

Book review for Acta Math. Sci. (Szeged) on

**Wolfgang Woess:** *Random Walks on Infinite Graphs and Groups* (Cambridge Tracts in Mathematics Vol. 138), xi+334 pages, Cambridge University Press, Cambridge, 2000.

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The theme of this book is the interplay between the behaviour of random walks and the properties of their countable state space. This underlying state space is a (usually locally finite) infinite graph; the most important examples are the Cayley graphs of finitely generated groups and plane tilings. Note that a question on such random walks can always be translated into harmonic analysis on infinite electrical networks.

The main interest is in probabilistic properties of the random walk which reflect the *coarse geometric* structure of the underlying graph — this means properties that are invariant under quasi- (also called rough-) isometries of the graph. For example, having polynomial growth of degree  $d$  is such a geometric property of a graph, and a celebrated theorem of Gromov in geometric group theory says that for groups this is equivalent to being almost-nilpotent (having a nilpotent subgroup of finite index). However, the geometric point of view on random walks does not come from enjoying the comfort of having nice structural theorems for graphs, but is a result of asking natural probabilistic questions. The development of the book reflects this idea very well: each of the four chapters is built around one specific type of question about the behaviour of random walks, and the answers always turn out to be of geometric nature. Besides general results on the underlying graph, more exact answers are always given for various different graphs, such as integer lattices, trees, Cartesian and free products, nilpotent groups, Gromov-hyperbolic groups, plane tilings.

Chapter I (**The type problem**) begins with a nice introduction to random walks and infinite networks, and continues with answering the most natural question: which graphs can carry a recurrent random walk? The whole chapter is basically a well-organized proof of the following answer. *Let  $X$  be a quasi-transitive infinite graph, i.e.  $\text{Aut}(X)$  acts with finitely many orbits on  $X$ . If some quasi-transitive random walk  $(X, P)$  is recurrent, then  $X$  is quasi-isometric to the one- or two-dimensional grid; moreover,  $\text{Aut}(X, P)$  has a discrete subgroup isomorphic to  $\mathbb{Z}$  or  $\mathbb{Z}^2$  which acts quasi-transitively and fixed-point-freely. In this case, every strongly reversible, quasi-transitive random walk on  $X$  with finite second moment is recurrent. Otherwise,  $X$  contains a transient subtree.* The generalization from Varopoulos' characterization of recurrent Cayley graphs to the above result on quasi-transitive graphs is mainly done by pure graph theory. Note that this is a real generalization: while recurrence is easily seen to be quasi-isometry invariant, it is a conjecture that *not every* quasi-transitive graph is quasi-isometric to a Cayley graph. The more subtle results for particular classes of graphs are about generalized lattices and trees, there is a characterization of recurrent circle packings on the plane, and we can find the beautiful result that every quasi-regular plane tiling is recurrent.

If the **spectral radius** of a random walk  $P$  is  $\rho(P) < 1$ , then it is transient; the converse is not true. Chapter II is devoted to an analysis of this finer parameter. The rate of escape is examined for the case  $\rho(P) < 1$ , Green function computations are presented to determine the spectral radius, and connections to strong isoperimetric inequalities are shown. In particular, a graph is amenable (i.e. does not satisfy a strong isoperimetric inequality) iff the simple random walk on it has  $\rho(P) = 1$ . Clearly, amenability is the most important coarse geometric property of graphs.

Chapter III deals with an even more subtle problem: **the asymptotic behaviour of transition probabilities**  $p^{(n)}(x, y)$ ,  $x, y \in X$ . It starts with the local central limit theorem about general measures on  $\mathbb{Z}^d$ , then gives lower and upper bounds on the transition probabilities in terms of isoperimetric inequalities. These bounds are sharp for so-called quasi-homogeneous graphs of polynomial growth and for non-amenable graphs, and also for polycyclic groups and for the solvable Baumslag-Solitar groups (which are amenable groups with exponential growth). Examples for non-sharpness are investigated: the lamplighter groups and the Sierpiński graphs.

Local limit theorems for free and Cartesian products and homogeneous trees are also given.

Chapter IV is **an introduction to topological boundary theory**. This means that we consider a compactification  $\hat{X}$  of the underlying infinite graph  $X$ ,  $\partial X := \hat{X} \setminus X$  is called the boundary, and we would like to answer the following questions. 1. Given a transient random walk  $Z_n$  on  $X$ , does it converge to a random variable  $Z_\infty$  on  $\partial X$ ? 2. *Dirichlet problem at infinity*: given a continuous function  $\partial X \rightarrow \mathbb{R}$ , does it always have a continuous extension on  $X$  which is harmonic with respect to a given random walk? 3. *Identification of the Martin boundary*: does every positive harmonic function have an integral representation over  $\partial X$ ? Each of these questions is finer than the previous one, and answers are given for different classes of graphs, with respect to end compactifications, to Gromov-hyperbolic boundaries, and to the natural boundary for circle packings of the unit disk. Note that these boundaries are quasi-isometric invariant, but there are two quasi-isometric graphs such that one of them has non-constant bounded harmonic functions, while the other does not. We also remark that the word ‘topological’ in the title is to distinguish from the measure theoretic Poisson-Furstenberg-Kaimanovich boundary theory, which is the study of the bounded harmonic functions instead of all positive harmonic ones, and which is considered only marginally in this book.

The organization of the book by the four probabilistic themes instead of saying first ‘everything’ about random walks on integer lattices, then nilpotent groups, trees, hyperbolic groups, etc. made me a little bit difficult to find particular results I was looking for, but makes the ideas and methods very transparent. This feature, together with the plenty of examples and the clarity of the exposition, make the book very suitable for graduate study. For researchers, the respectable list of 357 references, and the historical notes helping to navigate among them, are useful. The main disadvantage of the book is that it contains very little about applications of the results and methods to stochastic problems other than random walks, which connections have been very important since the emergence of a strong group of random walkers (and beyond) in Israel and the USA (Benjamini, Lyons, Pemantle, Peres, Schramm) in the 90’s. The author himself apologizes for this gap a few times, and refers the Reader to the forthcoming book by Lyons and Peres (<http://www.stat.berkeley.edu/~peres>).

The author, who is an acknowledged researcher in the field, ends his preface with the hope that the fun he found in writing this book will infect some of the readers, too. I think that this is one of the most important achievements a book can reach, and I am very glad to recognize that at least one reader has been infected.