

## ERRATA

The preparation of the paperback edition gave me the opportunity to prepare 3 pages containing the corrections of a few misprints (many more will have remained) and two “true” mistakes, as well as a missing reference. Another correction was added in January, 2012 and further improved in January, 2015.

Graz (Austria), September 2007 + January 2012 + January 2015  
Wolfgang Woess

**Page 4, lines 13–14 from top.** If  $n \rightarrow \infty$  then  $q_n/n \rightarrow 1/m$  [in the proof of Lemma 1.9]

**Page 63, line 7 from bottom.** Reference to (6.10) [instead of (6.9)].

**Page 82, lines 12–15 from top.** In (2), (3) and (4):  $\rho(P) < 1$  [instead of  $\rho(P) > 1$ ].

**Page 140, line 7 from bottom.** As was pointed out by L. Saloff-Coste, the application of Green’s identity as stated here is wrong for generic  $\mathbf{y} \in \mathbb{R}^d$ , because then the integrand is not  $2\pi$ -periodic (unless  $\mathbf{m}$  is integer). [Indeed, I copied this mistake from the classical paper by Ney and Spitzer!] The following changes are proposed.

We expand and apply Green’s second identity:

$$\begin{aligned} & \int_{\mathbb{W}_d} \left( \psi(\mathbf{x})^n \Delta^r e^{-i\mathbf{x}\cdot\mathbf{y}} - e^{-i\mathbf{x}\cdot\mathbf{y}} \Delta^r \psi(\mathbf{x})^n \right) d\mathbf{x} \\ &= \sum_{k=0}^{r-1} \int_{\mathbb{W}_d} \left( \Delta^k \psi(\mathbf{x})^n \Delta^{r-k} e^{-i\mathbf{x}\cdot\mathbf{y}} - \left( \Delta^{k+1} \psi(\mathbf{x})^n \Delta^{r-k-1} e^{-i\mathbf{x}\cdot\mathbf{y}} \right) \right) d\mathbf{x} \\ &= \sum_{k=0}^{r-1} I(k, r - k - 1), \end{aligned}$$

where for  $k, l \in \mathbb{N}_0$

$$I(k, l) = \int_{\partial\mathbb{W}_d} \left( \Delta^k \psi(\mathbf{x})^n \frac{\partial}{\partial \mathbf{n}} \Delta^l e^{-i\mathbf{x}\cdot\mathbf{y}} - \Delta^l e^{-i\mathbf{x}\cdot\mathbf{y}} \frac{\partial}{\partial \mathbf{n}} \Delta^k \psi(\mathbf{x})^n \right) d_{\partial} \mathbf{x}.$$

Here,  $\frac{\partial}{\partial \mathbf{n}}$  is the outer normal derivative and  $d_{\partial}$  is Lebesgue measure on the boundary  $\partial\mathbb{W}_d$ . By induction on  $k$ ,

$$\Delta^k \psi(\mathbf{x})^n = \varphi_{n,k}(\mathbf{x}) e^{-i\mathbf{n}\cdot\mathbf{x}}, \quad \text{and} \quad \Delta^l e^{-i\mathbf{x}\cdot\mathbf{y}} = (-1)^l |\mathbf{y}|^{2l} e^{-i\mathbf{x}\cdot\mathbf{y}},$$

where  $\varphi_{n,k}(\mathbf{x})$  is  $2\pi$ -periodic in each variable. Now we decompose  $I(k, l)$  as the sum of  $d$  pairs of integrals over opposite faces of  $\mathbb{W}_d$ . We consider the two faces corresponding to  $x_d = \pm\pi$ . Their respective contribution is

$$(-1)^l |\mathbf{y}|^{2l} \times \pm \int_{\mathbb{W}_{d-1}} e^{-i(\mathbf{x}', \pm\pi) \cdot (n\mathbf{m} + \mathbf{y})} \left( i(nm_d - y_d) \varphi_{n,k}(\mathbf{x}', \pm\pi) - \frac{\partial}{\partial x_d} \varphi_{n,k}(\mathbf{x}', \pm\pi) \right) d\mathbf{x}'$$

If  $n\mathbf{m} + \mathbf{y} = \mathbf{k} \in \mathbb{Z}^d$  then those two contributions cancel by periodicity of  $\varphi_{n,k}$ , and the same is true for all other pairs of opposite faces, whence for such  $\mathbf{y}$ ,

$$|\mathbf{y}|^{2r} \int_{\mathbb{W}_d} \psi(\mathbf{x})^n e^{-i\mathbf{x} \cdot \mathbf{y}} d\mathbf{x} = (-1)^r \int_{\mathbb{W}_d} e^{-i\mathbf{x} \cdot \mathbf{y}} \Delta^r \psi(\mathbf{x})^n d\mathbf{x}.$$

Replacing  $\mathbf{x}$  with  $\frac{1}{\sqrt{n}}\mathbf{x}, \dots$

Now everything continues as on page 140, line 5 from bottom.

**Page 167, line 4 from bottom.** for all  $r, n \in \mathbb{N}$  [instead of  $m, n \in \mathbb{N}$ ].

**Page 170, proof of Theorem 15.15.** The mistake is that the measure  $\mu$  on line 7 is not symmetric. The proof should start as follows.

Let  $\mu_0$  and  $\nu_0$  be the equidistributions on  $\{\mathbf{0}, \pm\mathbf{e}_i : i = 1, \dots, d\} \subset \mathbb{Z}^d$  and on  $\mathfrak{A}$ , respectively. Via the embedding of  $\mathbb{Z}^d$  and  $\mathfrak{A}$  into  $\mathbb{Z}^d \wr \mathfrak{A}$ , both are also considered as measures on the wreath product. For the proof, in view of Corollary 15.5, it is sufficient to consider the random walk on  $\mathbb{Z}^d \wr \mathfrak{A}$  whose law is  $\mu = \nu_0 * \mu_0 * \nu_0$  that is,

$$\mu(y, \eta) = \begin{cases} \mu_0(y)/|\mathfrak{A}|^2, & \text{if } \eta \in \{\eta_a + T_y \eta_b : a, b \in \mathfrak{A}\}, \\ 0, & \text{otherwise.} \end{cases}$$

Since  $\nu_0 * \nu_0 = \nu_0$ , we have  $\mu^{(n)} = (\nu_0 * \mu_0)^{(n)} * \nu_0$ . Consider i.i.d. random variables  $(K_n, V_n)$ , where  $K_n \in \mathbb{Z}^d$  has distribution  $\mu_0$  and the  $\mathfrak{B}$ -valued random variables  $V_n$  are all equidistributed on the set of configurations  $\eta \in \mathfrak{B}$  with  $\text{supp } \eta \subset \{\mathbf{0}\}$ , and  $K_n$  and  $V_n$  are independent. Then  $\mu^{(n)}$  is the distribution of

$$\left( S_n, \sum_{j=1}^{n+1} T_{S_{j-1}} V_j \right) \in \mathbb{Z}^d \wr \mathfrak{A},$$

where  $S_n = K_1 + \dots + K_n$  is the random walk on  $\mathbb{Z}^d$  with law  $\mu_0$ , with  $S_0 = \mathbf{0}$ .

The proof of the lower bound is then precisely as on page 170, taking into account that on line 10 from bottom, the middle term of the inequality has to be  $\mathbb{P}_0[\max\{|S_j| : j \leq n\} \leq r]^2/|A_r|$  [the square was missing].

For the upper bound, the summation over  $j$  on page 171, lines 2–4 from top, should go up to  $n + 1$  instead of  $n$ , so that line 4 becomes

$$= \mathbb{E}_0(|\mathfrak{A}|^{-|D_n|} \mid S_n = \mathbf{0}) \mathbb{P}_0[S_n = \mathbf{0}] = \mathbb{E}_0(|\mathfrak{A}|^{-|D_n|} \mathbf{1}_{[S_n=\mathbf{0}]}) ,$$

after which the proof concludes as before.

**Page 185, line 6 from bottom.** The last term of the sum is  $\dots + C(x|\mathbf{r})\sqrt{\mathbf{r} - z/\xi_\ell}^5$  [coefficient  $C(x|\mathbf{r})$  instead of  $C(x|\mathbf{r}/\xi_\ell)$ ].

**Page 215, line 5 from top.**  $p^{(n)}(x, y) \sim A \left(1 + \frac{q-1}{q+1}d(x, y)\right) \dots$  [asymptotic equivalence instead of equality].

**Page 294–295, Proof of Theorem 27.1.** As pointed out by the late Martine Babillot, there is a mistake in the proof on page 295, lines 8–9: it does not follow from the preceding arguments that  $\frac{1}{1-c_{n-1}}(h_1 - c_{n-1} \cdot h_2) \in \mathcal{C}_\xi$ . We explain how the proof can be repaired by re-ordering the material.

The initial piece remains the same until the displayed formula on page 194, lines 4–3 from bottom, which contains some misprints. The material starting with this formula and ending on page 295, line 11 should be replaced by the following:

$$\begin{aligned} \frac{K(x, y_n)}{K(x, y'_n)} &= \frac{F(x, y_n)F(o, y'_n)}{F(o, y_n)F(x, y'_n)} \\ &\geq \frac{F(x, v)F(v, y_n) F(o, v)F(v, y'_n)}{C(2\delta)F(o, v)F(v, y_n) C(2\delta)F(x, v)F(v, y'_n)} = \frac{1}{C(2\delta)^2} . \end{aligned}$$

Having proved (27.15), we now let  $L_\xi$  be the set of all limit points in the Martin boundary  $\mathcal{M}(P)$  of sequences in  $X$  which converge to  $\xi$  in the hyperbolic topology. Bounded range implies that  $K(\cdot, \alpha) \in \mathcal{H}^+(P)$  for every  $\alpha \in L_\xi$ . By (27.15),  $K(\cdot, \alpha) \geq \varepsilon_1 K(\cdot, \beta)$  for all  $\alpha, \beta \in L_\xi$ .

We next show in Step 2 that there is  $\alpha \in L_\xi$  such that  $K(\cdot, \alpha)$  is minimal harmonic. Then the last inequality will imply that  $K(\cdot, \beta) = K(\cdot, \alpha)$  for all  $\beta \in L_\xi$ , that is,  $L_\xi$  consists of the single point  $\alpha$ . The latter is then the natural image of  $\xi$ , completing Step 1.

*Step 2.* Let  $\pi(o, \xi)$  be a geodesic from  $o$  to  $\xi$ . There must be a sequence  $(x_n)$  of points on  $\pi(o, \xi)$  such that  $|x_{n+1}| > |x_n|$  and  $x_n \rightarrow \alpha \in L_\xi$  in the Martin topology. We define  $\mathcal{H}_\alpha = \{h \in \mathcal{H}^+ : \sup_x h(x)/K(x, \alpha) = 1\}$ . If we can show that  $\mathcal{H}_\alpha = \{K(\cdot, \alpha)\}$  then minimality of  $K(\cdot, \alpha)$  follows.

Setting  $\varepsilon = 1/C(0)$ , Theorem 27.12 yields  $K(x_k, x_n) \geq \varepsilon/F(o, x_k)$  whenever  $0 \leq k \leq n$ . Therefore

$$F(x, x_k)K(x_k, \alpha) \geq \varepsilon K(x, x_k) \quad \text{for all } x \in X .$$

If  $h \in \mathcal{H}^+$  is arbitrary then – using Lemma 27.5 – for all  $x$

$$(27.16) \quad h(x) \geq F(x, x_k) h(x_k) \geq \varepsilon K(x, x_k) \frac{h(x_k)}{K(x_k, \alpha)}.$$

Now let  $h \in \mathcal{H}_\alpha$ , and apply (27.16) to  $h' = K(\cdot, \alpha) - h$ . Then

$$h'(x) \geq \varepsilon K(x, \alpha) \limsup_{k \rightarrow \infty} \frac{h'(x_k)}{K(x_k, \alpha)}.$$

As  $\inf_X (h'/K(\cdot, \alpha)) = 0$ , we must have  $\lim_k (h(x_k)/K(x_k, \alpha)) = 1$ . We use this fact, and apply (27.16) to our  $h \in \mathcal{H}_\alpha$ . Letting  $k \rightarrow \infty$ , we infer  $h \geq \varepsilon K(\cdot, \alpha)$ . This holds for every  $h \in \mathcal{H}_\alpha$ .

Set  $c_n = \varepsilon(1 + (1 - \varepsilon) + \cdots + (1 - \varepsilon)^n)$ . We show inductively that  $h \geq c_n K(\cdot, \alpha)$  for all  $n \geq 0$ . This is true for  $n = 0$ . Suppose it holds for  $n - 1$ . Then the function  $\frac{1}{1 - c_{n-1}}(h - c_{n-1} K(\cdot, \alpha))$  is also an element of  $\mathcal{H}_\alpha$  and  $\geq \varepsilon K(\cdot, \alpha)$ . This yields  $h \geq (c_{n-1} + \varepsilon(1 - c_{n-1}))K(\cdot, \alpha) = c_n K(\cdot, \alpha)$ . Letting  $n \rightarrow \infty$ , we get  $h \geq K(\cdot, \alpha)$ . Therefore  $h = K(\cdot, \alpha)$  for every  $h \in \mathcal{H}_\alpha$ . This concludes the proof of minimality of  $K(\cdot, \alpha)$ , and completes Step 2 and thus also Step 1.

At this point follows – without any change – the old Step 2, which now becomes Step 3, after which the proof is complete. (The old Step 3 has been modified and incorporated into what is now Step 2 above.)

**Missing reference.** It is unforgivable that in the Preface there is no reference to the following book.

Guivarc'h, Y., Keane, M., and Roynette, B.: *Marches Aléatoires sur les Groupes de Lie*, Lect. Notes in Math. **624**, Springer, Berlin, 1977.

Indeed, while I did not use any specific material from that volume in the present monograph, it documents an important phase in the development of the theory of random walks on groups – not discrete ones, but Lie groups.