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★**Random walks on infinite graphs and groups. (English. English summary)**

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A random walk is a time-homogeneous Markov chain $(Z_n)_{n\geq 0}$ whose transition probabilities $p(x,y)$ are adapted in some way to the structure of the underlying state space X. In this book, X is (usually) a locally finite infinite graph. One motivating example is the simple random walk on (the vertex set of) X, with $p(x,y)=1/\deg(x)$ if y is a neighbour of x, and $p(x,y)=0$ otherwise, $\deg(x)$ being the number of neighbours of x. The book discusses many different sorts of adaptedness. For example, if Γ is a group of graph automorphisms of X, it is natural to consider transition probabilities $p(x,y)$ which satisfy $p(\gamma x,\gamma y)=p(x,y)$ for all $x,y\in X$ and $\gamma\in\Gamma$. Besides the simple random walk, Cayley graphs provide another motivating example: the vertex set of X is a group Γ which has a finite generating set S which satisfies $S^{-1}=S$ and $1\notin S$; the neighbours of $x\in\Gamma$ are the elements xs, $s\in S$. If μ is a probability measure on Γ, then $p(x,y)=\mu(x^{-1}y)$ defines transition probabilities which are invariant under the left action of Γ on itself. This action is transitive. A common hypothesis in this book is that, more generally, the $p(x,y)$ are invariant under a group of graph automorphisms which acts quasi-transitively, i.e., with only finitely many orbits.

The book is divided into four chapters. The first chapter is devoted to the question of whether a random walk is transient or recurrent. It starts from the results of Polya's 1921 paper on the simple random walk on $\mathbb{Z}^d$. One highlight is Varopoulos' characterization of the groups Γ carrying a probability measure μ for which the corresponding random walk is recurrent. This is also presented in the more general context of $p(x,y)$ invariant under a quasi-transitive group. Techniques used include Nash-Williams' criterion in terms of electrical networks, rough isometries, growth functions, and isoperimetric inequalities.

The spectral radius $\rho(P)=\limsup_{n\to\infty}p^{(n)}(x,y)^{1/n}$ is the focus of the second chapter. Here P is the matrix $(p(x,y))_{x,y\in X}$, and $(p^{(n)}(x,y))_{x,y\in X}$ is its n-th power. If $\rho(P)<1$, then the random walk (Z_n) is transient, and indeed, under mild conditions there are constants $c_1,c_2>0$ such that, with probability 1, $c_1\leq \operatorname{dist}(Z_0,Z_n)/n\leq c_2$ for large n. The Green function $G(x,y|z)=\sum_{n=0}^{\infty}p^{(n)}(x,y)z^n$ has radius of convergence $1/\rho(P)$, and the operator $f\mapsto Pf$, where

$(Pf)(x) = \sum_{y \in X} p(x,y)f(y)$, acting on an appropriate Hilbert space, has norm $\rho(P)$ in the case when P is reversible. The connection of $\rho(P)$ with isoperimetric inequalities and amenability is studied.

In Chapter 3, the author studies the asymptotic behaviour of $p^{(n)}(x,y) = P(Z_n = y|\ Z_0 = x)$. One of several highlights is Lalley's theorem that $p^{(n)}(e,x) = \mu^{*n}(x)$ is asymptotic to $C_x \rho(P)^n n^{-3/2}$ for some $C_x > 0$, when $p(x,y) = \mu(x^{-1}y)$ for some suitable probability measure μ on a free group.

Finally, in Chapter 4, the author provides a thorough introduction to topological boundary theory of random walks. The idea is to refine the analysis of transient random walks, and find compactifications $\widehat{X}$ of X such that, with probability 1, Z_n converges to a point on the boundary $\partial X = \widehat{X} \setminus X$. This is closely related to the Dirichlet problem: Given a continuous function on ∂X, can it be extended to a continuous function on $\widehat{X}$ which is harmonic on X, i.e., satisfies $h(x) = \sum_{y \in X} p(x,y)h(y)$ for all $x \in X$? Topics covered include: the end compactification, hyperbolic graphs and groups, the hyperbolic compactification of these, the Martin compactification.

The book is full of extensive examples which illustrate the general problems and results of the four chapters. It provides a succinct and thorough analysis of the great progress on these questions up to recent years. It is clearly and carefully written, the expositions of results often much improved on the original. There are extensive notes at the end of each chapter with historical remarks and which lead to further reading. I found only a very small number of typographical errors. It makes a very valuable addition to the literature on this fascinating and important subject. *Donald I. Cartwright* (5-SYD-PM)