papers with both. Currently, I am collaborationg with Saloff-Coste on a project of *computation of norms of transition operators on non-discrete homogeneous spaces* which is very closely related with the present project but will hopefully be concluded soon. In the last decade, both Kaimanovich and Saloff-Coste have emerged as leading mathematicians. While Kaimanovich is an excellent interlocutor for all questions regarding boundary theory, Saloff-Coste has become an expert (among many other topics) on IDLA, having supervised Ph.D. students working on this subject. It will be desirable to send each of the project collaborators to Cornell University for a couple of weeks (also G. Lawler has moved there recently) so that they can interact with these leading reserachers and learn from them. Hopefully both Kaimanovich and Saloff-Coste will have the opportunity to visit Graz in the next years in order to collaborate with us in the present project. (Kaimanovich has already come to Graz as a visiting professor in May 2001, when we prepared the second part of the "Random Walk" semester at the Erwin Schrödinger Insitute in Vienna.)

Regarding methods of complex analysis for studying random walks (one of my fields of expertise, I believe), in the past I have shared my preference for this type of approach with D. I. Cartwright (Sydney University). He has been in Graz as a visiting professor in May-June 2000, and it is likely that I will visit him in 2002, but at the present stage I am not applying for specific funding within this project.

In the last 8 years, there has been a fruitful exchange of ideas and results with a group of theretical physicists from the University of Parma (Italy), lead by Prof. D. Cassi. Athough we have not (yet) written joint papers, this has proven to be very stimulating. On one hand the physicists have many rapid ideas, on the other hand, their methods are not guided by the rigour that mathetmatics usually requires. This has been leading to interesting mathematical research. For example, the papers by Grabner and Woess (1997) and by my former Ph.D. students Bertacchi and Zucca (1999) have been motivated by heuristic arguments that I found in papers pointed out to me by Cassi, and B. Steinsky is actually working on a problem motivated by a paper by R. Burioni, D. Cassi, A. Pirati and S. Regina (1998). Recently, Cassi and myself have applied for a project with the rather generic title "Random walks in Mathematics and Theoretical Physics" within the Italian - Austrian scientific exchange program. There has been some bureaucratical delay, but I hope this will begin soon and give rise to fruitful interaction with the present project. Since the collaboration with the group from Parma is likely to be supported by different sources, I am not applying for specific funding within the present project.

Three very bright younger researchers whom I plan to bring to Graz in the next years within the present project are L. Bartholdi (from Geneva, currently at Berkeley), T. Nagnibeda (Stockholm) and A. Zuk (Lyon/Chicago). Bartholdi is an expert of generating function techniques related with graphs and knows a lot about geometric group theory - a field that is basic for working on random walks. Zuk is an expert of spectral theory and related subjects. Both have recently written very significant papers with R. I. Grigorchuk on spectral theory on groups, related to random walks: Bartholdi and Grigorchuk (2000), Grigorchuk and Zuk (2001). Nagnibeda has collaborated with Grigorchuk on the topic of growth of groups (which has a strong interpaly with random walk theory), see Grigorchuk and Nagnibeda (1997); as mentioned in §3.A, we have written a joint paper on random walks on "automatic" trees, and she will come to Graz as a visiting professor in October-November 2001. It will be highly desirable / necessary to interact with these persons within the present project.

8. Research plan

In the first two years, the postdoc research assistant (i.e., S. Brofferio) should concentrate on the topics outlined in §3 **A, E**, while the topics that are suitable for Ph. D. are §3 **B, C**.

At a later stage (end of second and third year), the focus should move to the topics of §3 **D**.

9. Requested funding

The requested funding is basically for a Post Doc fellowship for three years (S. Brofferio) and a two year Ph.D. + one year Post Doc fellowship (N.N.).

Furthermore, I plan to invite the following persons to Graz for a period of two weeks each: L. Saloff-Coste (spring/summer 2003; collaboration regarding IDLA and Green function estimates related with §3 **A, E**), L. Bartholdi (summer/autumn 2003; collaboration regarding walk generating functions related with §3 **B, C, E**), A. Zuk (spring/summer 2004; collaboration regarding the spectral theory of random walk operators related with §3 **B, C, E**) T. NAGNIBEDA (autumn/winter 2004; collaboration regarding generating function techniques related with §3 **A, B, C**) and V. A. KAIMANOVICH (spring 2005; collaboration regarding boundary theory related with §3 **D, E**).

For each of these five persons, I am estimating travel costs only within Europe at an amount of 400 EURO (5500 ATS). Regarding Saloff-Coste, Bartholdi and Zuk this implies that their visits will have to be coordinated with stays in other European countries. The costs for their stays are estimated on the basis of the official "Reisegebührenvorschrift" (guidelines for travel refunding) (10266 ATS = 741,70 EURO for Saloff-Coste, Zuk and Kaimanovich; 8316 ATS = 604,35 EURO for Bartholdi and Nagnibeda).

Conversely, towards the end of the first year of the project (summer/autumn 2003), S. BROFFERIO should visit Cornell University in order to learn from and interact with L. SALOFF-COSTE, his students, and possibly also with G. LAWLER (collaboration related with §3 **A, E**). Regarding the other project collaborator (N.N.; Ph.D. grant) it seems reasonable to wait with an analogous trip until the end of the second or beginning of the third year (autumn 2004).

Analogous visits to Rennes can probably funded from different sources (e.g. Socrates program for students).

For the two trips to the USA, I am estimating travel costs of 800 EURO (11000 ATS) each, the costs for the stays being "canonical" (13986 ATS = 1016,40 EURO for two weeks).

(in EURO)	1st year	2nd year	3rd year
PostDoc (Brofferio)	44.270, 00	44.270, 00	44.270, 00
PhD (N.N.)	26.750, 00	26.750, 00	
PostDoc (N.N.)			44.270, 00
PostDoc to Cornell U.	1.815, 80		
PhD to Cornell U.		1.815, 80	
Saloff-Coste to Graz	1.145, 76		
Bartholdi to Graz	1.004, 05		
Zuk to Graz		1.145, 76	
Nagnibeda to Graz			1.004, 05
Kaimanovich to Graz			1.145, 76
Total	74.985, 61	73.981, 56	90.689, 81

Grand total: 239.656, 98 EURO

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